Cantor-connectedness revisited

R. Lowen

Abstract. Following Preuss' general connectedness theory in topological categories, a connectedness concept for approach spaces is introduced, which unifies topological connectedness in the setting of topological spaces, and Cantor-connectedness in the setting of metric spaces.

Keywords: connected, Cantor-connected, metric space, topological space, approach space

Classification: 54A05, 54B30, 54D05

1. Introduction.

Applying G. Preuss' general theory of connectedness in topological categories [19], [20], we show that connectedness in **TOP** and Cantor-connectedness in p-$\mathbf{MET}^\infty$ [3], are instances of a unifying concept of connectedness in **AP**, the category of approach spaces [14]. **AP** is a topological category in which both **TOP** and p-$\mathbf{MET}^\infty$ are embedded as full isomorphism-closed subcategories.

One of the advantages of the category of approach spaces is that it makes the relationship between topological spaces and metric spaces considerably nicer. An uncountable product of metrizable topological spaces need not be metrizable and a countable product is metrizable only by ad-hoc procedures. In **AP** every product of metric spaces is canonically endowed with an approach structure, i.e. a point-set distance rather than a point-point distance. This structure then has as **TOP**-coreflection precisely the product topology. For details concerning this we refer to [14], [12]. The category of approach spaces also seems to be the right setting for the work of E.V. Sčepin and I. Isiwata [9], [21] on generalizations of metrizability. In [13] we showed that **AP** allowed for a categorically right formulation of C. Kuratowski's concept of measure of compactness [11], and in [15] we showed that it was also relevant to concepts predominantly used in approximation theory. Apart from unifying different types of connectedness we now also show that there exists a measure of connectedness not unsimilar to the measure of compactness and which may be useful in fixpoint theory [4], [6], [5], [10].

2. Approach spaces.

The purpose of this section is to recall those basic concepts introduced in [14] which are required for this paper.

Let X be an arbitrary set, $\mathbb{P} = [0, \infty]$ and $\mathbb{P}_0 =]0, \infty]$. A map

$$\delta : X \times 2^X \to \mathbb{P}$$

is called a distance if it fulfills

(D1) $\forall\, A \in 2^X,\, \forall\, x \in X : x \in A \Rightarrow \delta(x, A) = 0$,
(D2) $\forall\, x \in X : \delta(x, \emptyset) = \infty$,
(D3) $\forall\, A, B \in 2^X,\, \forall\, x \in X : \delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B)$,
(D4) $\forall\, A \in 2^X,\, \forall\, x \in X,\, \forall\, \varepsilon > 0 : \delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon$

where $A^{(\varepsilon)} = \{x \mid \delta(x, A) \leq \varepsilon\}$.

Functions giving distances between points and sets were also already considered by E.V. Sčepin [21] and I. Isiwata [9] in studies on generalizations of metrizability.

A set equipped with a distance is called an approach space. Given approach spaces (X, δ) and (X', δ') a map $f : X \to X'$ is called a contraction if

(C) $\forall\, A \in 2^X,\, \forall\, x \in X : \delta'(f(x), f(A)) \leq \delta(x, A)$.

In [14] we showed that approach spaces and contractions determine a topological category in the sense of H. Herrlich [7]. We denote this category **AP**. **TOP** is embedded as a full isomorphism-closed, bireflective and bicoreflective subcategory by

$$\begin{array}{rcl} \mathbf{TOP} & \to & \mathbf{AP} \\ (X, \mathcal{T}) & \to & (X, \delta_{\mathcal{T}}) \\ f & \to & f \end{array}$$

where

$$\delta_{\mathcal{T}}(x, A) = \begin{cases} 0 & \text{if } x \in \overline{A} \\ \infty & \text{if } x \notin \overline{A} \end{cases}$$

and this for all $x \in X$, $A \subset X$.

Given an approach space (X, δ), its **TOP**-coreflection is given by $(X, \mathcal{T}_\delta)$ where $\mathcal{T}_\delta$ is the topology with closure operator

$$\overline{A} = \{x \mid \delta(x, A) = 0\}$$

for all $A \subset X$.

Analogously p-$\mathbf{MET}^\infty$, the category of all extended pseudo-metric spaces and non-expansive maps, is embedded as a full isomorphism closed bicoreflective subcategory by

$$\begin{array}{rcl} p\text{-}\mathbf{MET}^\infty & \to & \mathbf{AP} \\ (X, d) & \to & (X, \delta_d) \\ f & \to & f \end{array}$$

where

$$\delta_d(x, A) = \inf_{a \in A} d(x, a)$$

and this for all $x \in X$, $A \subset X$.

Given an approach space (X,δ), its p-$\mathbf{MET}^\infty$-coreflection is given by (X,d_δ) where

$$d_\delta(x,y)=\delta(x,\{y\})\vee\delta(y,\{x\})$$

for all $x,y\in X$.

When required, we shall denote the **TOP**-coreflection of a space X by $c_{\mathrm{T}}(X)$ and the p-$\mathbf{MET}^\infty$-coreflection by $c_{\mathrm{M}}(X)$.

3. Connectedness in AP.

Given a topological category $\mathcal{C}$ and a class $\mathcal{D}$ of $\mathcal{C}$-objects then following G. Preuss [19], [20] we say that $X\in|\mathcal{C}|$ is $\mathcal{D}$-connected if all morphisms in $\bigcup_{D\in\mathcal{D}}\mathrm{Hom}(X,D)$ are constant.

The following proposition is clear.

Proposition 3.1. *If $\mathcal{C}$ is a topological category and $\mathcal{D}$ is a set of $\mathcal{C}$-objects then $\mathcal{D}$-connectedness coincides with $\{\prod_{D\in\mathcal{D}}D\}$-connectedness.* $\square$

In the sequel, if $\mathcal{D}=\{D\}$ we write D-connected for $\mathcal{D}$-connected. We recall the following theorem from G. Preuss [20]. The terminology used in this theorem is mainly due to A. Arhangel'skii and R. Wiegandt [1].

Proposition 3.2 (Preuss)**.** *If $\mathcal{C}$ is a topological category and $\mathcal{D}$ is a class of $\mathcal{C}$-objects then the full subcategory $c(\mathcal{D})$ consisting of all $\mathcal{D}$-connected objects fulfills the following properties:*

(1) *$c(\mathcal{D})$ is morphism-closed i.e. if $X\in|c(\mathcal{D})|$, $Y\in|\mathcal{C}|$ and $f\in\mathrm{Hom}(X,Y)$ then $f(X)\in|c(\mathcal{D})|$,*
(2) *$c(\mathcal{D})$ is second-additive i.e. if $X\in|c(\mathcal{D})|$ and $(X_j)_{j\in J}$ is a family of subspaces of X with non-empty intersection and such that each of them is $\mathcal{D}$-connected then $\bigcup_{j\in J}X_j$ is $\mathcal{D}$-connected,*
(3) *$c(\mathcal{D})$ contains all one-point spaces.* $\square$

Proposition 3.3. *Under the same conditions as the foregoing proposition $c(\mathcal{D})$ is quotient-reversible i.e. if $Y\in|c(\mathcal{D})|$, $X\in|\mathcal{C}|$, $f\in\mathrm{Hom}(X,Y)$ is a quotient-map and $f^{-1}(\{y\})\in|c(\mathcal{D})|$ for all $y\in Y$ then $X\in|c(\mathcal{D})|$.*

Proof: Consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[dr, "f"] & D \\
F_y \arrow[u, "i"] \arrow[r, "\hat{y}"'] & Y \arrow[u, "h"']
\end{tikzcd}$$

where $D\in\mathcal{D}$, g is a given morphism, $y\in Y$, F_y stands for $f^{-1}(\{y\})$, $\hat{y}$ is the obvious constant map and i is the canonical embedding. Since $F_y\in|c(\mathcal{D})|$, g is constant on F_y for each $y\in Y$. Thus there exists a factorization h which, since $Y\in|c(\mathcal{D})|$, is also constant. Consequently g is constant and we are done. $\square$

Now let us turn to the case of approach spaces. In the sequel, given $(X,\delta)\in|\mathbf{AP}|$, an overline will denote closure in the **TOP**-coreflection of (X,δ), i.e. $x\in\overline{A}$ iff $\delta(x,A)=0$.

Proposition 3.4. *If $\mathcal{E}$ is a class of approach spaces such that for each $(E, \delta_E) \in \mathcal{E}$, its* **TOP***-coreflection is* T_1*, if moreover $X \in |\mathbf{AP}|$, $Y \subset Z \subset \overline{Y} \subset X$ and Y is $\mathcal{E}$-connected, then Z is $\mathcal{E}$-connected.*

Proof: Let $(E, \delta_E) \in \mathcal{E}$ and let $f \in \mathrm{Hom}(Z, E)$. Then there exists $e \in E$ such that $f(Y) = \{e\}$. If $z \in Z$ then $\delta(z, Y) = 0$ which implies $\delta_E(f(z), \{e\}) = 0$, i.e. f is constant, and we are done. □

Theorem 3.5. *If $\mathcal{E}$ has the properties required in the foregoing proposition then a product of* **AP***-objects is $\mathcal{E}$-connected if and only if each factor is $\mathcal{E}$-connected.*

Proof: The only-if-part follows from 3.2. (1), and the if-part follows from 3.2. (2) and 3.3 in virtually the same way as in **TOP**. □

Remark 3.6. Spaces fulfilling the conditions of the foregoing proposition are in particular T_0-approach spaces according to T. Marny [16]. Indeed, if a space $X \in |\mathbf{AP}|$ has a T_1 **TOP**-coreflection then for any two-point indiscrete object $I \in |\mathbf{AP}|$, $\mathrm{Hom}(I, X)$ only contains constant functions.

We now consider the following set $\mathcal{D}$ of extended metric spaces. For each $\alpha \in \mathbb{P}_0$, let D_α be the two-point space $\{0, \infty\}$, equipped with the extended metric d_α where $d_\alpha(0, \infty) = \alpha$. Further let (D, δ_D) stand for the **AP**-product of the spaces D_α. We recall that then for any $x \in D$, $A \subset D$ we have

$$\delta_D(x, A) = \sup_{\substack{P \in 2^{\mathbb{P}_0} \\ P \text{ finite}}} \inf_{a \in A} \sup_{\alpha \in P} d_\alpha(pr_\alpha(x), pr_\alpha(a)).$$

Since $\mathbb{P}_0$ is infinite we know that (D, δ_D) is not metric [14].

An object in **AP** which is $\mathcal{D}$-connected (or D-connected) shall simply be called connected. The full subcategory of **AP** with objects all $\mathcal{D}$-connected spaces shall be denoted c-**AP**. Clearly all results proved in this section as well as results in the general theory of G. Preuss are valid for this concept of connectedness.

4. Cantor-connectedness and topological connectedness.

Cantor's definition of connectedness (nowadays more often referred to as uniform connectedness) goes back to 1883 [3]. He originally defined this concept with so-called ε-chains, where $\varepsilon > 0$. Given two points x, y in a metric space, an ε-chain connecting these points is a finite set $\{x_1, x_2, \ldots, x_n\}$ such that $x_1 = x$, $x_n = y$ and $d(x_i, x_{i+1}) \leq \varepsilon$ for all $i \in \{1, \ldots, n-1\}$. Cantor then defined X to be chain-connected if any two points can be ε-chain connected for any $\varepsilon > 0$. It is easily seen that this is equivalent to saying that there does not exist a subset A of X such that $d(A, X \setminus A) > 0$ (see e.g. H. Herrlich [8]).

Theorem 4.1. *If $(X, d) \in |p\text{-}\mathbf{MET}^\infty|$ then the following are equivalent:*

(1) *(X, d) is Cantor-connected,*
(2) *$(X, \delta_d) \in |c\text{-}\mathbf{AP}|$.*

PROOF: To see that (1) $\Rightarrow$ (2) let (X, δ_d) be not connected. Then there exists $\alpha \in \mathbb{P}_0$ and a contraction $f : X \to D_\alpha$ which is not constant. Put $X_0 = f^{-1}(\{0\})$ and $X_\infty = f^{-1}(\{\infty\})$. It is immediately verified that $d(X_0, X_\infty) \geq \alpha$ and thus (X, d) is not Cantor-connected.

To see that (2) $\Rightarrow$ (1) let (X, d) be partitioned in X_0 and X_∞ such that $d(X_0, X_\infty) \geq \alpha$ for some $\alpha > 0$. It is immediately verified that the map $f : X \Rightarrow D_\alpha$, defined by $f(X_0) = \{0\}$ and $f(X_\infty) = \{\infty\}$, is a contraction, and thus (X, δ_d) is not connected.

$\square$

Theorem 4.2. *If $(X, \mathcal{T}) \in |\mathbf{TOP}|$ then the following are equivalent:*

(1) *$(X, \mathcal{T})$ is connected,*
(2) *$(X, \delta_{\mathcal{T}}) \in |c\text{-}\mathbf{AP}|$.*

PROOF: To see that (1) $\Rightarrow$ (2) let $(X, \delta_{\mathcal{T}})$ be not connected. Then there exists $\alpha \in \mathbb{P}_0$ and a contraction $f : X \to D_\alpha$ which is not constant. Put $X_0 = f^{-1}(\{0\})$ and $X_\infty = f^{-1}(\{\infty\})$ and let $x \in X_0$. Then $\alpha \leq \delta_{\mathcal{T}}(x, X_\infty)$ and thus $x \notin \overline{X}_\infty$, which proves that both X_0 and X_∞ are open and thus $(X, \mathcal{T})$ is not connected.

To see that (2) $\Rightarrow$ (1) let X be partitioned in the open sets X_0 and X_∞. Then it is immediately verified that for any $\alpha \in \mathbb{P}_0$, the map $f : X \Rightarrow D_\alpha$, defined by $f(X_0) = \{0\}$ and $f(X_\infty) = \{\infty\}$, is a contraction and thus (X, δ_d) is not connected.

$\square$

The foregoing results prove that the connectedness in **AP** unifies Cantor-connectedness for metric spaces and connectedness for topological spaces.

5. Measure of connectedness.

Measures of compactness are useful tools in e.g. theory of operator equations and in fixed point theory [2], [4]. Two such measures are Kuratowski's measure [11], and a rather more used variant, the so-called Hausdorff measure.

In [13] we showed that these measures can be recaptured as canonical concepts in **AP**. In this section we want to show that it is very easy to extend Kuratowski's ideas to connectedness.

Given an approach space (X, δ) we define the measure of connectedness of (X, δ) as

$$\operatorname{conn}(X) = \inf\{\alpha \in \mathbb{P} \mid X \text{ is } D_\delta\text{-connected}\}.$$

From this definition it follows immediately that a space is connected iff $\operatorname{conn}(X) = 0$. Also it is worthwhile to notice that for any $\alpha \in \mathbb{P}_0$, we have $\operatorname{conn}(D_\alpha) = \alpha$, and so this measure, even when restricted to $p\text{-}\mathbf{MET}^\infty$, takes on all possible values.

Theorem 5.1. *If $X \in |\mathbf{AP}|$, $Y \in |\mathbf{AP}|$ and $f \in \operatorname{Hom}(X, Y)$ then:*

(1)
$$\operatorname{conn}(f(X)) \leq \operatorname{conn}(X),$$

(2) *if $(X_j)_{j \in J}$ is a family of subspaces of X with non-empty intersection then*

$$\operatorname{conn}\Big(\bigcup_{j \in J} X_j\Big) \leq \sup_{j \in J} \operatorname{conn}(X_j),$$

(3) *if f is a quotient map then*

$$\mathrm{conn}(X) \leq \mathrm{conn}(Y) \vee \left(\sup_{y\in Y}(f^{-1}(y))\right).$$

Proof: In all cases, the proof is obtained by considering spaces which occur on the right hand side of an inequality and supposing that for some $\alpha \in \mathbb{P}$ they are D_α-connected. An application of foregoing results then allows one to conclude that the space on the left hand side is also D_α-connected. □

Trivial examples show that all inequalities in the foregoing theorem are strict in general.

Theorem 5.2. *If $(X_j, \delta_j)_{j\in J}$ is a family of approach spaces then*

$$\mathrm{conn}\Big(\prod_{j\in J} X_j\Big) = \sup_{j\in J} \mathrm{conn}(X_j).$$

Proof: For any $X \in |\mathbf{AP}|$ let us put

$$\mathrm{K}(X) = \{\alpha \in \mathbb{P} \mid X \text{ is } D_\alpha\text{-connected}\}.$$

Then obviously $\mathrm{K}(X)$ is an unbounded interval in $\mathbb{P}$. From this and upon applying 3.5 we obtain

$$\begin{aligned}\mathrm{conn}\Big(\prod_{j\in J} X_j\Big) &= \inf \mathrm{K}\Big(\prod_{j\in J} X_j\Big)\\ &= \inf \bigcap_{j\in J} \mathrm{K}(X_j)\\ &= \sup_{j\in J} \mathrm{conn}(X_j)\end{aligned}$$

and we are done. □

Applying the foregoing results to the special case of topological spaces of course gives all well known invariance properties of topological connectedness. In the same way applying them to p-$\mathbf{MET}^\infty$, and keeping in mind that p-$\mathbf{MET}^\infty$ is coreflectively embedded in $\mathbf{AP}$, we obtain the following corollary.

Corollary 5.3. *If $X \in |p\text{-}\mathbf{MET}^\infty|$, $Y \in |p\text{-}\mathbf{MET}^\infty|$ and $f : X \to Y$ is non-expansive then:*

(1) *if X is Cantor-connected, then Y is Cantor-connected,*
(2) *if $(X_j)_{j\in J}$ is a family of Cantor-connected subspaces of X with non-empty intersection then $\bigcup_{j\in J} X_j$ is Cantor-connected,*
(3) *if Y is Cantor-connected, $f : X \to Y$ is a quotient map and $f^{-1}(\{y\})$ is Cantor-connected for each $y \in Y$, then X is Cantor-connected.* □

Examples 5.4. (1) It is well-known that $\mathbb{Q}$ equipped with its usual metric is Cantor-connected and thus $\mathrm{conn}(\mathbb{Q}) = 0$. If we take the **TOP**-coreflection, i.e. $\mathbb{Q}$ equipped with its usual topology then it is no longer connected and so $\mathrm{conn}(c_{\mathrm{T}}(\mathbb{Q})) = \infty$. The converse situation of course cannot occur since (in **AP**) a topological space is endowed with a finer structure than any metric which underlies it.

(2) An interesting ("genuine") approach structure on $\mathbb{P}$ is given by

$$\delta(x, A) = \begin{cases} \inf_{a \in A} |x - a| & \text{if } x < \infty \\ 0 & \text{if } x = \infty,\ \sup A = \infty \\ \infty & \text{if } x = \infty,\ \sup A < \infty. \end{cases}$$

Although the Alexandroff compactification of $[0, \infty[$ is metrizable, it is of course not metrizable by a metric which extends the usual metric. Now δ is the unique distance which does extend this metric and which moreover "distancizes" the Alexandroff topology on $\mathbb{P}$, i.e. which is such that its **TOP**-coreflection coincides with it. As for connectedness we have $\mathrm{conn}(\mathbb{P}) = 0$ since δ is finer than the Alexandroff topology, but $\mathrm{conn}(c_{\mathrm{M}}(\mathbb{P})) = \infty$ since for the p-$\mathbf{MET}^{\infty}$-coreflection the point ∞ is at distance ∞ of all other points.

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University of Antwerp, RUCA, Department of Mathematics and Computer Science, Groenenborgerlaan 171, B-2020 Antwerpen, Belgium

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