# Evolution inclusions of the subdifferential type depending on a parameter

Dimitrios Kandilakis, Nikolaos S. Papageorgiou

*Abstract.* In this paper we study evolution inclusions generated by time dependent convex subdifferentials, with the orientor field $F$ depending on a parameter. Under reasonable hypotheses on the data, we show that the solution set $S(\lambda)$ is both Vietoris and Hausdorff metric continuous in $\lambda \in \Lambda$. Using these results, we study the variational stability of a class of nonlinear parabolic optimal control problems.

*Keywords:* subdifferential, compact type, Vietoris topology, Hausdorff metric, parabolic optimal control problem

*Classification:* 34G20, 49A20

## 1. Introduction.

In this paper we consider the following family of evolution inclusions, defined on a separable Hilbert space $H$ and parametrized by a parameter $\lambda \in \Lambda$, $\Lambda$ being a complete metric space:

$$(\underline{\underline{1}}) \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(t, x(t)) + F(t, x(t), \lambda) \, a.e. \\ x(0) = x_0(\lambda). \end{array} \right\}$$

Here $\varphi(t, \cdot)$ is a proper convex function, $\partial\varphi(t, x)$ denotes its convex subdifferential and $F(t, x, \lambda)$ is a parametrized set-valued perturbation. Let $S(\lambda) \subseteq C(T, H)$ be the set of strong solutions of ($\underline{\underline{1}}$) (see Section 2). In this paper we study the continuity properties of the multifunction $\lambda \to S(\lambda)$. Previously, such continuous dependence results were obtained by Vasilev [19] and Lim [7] for differential inclusions in $\mathbb{R}^n$ and by Tolstonogov [17] and Papageorgiou [9], who considered differential inclusions in Banach space, but without subdifferential operators present. In fact, their formulation of the problem precludes the applicability of their work to multivalued partial differential equations and distributed parameter control systems.

In Section 4, we use our continuous dependence results to study the variational stability of a class of nonlinear, parabolic optimal control problems. Such sensitivity analysis is important from both the theoretical and applied viewpoints, because it produces useful continuous dependence results, it suggests ways to solve parametric problems, it gives us important information on what tolerances are permitted in the specification of the mathematical model and it produces efficient algorithms for the computational analysis of the problem. Our results in Section 4 extend the works of Stassinopoulos-Vinter [16], who studied finite dimensional systems and of Przyluski [14], who examined linear, quadratic optimal control problems, with the parameter appearing only in the control constraint set.

## 2. Preliminaries.

Let $T = [0,r]$ equipped with the Lebesgue measure and the $\sigma$-field of the Lebesgue measurable sets and $X$ a separable Banach space. Throughout this note we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{ nonempty, closed (and convex)}\}$$

and $$P_{(w)k(c)}(X) = \{A \subseteq X : \text{ nonempty, (weakly-) compact, (convex)}\}.$$

A multifunction $F : T \to P_f(X)$ is said to be measurable, if $t \to d(x, F(t)) = \inf\{\|x - z\| : z \in F(t)\}$ is measurable for every $x \in X$. By $S_F^p$, $1 \leq p \leq \infty$, we will denote the set of measurable selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(X)$; i.e. $S_F^p = \{f \in L^p(X) : f(t) \in F(t)\, a.e.\}$. This set may be empty. For a measurable multifunction, it is nonempty if and only if $\omega \to \inf\{\|x\| : x \in F(t)\} \in L_+^p$.

Let $\Lambda$ be a complete metric space and $G : \Lambda \to 2^X \setminus \{\emptyset\}$ a multifunction. We say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)) if and only if for all $C \subseteq X$ closed, the set $G^-(C) = \{\lambda \in \Lambda : G(\lambda) \cap C \neq \emptyset\}$ (resp. $G^+(C) = \{\lambda \in \Lambda : G(\lambda) \subseteq C\}$) is closed. A multifunction $G(\cdot)$ which is both upper and lower semicontinuous, is said to be Vietoris continuous, to emphasize the fact that $G(\cdot)$ is continuous when we endow the hyperspace $2^X \setminus \{\emptyset\}$ with the Vietoris topology. For further details we refer to Klein-Thompson [6].

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by setting for $A, B \in P_f(X)$

$$h(A,B) = \max\left[\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right].$$

Recall that $(P_f(X), h)$ is a complete metric space. A multifunction $G : \Lambda \to P_f(X)$ is said to be Hausdorff continuous ($h$-continuous) if it is continuous from $\Lambda$ into $(P_f(X), h)$. Also it is said to be Hausdorff Lipschitz ($h$-Lipschitz) with constant $k$, if $h(G(\lambda), G(\lambda')) \leq k d_\Lambda(\lambda, \lambda')$ for all $\lambda, \lambda' \in \Lambda$ (here $d_\lambda(\cdot,\cdot)$ denotes the metric on $\Lambda$). In general, Vietoris and Hausdorff continuity are disjoint notions. However, since on $P_k(X)$ the Vietoris and Hausdorff topologies coincide (see Klein-Thompson [6, Corollary 4.2.3, p. 41]), we deduce that a multifunction $G : \Lambda \to P_k(X)$ is Vietoris continuous if and only if it is $h$-continuous.

Let $\{A_n, A\}_{n\geq 1} \subseteq 2^X \setminus \{\emptyset\}$. We define the following limit sets:

$$s\text{-}\underline{\lim} A_n = \{x \in X : \lim d(x, A_n) = 0\} = \{x \in X : x = s\text{-}\lim x_n,\ x_n \in A_n,\ n \geq 1\}$$
$$s\text{-}\overline{\lim} A_n = \{x \in X : \underline{\lim} d(x, A_n) = 0\} = \{x \in X : x = $$
$$= s\text{-}\lim x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < \dots < n_k < \dots\}$$

and

$$w\text{-}\overline{\lim} A_n = \{x \in X : x = w\text{-}\lim x_{n_k},\ x_{n_k} \in A_{n_k},\ n_1 < n_2 < n_3 < \dots < n_k < \dots\},$$

where $s$- denotes the strong topology on $X$ and $w$- the weak topology on $X$. It is clear from the above definitions that we always have

$$s\text{-}\underline{\lim} A_n \subseteq s\text{-}\overline{\lim} A_n \subseteq w\text{-}\overline{\lim} A_n.$$

We say that $A_n$'s converge to $A$ in the Kuratowski sense, denoted by $A_n \xrightarrow{K} A$, if $s\text{-}\underline{\lim} A_n = s\text{-}\overline{\lim} A_n = A$. We say that $A_n$'s converge to $A$ in the Kuratowski-Mosco sense, denoted by $A_n \xrightarrow{K-M} A$, if $s\text{-}\underline{\lim} A_n = w\text{-}\overline{\lim} A_n = A$. If $G : \Lambda \to P_k(X)$ is a multifunction s.t. $\overline{G(\Lambda)} \in P_k(X)$, then $G(\cdot)$ is Vietoris continuous if and only if for $\lambda_n \to \lambda$, we have that $G(\lambda_n) \xrightarrow{K} G(\lambda)$ (see DeBlasi-Myjak [4, the remarks 1.6 and 1.7]).

Now let $H$ be a separable Hilbert space and $\varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We say that $\varphi(\cdot)$ is proper, if it is not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and l.s.c. (usually this family of $\overline{\mathbb{R}}$-valued functions is denoted by $\Gamma_0(H)$). By $\operatorname{dom}\varphi$, we will denote the effective domain of $\varphi(\cdot)$; i.e. $\operatorname{dom}\varphi = \{x \in H : \varphi(x) < \infty\}$. The subdifferential of $\varphi(\cdot)$ at $x \in H$ is the set $\partial\varphi(x) = \{x^* \in H : (x^*, y-x) \le \varphi(y) - \varphi(x)$ for all $y \in \operatorname{dom}\varphi\}$ (here $(\cdot,\cdot)$ denotes the inner product in $H$). If $\varphi(\cdot)$ is Gateaux differentiable, then $\partial\varphi(x) = \{\varphi'(x)\}$. We will say that $\varphi(\cdot)$ is of compact type, if for every $\theta \in \mathbb{R}$, the level set $\{x \in H : \|x\|^2 + \varphi(x) \le \theta\}$ is compact.

By a strong solution of (1), we understand a function $x(\cdot) \in C(T, H)$ s.t. $x(\cdot)$ is strongly absolutely continuous on $(0, b)$, $x(t) \in \operatorname{dom}\varphi(t,\cdot)$ a.e. and satisfies $-\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t)$ a.e. with $f \in L^2(h)$, $f(t) \in F(t, x(t))$ a.e. Recall that a strongly absolutely continuous function from $T$ into $H$ is almost everywhere differentiable.

The following hypothesis concerning $\varphi(t, x)$ will be valid throughout this paper, and is originally due to Yotsutani [23].

<u>$H(\varphi)$ :</u> $\varphi : T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function s.t.

(1) for every $t \in T$, $\varphi(t,\cdot)$ is proper, convex, l.s.c. (i.e. $\varphi(t,\cdot) \in \Gamma_0(H)$) and of compact type,

(2) for any positive integer $r'$, there exists a constant $K_{r'} > 0$, an absolutely continuous function $g_{r'} : T \to \mathbb{R}$ with $\dot{g}_{r'} \in L^{\dot{p}}(T)$ and a function of bounded variation $h_{r'} : T \to \mathbb{R}$ s.t., if $t \in T$, $x \in \operatorname{dom}\varphi(t,\cdot)$ with $\|x\| \le r'$ and $s \in [t, b]$, then there exists $\hat{x} \in \operatorname{dom}\varphi(s,\cdot)$ satisfying

$$\|\hat{x} - x\| \le |g_{r'}(s) - g_{r'}(t)|(\varphi(t,x) + K_{r'})^{\alpha}$$

and $\varphi(s,\hat{x}) \le \varphi(t,x) + |h_{r'}(s) - h_{r'}(t)|(\varphi(t,x) + K_{r'})$ where $\alpha \in [0,1]$, and $\beta = 2$ if $\alpha \in [0, 1/2]$ or $\beta = 1/1-\alpha$ if $\alpha \in [1/2, 1]$.

The following existence theorem is due to Yotsutani [23] and extends earlier important works due to Watanabe [21] and Yamada [22]. In particular, Yamada [22] was the first to consider an interesting application on nonlinear, partial differential equations.

**Theorem 2.1.** *If hypothesis $H(\varphi)$ above holds, then the Cauchy problem $-\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t)$ a.e., $x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot)$ has a unique solution $x(\cdot) = p(f)(\cdot)$, for every $f \in L^2(H)$.*

Let $p : L^2(H) \to C(T, H)$ be the solution map; i.e. $p(f)(\cdot)$ is the unique solution of $-\dot{x}(t) \in \partial\varphi(t, x(t)) + f(t)$ a.e., $x(0) = x_0$. We know from the proof of Theorem 4.1 of [13] that $p(\cdot)$ is sequentially continuous from $L^2(H)$ equipped with the weak topology, into $C(T, H)$ equipped with the strong topology.

The next theorem was first proved by the second author in [9] and was recently improved by Rybinski [15], who relaxed the hypothesis on the Banach space $X$ and also removed the uniform boundedness by a weakly compact set hypothesis (see Theorem 3.1 of [9] and Theorem 1 of [15]). Here we will use the improved version due to Rybinski [15].

**Theorem 2.2.** *If $X$ is a Banach space, $W \in P_{fc}(X)$, $F_n, F : W \to P_{wkc}(X)$ are $h$-Lipschitz multifunctions with the same constant $k \in (0, 1)$ (i.e. $h(F_n(x), F_n(x'))$, $h(F(x), F(x')) \le k\|x - x'\|$) and for $x_n \xrightarrow{s} x$, we have $F_n(x_n) \xrightarrow{K-M} F(x)$, then if $L_n = \{x \in X : x \in F_n(x)\}$ and $L = \{x \in X : x \in F(x)\}$, we have $L_n \xrightarrow{K} L$.*

Note that the sets $L_n, L, n \ge 1$ are nonempty by Nadler's fixed point theorem [8].

The following lemma can be found in [12, Lemma $\delta$].

**Lemma 2.1.** *If $\Lambda$ is a metric space, $X$ is a Banach space, $F : \Lambda \to P_k(X)$ is a multifunction s.t. for every $K \subseteq \Lambda$ compact, $F|_K$ is u.s.c., then $F(\cdot)$ is u.s.c.*

## 3. Continuous dependence results.

Let $S(\lambda) \subseteq C(T, H)$ be the solution set of (1). We know from Theorem 3.1 of [13], that $S(\lambda) \in P_k(C(T, H))$. In this section, we study the continuity properties of multifunction $S : \Lambda \to P_k(C(T, H))$. We will need the following hypothesis on the orientor field $F(t, x, \lambda)$:

$\underline{H(F)}$: $F : T \times H \times \Lambda \to P_{fc}(H)$ is a multifunction s.t.

(1) $t \to F(t, x, \lambda)$ is measurable,
(2) $h(F(t, x, \lambda), F(t, x', \lambda)) \le k_B(t)\|x - x'\|$ a.e. with $k_B(\cdot) \in L^1_+$ for all $\lambda \in B \subseteq \Lambda$, $B$ compact,
(3) $\lambda \to F(t, x, \lambda)$ is $d$-continuous (i.e. $\lambda \to d(y, F(t, x, \lambda))$ is continuous for every $y \in H$),
(4) $|F(t, x, \lambda)| = \sup\{\|y\| : y \in F(t, x, \lambda)\} \le \alpha_B(t) + \beta_B(t)\|x\|$ a.e. with $\alpha_B, \beta_B \in L^2_+$, $\lambda \in B \subseteq \Lambda$, $B$ compact.

Also we will make the following hypothesis for the initial conditions.

$\underline{H_0}$: $x_0 : \Lambda \to H$ is continuous and for all $\lambda \in \Lambda$, $x_0(\lambda) \in \operatorname{dom}\varphi(0, \cdot)$.

**Theorem 3.1.** *If the hypotheses $H(\varphi)$, $H(F)$ and $H_0$ hold, then $S : \Lambda \to P_k(C(T, H))$ is Vietoris continuous.*

Proof: Let $B \subseteq \Lambda$ nonempty and compact. We will obtain an a priori bound for the elements in $\bigcup_{\lambda \in B} S(\lambda)$. To this end, let $x(\cdot) \in \bigcup_{\lambda \in B} S(\lambda)$. So $x(\cdot) \in S(\lambda)$,

$\lambda \in B$. Also let $u(\cdot) \in C(T, H)$ be the unique strong solution of the unperturbed Cauchy problem $-\dot{u}(t) \in \partial\varphi(t, u(t))$ a.e., $u(0) = x_0(\lambda)$. Exploiting the monotonicity of the subdifferential operator, we have

$$(-\dot{x}(t) + \dot{u}(t), u(t) - x(t)) \le (f(t), u(t) - x(t)) \text{ a.e.}$$

for some $f \in L^2(H)$, $f(t) \in F(t, x(t), \lambda)$ a.e. Then we have:

$$\frac{1}{2}\frac{d}{dt}\|x(t) - u(t)\|^2 \le \|f(t)\| \cdot \|x(t) - u(t)\| \text{ a.e.}$$
$$\Rightarrow \frac{1}{2}\|x(t) - u(t)\|^2 \le \int_0^t \|f(s)\| \cdot \|x(s) - u(s)\|\, ds.$$

Applying Lemma A.5, p. 157 of Brezis [3], we get

$$\|x(t) - u(t)\| \le \int_0^t \|f(s)\|\, ds \le \int_0^t (\alpha_B(s) + \beta_B(s)\|x(s)\|)\, ds$$
$$\Rightarrow \|x(t)\| \le \|u\|_{C(T,H)} + \int_0^t (\alpha_B(s) + \beta_B(s)\|x(s)\|)\, ds,\ t \in T.$$

Invoking Gronwall's lemma, we deduce that there exists $M_B > 0$ s.t. for all $x \in \bigcup_{\lambda\in B} S(\lambda)$, we have

$$\|x\|_{C(T,H)} \le M_B.$$

So without any loss of generality, we may assume that for all $\lambda \in B$

$$|F(t, x, \lambda)| = \sup\{\|y\| : y \in F(t, x, \lambda)\} \le \psi_B(t) = \alpha_B(t) + \beta_B(t)M_B \text{ a.e.},$$

with $\psi_B(\cdot) \in L^2_+$ (see hypothesis $H(F)$ (4)).

On $L^1(H)$ consider the equivalent norm $\|g\|_B = \int_0^r \exp[-L\int_0^t k_B(s)\, ds]\|g(s)\|\, ds$. We will show that the multifunctions $g \to R(g, \lambda) = S^1_{F(\cdot,p(g)(\cdot),\lambda)}$ are $h_B$-Lipschitz on $W_B = \{g \in L^1(H) : \|g(t)\| \le \psi_B(t) \text{ a.e}\}$ with the same Lipschitz constant $\hat{k}_L \in (0, 1)$ for $L > 1$. So let $g_1, g_2 \in W_B$ and let $v_1 \in R(g_1, \lambda)$. Then let $\Gamma(t) = \{z \in F(t, p(g_2)(t), \lambda) : d_B(v_1(t), F(t, p(g_2)(t), \lambda)) = \|v_1(t) - z\|\}$. We have $\Gamma(t) \ne \emptyset$ for all $t \in T$ and $Gr\Gamma \in \Sigma \times B(H)$, $B(H)$ being the Borel $\sigma$-field of $H$ (see the hypotheses $H(F)$ (1) and (2) and use Theorem 3.3 of [11]). Apply Aumann's selection theorem (see Wagner [20, Theorem 5.10]), to get $w : T \to H$ measurable s.t. $w(t) \in \Gamma(t)$, $t \in T$. Then we have $d(v_1(t), F(t, p(g_2)(t), \lambda)) = \|v_1(t) - w(t)\|$, $t \in T$ and so

$$\begin{aligned}
&d_B(v_1, R(g_2, \lambda)) \le \|v_1 - w\|_B \\
&= \int_0^r \exp\left[-L\int_0^t k_B(s)\, ds\right] \|v_1(t) - w(t)\|\, dt \\
&\le \int_0^r \exp\left[-L\int_0^t k_B(s)\, ds\right] h(F(t, p(g_1)(t), \lambda), F(t, p(g_2)(t), \lambda))\, dt \\
&\le \int_0^r \exp\left[-L\int_0^t k_B(s)\, ds\right] k_B(t)\|p(g_1)(t) - p(g_2)(t)\|\, dt \\
&= -\frac{1}{L}\int_0^r \|p(g_1)(t) - p(g_2)(t)\|\, d\left[\exp(-L\int_0^t k_B(s)\, ds)\right].
\end{aligned}$$

Exploiting the monotonicity of the subdifferential, we can check that

$$\|p(g_1)(t) - p(g_2)(t)\| \leq \int_0^t \|g_1(s) - g_2(s)\|\, ds,\ t \in T.$$

So we have:

$$\begin{aligned} d_B(v_1, R(f_2, \lambda)) &\leq -\frac{1}{L} \int_0^r (\int_0^t \|g_1(s) - g_2(s)\|\, ds)\, d \left[\exp(-L \int_0^t k_B(s)\, ds\right] \\ &= \frac{1}{L} \int_0^r \exp\left[-L \int_0^t k_B(s)\, ds\right] \|g_1(t) - g_2(t)\|\, dt \leq \frac{1}{L} \|g_1 - g_2\|_B\,. \end{aligned}$$

Similarly if $v_2 \in R(g_2, \lambda)$ , we get that

$$d_B(v_2, R(g_1, \lambda)) \leq \frac{1}{L} \|g_1 - g_2\|_B\,.$$

Therefore , we conclude that

$$h(R(g_1, \lambda), R(g_2), \lambda)) \leq \frac{1}{L} \|g_1 - g_2\|_B\,,\ L > 1.$$

Next we will show that if $[f_n, \lambda_n] \to [f, \lambda]$ in $(W_B, \|\cdot\|_B) \times B$, then $R(f_n, \lambda_n) \xrightarrow{K-M} R(f, \lambda)$. So let $g \in R(f, \lambda)$ and set $\gamma_n(t) = d(g(t), F(t, p(f_n)(t), \lambda_n))$. We have:

$$\begin{aligned} \gamma_n(t) &= d(g(t), F(t, p(f_n)(t), \lambda_n)) \\ &\leq d(g(t), F(t, p(f)(t), \lambda_n)) + h(F(t, p(f_n)(t), F(t, p(f)(t), \lambda_n)) \\ &\leq d(g(t), F(t, p(f)(t), \lambda_n)) + k_B(t) \|p(f_n)(t) - p(f)(t)\| \text{ a.e.} \end{aligned}$$

Because of the hypothesis $H(F)$ (3), we have

$$d(g(t), F(t, p(f)(t), \lambda_n)) \to 0 \text{ as } n \to \infty,$$

while from the continuity of the solution map $p(\cdot)$, we have

$$\|p(f_n)(t) - p(f)(t)\| \to 0 \text{ as } n \to \infty.$$

Thus we get $\gamma_n(t) \to 0$ a.e. as $n \to \infty$.

Let $H_n(t) = \{v \in F(t, p(f_n)(t), \lambda_n) : \|v - g(t)\| \leq \gamma_n(t) + \frac{1}{n}\} \neq \emptyset$. As above, using the hypotheses $H(F)$ (1) and (2) and Theorem 3.3 of [11], we can get that

$$GrH_n \in B(T) \times B(H).$$

Apply Aumann's selection theorem to get $g_n : T \to H$, $n \geq 1$ measurable functions s.t. $g_n(t) \in F(t, p(f_n)(t), \lambda_n)$ a.e. $\|g_n(t) - g(t)\| \leq \gamma_n(t) + \frac{1}{n} \to 0$ a.e. as

$n \to \infty \Rightarrow g_n \to g$ in $(W_B, \|\cdot\|_B)$. Since $g_n \in R(f_n, \lambda_n)$, $n \geq 1$, we have established that

$$(\underline{\underline{2}}) \qquad R(f, \lambda) \subseteq s\text{-}\underline{\lim} R(f_n, \lambda_n).$$

Next let $g \in w\text{-}\overline{\lim} R(f_n, \lambda_n)$. Denoting subsequences with the same index as the original sequences, we know that we can find $g_n \in R(f_n, \lambda_n)$ s.t. $g_n \xrightarrow{w} g$ in $L^1(H)$ and clearly because $\|g_n(t)\| \leq \psi_B(t)$ a.e. with $\psi_B(\cdot) \in L^2_+$, we also have $g_n \xrightarrow{w} g$ in $L^2(H)$. Apply Theorem 3.1 of [10] to get

$$g(t) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\{f_n(t)\}_{n \geq 1} \subseteq \overline{\text{conv}}\, w\text{-}\overline{\lim} F(t, p(f_n)(t), \lambda_n) \text{ a.e.}$$

Observe that for any $v \in H$, we have

$$\begin{aligned}
&d(v, F(t, p(f)(t), \lambda_n)) \\
&\leq d(v, F(t, p(f_n)(t), \lambda_n)) + h(F(t, p(f)(t), \lambda_n), F(t, p(f_n)(t), \lambda_n)) \\
&\leq d(v, F(t, p(f_n)(t), \lambda_n)) + k_B(t)\|p(f_n)(t) - p(f)(t)\|.
\end{aligned}$$

Passing to the limit as $n \to \infty$ and using the hypothesis $H(F)\,(\underline{3})$, we get $d(v, F(t, p(f)(t), \lambda)) \leq \underline{\lim} d(v, F(t, p(f_n)(t), \lambda_n))$.

From Theorem 2.2. (iv) of Tsukada [18], we deduce that

$$\begin{aligned}
&w\text{-}\overline{\lim} F(t, p(f_n)(t), \lambda_n) \subseteq F(t, p(f)(t), \lambda) \text{ a.e.} \\
&\Rightarrow g(t) \in F(t, p(f)(t), \lambda) \text{ a.e.; i.e. } g \in R(f, \lambda).
\end{aligned}$$

So we have proved that

$$(\underline{\underline{3}}) \qquad w\text{-}\overline{\lim} R(f_n, \lambda) \subseteq R(f, \lambda).$$

From $(\underline{\underline{2}})$ and $(\underline{\underline{3}})$ above, we deduce that

$$R(f_n, \lambda_n) \xrightarrow{K-M} R(f, \lambda).$$

Let $L(\lambda) = \{f \in W_B : f \in R(f, \lambda_n)\}$ and $L(\lambda) = \{f \in W_B : f \in R(f, \lambda)\}$. Then from Theorem 2.2, we have that $L(\lambda_n) \xrightarrow{K} L(\lambda)$ in $(W_B, \|\cdot\|_B) \Rightarrow p(L(\lambda_n)) \xrightarrow{K} p(L(\lambda))$ as $n \to \infty$ in $C(T, H)$. But $S(\lambda_n) = p(L(\lambda_n))$ and $S(\lambda) = p(L(\lambda))$. Hence $S(\lambda_n) \xrightarrow{K} S(\lambda)$ in $C(T, H)$. Since $\bigcup_{l \in B} S(\lambda) \subseteq p(W_B) \in P_k(C(T, H))$, we deduce (see Section 2) that $S\mid_B$ is Vietoris continuous. Note that by the remark 1.7 of DeBlasi-Myjak [4], $S(\cdot)$ is l.s.c. and from Lemma 2.1, $S(\cdot)$ is u.s.c. Therefore $S(\cdot)$ is Vietoris continuous. $\square$

Recalling (see Section 2) that on $P_k(C(T, H))$, the Vietoris and Hausdorff metric topologies coincide, we get:

**Theorem 3.2.** *If the hypotheses $H(\varphi)$, $H(F)$ and $H_0$ hold, then $S : P_k(C(T,H))$ is h-continuous.*

## 4. Sensitivity of optimal control problems.

Let $Z$ be a bounded domain in $\mathbb{R}^N$ with boundary $\Gamma = \partial Z$ and $T = [0,r]$. Also let $\Lambda$ be a complete metric space (the parameter space). We consider the following parametrized family of optimal control problems:

$$(\underline{4}) \quad \left| \begin{array}{c} \displaystyle\int_Z \eta(z, x(b,z), \lambda)\, dz \to \inf = m(\lambda) \\ \text{s.t. } \displaystyle\frac{\partial x}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial z_j}\big(a_{ij}(t,z)\frac{\partial x}{\partial z_i}\big) + \beta(x(t,z)) \ni f(t,z,x(t,z),\lambda)u(t,z) \\ x(0,z) = x_0(z,\lambda), x\,|_{T\times\Gamma} = 0,\ |u(t,z)| \le \theta(t,z,\lambda) \text{ a.e.,} \\ u(\cdot,\cdot) \ \text{ is measurable.} \end{array} \right|$$

We will need the following hypotheses on the data of ($\underline{4}$) above:

$\underline{H(a):}$ $a_{ij} \in L^\infty(T\times Z)$, $a_{ij} = a_{ji}$, $\sum_{i,j=1}^{N} a_{ij}(t,z)\eta_i\eta_j \ge c\|\eta\|^2$ for every $(t,z) \in T\times Z$ and every $\eta \in \mathbb{R}^N$ with $c > 0$ and that $|a_{ij}(t,z) - a_{ij}(t',z)| \le k|t-t'|$ a.e. on $Z$, with $k > 0$.

$\underline{H(\beta):}$ $\beta = \partial j$ with $j \in \Gamma_0(\mathbb{R}, \mathbb{R}_+)$.

$\underline{H(f):}$ $f : T\times Z\times\mathbb{R}\times\Lambda \to \mathbb{R}$ is a function s.t.

($\underline{1}$) $(t,z) \to f(t,z,x,\lambda)$ is measurable,
($\underline{2}$) $|f(t,z,x,\lambda) - f(t,z,y,\lambda)| \le k_B(t,z)|x-y|$ a.e. with $k_B(\cdot,\cdot) \in L^1(T\times Z)$, $\lambda \in B \subseteq \Lambda$, $B$ compact,
($\underline{3}$) $\lambda \to f(t,z,x,\lambda)$ is continuous,
($\underline{4}$) $|f(t,z,x,\lambda)| \le a_B(t,z) + c_B(t,z)|x|$ a.e. with $a_B \in L^2(T\times Z)$, $c_B \in L^\infty(T\times Z)$, $\lambda \in B \subseteq \Lambda$, $B$ compact.

$\underline{H(\theta):}$ $\theta(\cdot,\cdot,\lambda) \in L^\infty(T\times Z)$ and $\lambda \to \theta(t,z,\lambda)$ is continuous.

$\underline{H(\eta):}$ $\eta : Z\times\mathbb{R}\times\Lambda \to \mathbb{R}$ is an integrand s.t.

($\underline{1}$) $z \to \eta(z,x,\lambda)$ is measurable,
($\underline{2}$) $(x,\lambda) \to \eta(z,x,\lambda)$ is continuous,
($\underline{3}$) $|\eta(z,x,\lambda)| \le \psi_{1B}(z) + \psi_{2B}(z)|x|^2$ a.e. with $\psi_{1B}(\cdot) \in L^2_+$, $\psi_{2B}(\cdot) \in L^\infty_+$, $\lambda \in B \subseteq \Lambda$, $B$ compact.

$\underline{H_0:}$ $x_0(\cdot,\lambda) \in H^1_0(Z)$, $j(x_0(\cdot,\lambda)) \in L^1(Z)$ and $\lambda \to x_0(\cdot,\lambda)$ is continuous from $\Lambda$ into $L^2(Z)$.

Let $Q(\lambda) \subseteq C(T,H)$ be the set of optimal trajectories of ($\underline{4}$).

**Theorem 4.1.** *If the hypotheses $H(a)$, $H(\beta)$, $H(f)$, $H(\theta)$, $H(\eta)$ and $H_0$ hold, then for every $\lambda \in \Lambda$, $Q(\lambda) \ne \emptyset$, $Q : \Lambda \to P_k(C(T,H))$ is u.s.c. and $m : \Lambda \to \mathbb{R}$ is continuous.*

Proof: In this case $H = L^2(Z)$. Define $\varphi : T \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(t,z) = \begin{cases} \frac{1}{2}\sum_{i,j=1}^{N} \int_Z a_{ij}(t,z)\frac{\partial x}{\partial z_i}\frac{\partial x}{\partial z_j}\,dz + \int_Z j(x(z))\,dz & \text{if } x \in H_0^1(Z), \\ & j(x(\cdot)) \in L^1(Z) \\ +\infty \quad \text{otherwise.} & \end{cases}$$

As in [13] (see also Barbu [2]), we can check that $\varphi(t,x)$ satisfies hypothesis $H(\varphi)$ and furthermore

$$\partial\varphi(t,x) = \left\{ -\sum_{i,j=1}^{N} \frac{\partial}{\partial z_j}(a_{ij}(t,\cdot)\frac{\partial x}{\partial z_j}) + g(\cdot) : g \in L^2(Z),\ g(z) \in \beta(x(z)) \text{ a.e.} \right\}.$$

Let $\hat{f} : T \times H \times \Lambda \to H$ be defined by

$$\hat{f}(t,x,\lambda)(\cdot) = f(t,\cdot,x(\cdot),\lambda),$$

i.e. $\hat{f}$ is the Nemitsky (superposition) operator corresponding to $f$.

Also let $U(t,\lambda) = \{u \in L^2(Z) : |u(z)| \le \theta(t,z,\lambda) \text{ a.e.}\}$. Set

$$F(t,x,\lambda) = \hat{f}(t,x,\lambda)U(t,\lambda) = \{\hat{f}(t,x,\lambda)u : u \in U(t,\lambda)\} \in P_{wkc}(H).$$

Given $v \in H = L^2(Z)$, we have

$$\begin{aligned} d(v,F(t,x,\lambda)) &= \inf\{\|v - \hat{f}(t,x,\lambda)u\|_2 : u \in U(t,\lambda)\} \\ &= \inf\left[(\int_Z |v(z) - f(t,z,x(z),\lambda)u(z)|^2\,dz)^{1/2} : u \in U(t,\lambda)\right] \\ &= \left[\inf(\int_Z |v(z) - f(t,z,x(z),\lambda)u(z)|^2\,dz : u \in U(t,\lambda))\right]^{1/2} \\ &= \left[\int_Z \inf(|v(z) - f(t,z,x(z),\lambda)u|^2 : |u| \le \theta(t,z,\lambda))\,dz\right]^{1/2} \\ &\quad \text{(see Hiai-Umgaki [5, Theorem 2.2)]} \\ &= (\int_Z |v(z) - f(t,z,x(z),\lambda)\hat{u}_0(t,z)|^2\,dz)^{1/2} \\ &\quad \text{(via Aumann's selection theorem, } \hat{u}_0(\cdot,\cdot) \text{ measurable,} \\ &\qquad \hat{u}_0(t,\cdot) \in U(t,\lambda),\ t \in T) \\ &= \|v - \hat{f}(t,x,\lambda)\hat{u}_0(t,\cdot)\|_2\,. \end{aligned}$$

But note that $(\hat{f}(t,x,\lambda)\hat{u}_0(t,\cdot),h)_{L^2(Z)} = \int_Z f(t,z,x(z),\lambda)\hat{u}_0(t,z)h(z)\,dz$ is measurable in $t$ (Fubini's theorem), so that $t \to \hat{f}(t,x,\lambda)\hat{u}_0(t,\cdot)$ is weakly measurable and since $H = L^2(Z)$ is separable, we conclude from the Pettis measurability theorem that $t \to \hat{f}(t,x,\lambda)\hat{u}_0(t,\cdot)$ is measurable $\Rightarrow t \to d(v, F(t,x,\lambda))$ is measurable $\Rightarrow t \to F(t,x,\lambda)$ is measurable.

Also note that

$$\begin{aligned} &h(F(t,x,\lambda), F(t,y,\lambda)) \le \|\hat{f}(t,x,\lambda) - \hat{f}(t,y,\lambda)\| \; \|\theta\|_\infty \\ &\le \|k\|_1 \; \|\theta\|_\infty \sqrt{r}\|x-y\|_2 \qquad \text{(see the hypothesis } H(f)\,(\underline{2})\text{)}. \end{aligned}$$

Next we will show that for every $v \in H = L^2(Z)$, $\lambda \to d(v, F(t,x,\lambda))$ is continuous. To this end, let $\lambda_n \to \lambda$ and let $u \in U(t,\lambda)$. Clearly $U(t,\cdot)$ is $h$-continuous (see the hypothesis $H(\theta)$) and so we can find $u_n \in U(t,\lambda_n)$, $u_n \xrightarrow{s} u$ in $L^2(Z)$. We have:

$$\begin{aligned} &d(v, F(t,x,\lambda_n)) \le \|v - \hat{f}(t,x,\lambda_n)u_n\|_2 \\ &\Rightarrow \overline{\lim} d(v, F(t,x,\lambda_n)) \le \|v - \hat{f}(t,x,\lambda)u\|_2 \,. \end{aligned}$$

Since $u \in U(t,\lambda)$ was arbitrary, we get that

$$(\underline{5}) \qquad \overline{\lim} d(v, F(t,x,\lambda_n)) \le d(v, F(t,x,\lambda)).$$

On the other hand, let $u_n \in U(t,\lambda_n)$, $n \ge 1$ s.t.

$$d(v, F(t,x,\lambda_n)) = \|v - \hat{f}(t,x,\lambda_n)u_n\|_2 \,.$$

We may assume that $u_n \xrightarrow{w^*} u$ in $L^\infty(Z)$ (see the hypothesis $H(\theta)$). Then for every $w \in L^2(Z)$, we have

$$\begin{aligned} &(\hat{f}(t,x,\lambda_n)u_n, w)_{L^2(Z)} = \int_Z f(t,z,x(z),\lambda_n)u_n(z)w(z)\,dz \\ &\to \int_Z f(t,z,x(z),\lambda)u(z)w(z)\,dz = (\hat{f}(t,x,\lambda)u, w)_{L^2(Z)} \end{aligned}$$

$\Rightarrow \hat{f}(t,x_n,\lambda_n)u_n \xrightarrow{w} \hat{f}(t,x,\lambda)u$ in $L^2(H)$ and clearly $u \in U(t,\lambda)$. Using the fact that the norm is weakly l.s.c., we get

$$(\underline{6}) \qquad \begin{aligned} &\|v - \hat{f}(t,x,\lambda)u\|_2 \le \underline{\lim}\|v - \hat{f}(t,x,\lambda_n)u_n\|_2 \\ &\Rightarrow d(v, F(t,x,\lambda)) \le \underline{\lim} d(v, F(t,x,\lambda_n)). \end{aligned}$$

From ($\underline{5}$) and ($\underline{6}$) above, we deduce that $\lambda \to d(v, F(t,x,\lambda))$ is continuous.

Finally note that

$$|F(t,x,\lambda)| \le \|a_B(t,\cdot)\|_2\|\theta\|_\infty + \|c_B\|_\infty\|\theta\|_\infty|Z|^{1/2}\|x\|_{l^2(Z)}$$

with $|Z|$ denoting the volume (Lebesgue measure) of the domain $Z$.

So we have satisfied the hypothesis $H(F)$.

Next let $\hat{\eta} : H \times \Lambda \to \mathbb{R}$ be defined by

$$\hat{\eta}(x, \lambda) = \int_Z \eta(z, x(z), \lambda)\, dz.$$

Clearly $\hat{\eta}(\cdot, \cdot)$ is continuous (see the hypothesis $H(\eta)$).

Now rewrite $(**)$ in the following equivalent abstract form:

$$(\underline{7}) \qquad \left|\begin{array}{c} \hat{\eta}(x(b), \lambda) \to \inf = m(\lambda) \\ \text{s.t.} \quad -\dot{x}(t) \in \partial\varphi(t, x(t)) + F(t, x(t), \lambda) \text{ a.e.} \\ x(0) = x_0(\lambda). \end{array}\right|$$

Let $S(\lambda) \subseteq C(T, L^2(Z))$ be the set of admissible trajectories of $(\underline{7})$. We know that for every $\lambda \in \Lambda$, $S(\lambda)$ is compact in $C(T, L^2(Z))$. So since $\hat{\eta}(\cdot, \cdot)$ is continuous, we deduce that for every $\lambda \in \Lambda$, $Q(\lambda) \neq \emptyset$.

Next we will establish the continuity of the value function $m(\cdot)$. So let $\lambda_n \to \lambda$ in $\Lambda$ and take $x \in S(\lambda)$ s.t.

$$m(\lambda) = \hat{\eta}(x, \lambda) \quad (\text{i.e. } x \in Q(\lambda)).$$

From Theorem 3.1, we know that $S(\lambda_n) \xrightarrow{K} S(\lambda)$. So we can find $x_n \in S(\lambda_n)$, $n \geq 1$ s.t. $x_n \xrightarrow{s} x$ in $C(T, L^2(Z))$. Then we have:

$$(\underline{8}) \qquad \begin{array}{c} m(\lambda_n) \leq \hat{\eta}(x_n, \lambda_n) \\ \Rightarrow \overline{\lim}\, m(\lambda_n) \leq \lim \hat{\eta}(x_n, \lambda_n) = \hat{\eta}(x, \lambda) = m(\lambda). \end{array}$$

Also let $x_n \in S(\lambda_n)$ s.t. $m(\lambda_n) = \hat{\eta}(x_n, \lambda_n)$. Recalling that for all $n \geq 1$, we have

$$S(\lambda_n) \subseteq p(W_B) \in P_k(C(T, L^2(Z))) \quad \text{(see the proof of Theorem 3.1)}$$

we deduce that by passing to a subsequence, we may assume that $x_n \xrightarrow{s} x$ in $C(T, L^2(Z))$. Then

$$(\underline{9}) \qquad \begin{array}{c} \hat{\eta}(x_n, \lambda_n) \to \hat{\eta}(x, \lambda) \\ \Rightarrow m(\lambda) \leq \underline{\lim} m(\lambda_n). \end{array}$$

From $(\underline{8})$ and $(\underline{9})$ above, we get that $m(\cdot)$ is continuous.

Finally, using the continuity of $m(\cdot)$, we can easily check that

$$\overline{\lim} Q(\lambda_n) \subseteq Q(\lambda),$$

which implies that for any $B \subseteq \Lambda$ compact, $Q\mid_B$ has a closed graph, thus is u.s.c. (see DeBlasi-Myjak [4, the remark 1.6]). Then Lemma 2.1 gives us the desired upper semicontinuity of $Q(\cdot) : \Lambda \to P_k(C(T, L^2(Z)))$. $\square$

Let $K : T \to P_{fc}(\mathbb{R}^n)$ be a multifunction s.t. $h(K(t'), K(t)) \leq \int_t^{t'} \gamma(s)\, ds$. Let $\delta_{K(t)}(x) = 0$ if $x \in K(t)$, and $+\infty$ otherwise (the indicator function of the moving set $K(t)$). Then from the convex analysis, we know that $\partial\delta_{K(t)}(x) = N_{K(t)}(x) =$ the normal cone to the set $K(t)$ at $x \in \mathbb{R}^n$. It is easy to see that hypothesis $H(\varphi)$ is satisfied by $\delta_{K(t)}(\cdot)$ (take $\dot{g}_{r'} = \gamma$, $\beta = 1$, $h_{r'} = 0$). Then the problem ($\underline{1}$) takes the following special form:

$$(\underline{\underline{10}}) \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(t, x(t), \lambda) \text{ a.e.} \\ x(0) = x_0(\lambda) \in K(0). \end{array} \right\}$$

Evolution inclusions of this form arise in mathematical economics and theoretical mechanics and are also called "differential variational inequalities" (see Aubin-Cellina [1]). The work in this paper incorporates systems like ($\underline{\underline{10}}$) above.

## References


[1] Aubin J.-P., Cellina A., *Differential Inclusions*, Springer, Berlin, 1983.

[2] Barbu V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leiden, The Netherlands, 1976.

[3] Brezis H., *Operateurs Maximaux Monotones*, North Holland, Amsterdam, 1973.

[4] DeBlasi F., Myjak J., *On continuous approximations for multifunctions*, Pacific J. Math. **123** (1986), 9–31.

[5] Hiai F., Umgaki H., *Integrals, conditional expectations and martingales of multivalued functions*, J. Multiv. Anal. **7** (1977), 149–182.

[6] Klein E., Thompson A., *Theory of Correspondences*, Wiley, New York, 1984.

[7] Lim T.-C., *On fixed point stability for set-valued contractive mappings with applications to generalized differential equations*, J. Math. Anal. Appl. **110** (1985), 436–441.

[8] Nadler S.B., *Multivalued contraction mappings*, Pacific J. Math. **30** (1969), 475–488.

[9] Papageorgiou N.S., *A stability result for differential m inclusions in Banach spaces*, J. Math. Anal. Appl. **118** (1986), 232–246.

[10] ———, *Convergence theorems for Banach space valued integrable multifunctions*, International J. Math. and Math. Sci. **10** (1987), 433–442.

[11] ———, *On measurable multifunctions with applications to random multivalued equations*, Math. Japonica **34** (1989), 287–296.

[12] ———, *Infinite dimensional control systems with state and control constraints*, Proceedings of the Indian Academy of Sciences **100** (1990), 65–77.

[13] ———, *On evolution inclusions associated with time dependent convex subdifferentials*, Comment. Math. Univ. Carolinae **31** (1990), 517–527.

[14] Przyluski K.M., *Remarks on continuous dependence of an optimal control on parameters*, in: Game Theory and Mathematical Economics, eds. O. Moeschlin and D. Pallashke, North Holland, Amsterdam, 1981, pp. 331–327.

[15] Rybinski L., *A fixed point approach in the study of the solution sets of Lipschitzian functional-differential inclusions*, J. Math. Anal. Appl. **160** (1991), 24–46.

[16] Stassinopoulos G., Vinter R., *Continuous dependence of a differential inclusion on the right-hand side with applications to stability of optimal control problems*, SIAM J. Control and Optim. **17** (1979), 432–449.

[17] Tolstonogov A., *On the dependence on parameter of a solution of a differential inclusion with nonconvex second member*, Diff. Equations **18** (1982), 1105–1113.

[18] Tsukada M., *Convergence of best approximations in smooth Banach spaces*, J. Approx. Theory **40** (1984), 301–309.

[19] Vasilev A., *Continuous dependence of the solutions of differential inclusions on the parameter*, Ukrainian Math. J. **35** (1983), 520–524.

[20] Wagner D., *Survey of measurable selection theorems*, SIAM J. Control Optim. **15** (1977), 859–903.

[21] Watanabe J., *On certain nonlinear evolution equations*, J. Math. Soc. Japan **25** (1973), 446–463.

[22] Yamada Y., *On evolution inclusions generated by subdifferential operators*, J. Fac. Sci. Univ. Tokyo **23** (1976), 491–515.

[23] Yotsutani S., *Evolution equations associated with the subdifferentials*, J. Math. Soc. Japan **31** (1978), 623–646.

University of the Aegean, Department of Mathematics, Karlovassi, Samos 83000, Greece

National Technical University, Department of Mathematics, Zografou Campus, Athens 15773, Greece

(*Received* January 6, 1992)