Strong shape of the Stone-Čech compactification

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Abstract. J. Keesling has shown that for connected spaces X the natural inclusion $e : X \to \beta X$ of X in its Stone-Čech compactification is a shape equivalence if and only if X is pseudocompact. This paper establishes the analogous result for strong shape. Moreover, pseudocompact spaces are characterized as spaces which admit compact resolutions, which improves a result of I. Lončar.

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1. Introduction.

For every completely regular space X, there is a natural embedding $e : X \to \beta X$ of X in its Stone-Čech compactification βX. In 1975, K. Morita proved that, for pseudocompact spaces X, the shape $sh\,(X) = sh\,(\beta X)$ [11, Theorem 5.2 and Corollary 5.3]. In a survey article of J. Keesling, published in 1980, it is stated that, for connected spaces X, the embedding e is a shape equivalence if and only if X is pseudocompact [4, Theorem 1.2]. In the present paper we prove the analogous result for strong shape. Moreover, we improve a result of I. Lončar [8], who in the class of normal spaces characterized the countably compact spaces as spaces which admit a resolution consisting of metric compacta. Recall that for normal spaces countable compactness and pseudocompactness are equivalent properties (see [2, Theorem 3.10.20 and 3.10.21]).

Theorem 1. *For connected Tychonoff spaces X the following statements are equivalent.*

(i) *The natural embedding $e : X \to \beta X$ is a strong shape equivalence.*
(ii) *The natural embedding $e : X \to \beta X$ is a shape equivalence.*
(iii) *X is pseudocompact.*
(iv) *X admits a resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, which consists of compact polyhedra.*
(v) *X admits a resolution $\boldsymbol{p} : X \to \boldsymbol{X}$, which consists of compact spaces.*

In the sections which follow we will prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i). Only (ii) $\Rightarrow$ (iii) uses the assumption that X is connected. Basic facts about the Stone-Čech compactification can be found in [2] and [12]. For shape theory we use [10] and for strong shape [6], [7].

2. Shape-equivalence of e implies pseudocompactness of X.

(i) $\Rightarrow$ (ii). Recall that in shape theory one defines a shape category $\boldsymbol{Sh}$ and a shape functor $\boldsymbol{S} : \boldsymbol{HTop} \to \boldsymbol{Sh}$, where $\boldsymbol{HTop}$ denotes the homotopy category. In particular, this functor assigns to every mapping $f : X \to Y$ a shape morphism $S([f]) : X \to Y$, which depends only on the homotopy class $[f]$ of f. When we say that a mapping f is a shape equivalence, we mean that $S([f])$ is an isomorphism of $\boldsymbol{Sh}$. Similarly, in strong shape theory one defines a strong shape category $\boldsymbol{SSh}$ and a strong shape functor $\boldsymbol{S}_1 : \boldsymbol{HTop} \to \boldsymbol{SSh}$. Moreover, there is a functor $\boldsymbol{S}_2 : \boldsymbol{SSh} \to \boldsymbol{Sh}$ and $\boldsymbol{S}_2\boldsymbol{S}_1 = \boldsymbol{S}$. If we say that a mapping f is a strong shape equivalence, we mean that $S_1([f])$ is an isomorphism of $\boldsymbol{SSh}$. Clearly, the implication (i) $\Rightarrow$ (ii) is a consequence of the existence of the functor $\boldsymbol{S}_2$.

(ii) $\Rightarrow$ (iii) Assuming that e is a shape equivalence and X is connected, we will show that every mapping $\phi : X \to \mathbb{R}$ is bounded, i.e. X is pseudocompact. Consider the mapping $\exp : \mathbb{R} \to S^1 = \{z \in \mathbb{C} : |z| = 1\}$, defined by $\exp(t) = e^{2\pi it}$, $t \in \mathbb{R}$. Since S^1 is compact, the mapping $\exp\circ\phi : X \to S^1$ admits an extension $g : \beta X \to S^1$, so that

$$ge = \exp\circ\phi. \tag{1}$$

By the contractibility of $\mathbb{R}$, ϕ is homotopic to the constant map 0. Therefore, $ge = \exp\circ\phi$ is homotopic to the constant map 1, i.e.

$$ge \simeq i \circ 1, \tag{2}$$

where $1 : X \to \{1\}$ and $i : \{1\} \to S^1$ is the inclusion mapping. We claim that g, too, is homotopic to a constant mapping. Since S^1 and $\{1\}$ are polyhedra, it suffices to show that $S(g)$ factors in $\boldsymbol{Sh}$ through $\{1\}$. However, this follows from (2) because

$$S(g) = S(g)S(e)S(e)^{-1} = S(ge)S(e)^{-1} = S(i)S(1)S(e)^{-1}$$

and $S(1)S(e)^{-1} : \beta X \to \{1\}$. Since $\exp : \mathbb{R} \to S^1$ is a fibration and g is homotopic to a constant, the homotopy lifting property yields a mapping $\psi : \beta X \to \mathbb{R}$ such that $g = \exp\circ\psi$. It follows that $\exp\circ\phi = ge = \exp\circ\psi e$ or $\exp\circ(\phi - \psi e) = 1$. We conclude that $(\phi - \psi e)(X) \subseteq \mathbb{Z}$. Since X is connected and $\mathbb{Z}$ is discrete, it follows that $(\phi-\psi e)(X)$ is a single point. The set $\psi(\beta X)$ is compact and therefore bounded in $\mathbb{R}$. Consequently, $(\psi e)(X) \subseteq \psi(\beta X)$ is also bounded. This implies that $\phi(X)$ is bounded, too.

3. Resolutions.

We now recall the notion of resolution (see [9] and [10]). Let $\boldsymbol{X} = (X_a, p_{aa'}, A)$ be an inverse system. A morphism of $\boldsymbol{pro}$-$\boldsymbol{Top}$ (also called a *mapping of systems*) $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$ is a resolution of X, provided it possesses the following two properties:

(R1) For any polyhedron P, open covering $\mathcal{V}$ of P and mapping $f : X \to P$, there exist an $a \in A$ and a mapping $g : X_a \to P$, such that the mappings f and gp_a are $\mathcal{V}$-near, which we denote by $(f, gp_a) \prec \mathcal{V}$.

(R2) For any polyhedron P and open covering $\mathcal{V}$ of P, there exists an open covering $\mathcal{V}'$ of P such that, for any $a \in A$ and mappings $g, g' : X_a \to P$, which satisfy $(gp_a, g'p_a) \prec \mathcal{V}'$, there exists an $a' \geq a$ such that $(gp_{aa'}, g'p_{aa'}) \prec \mathcal{V}$.

A resolution is said to be cofinite provided each element of the index set A has only finitely many predecessors.

(iii) $\Rightarrow$ (iv) Let X be pseudocompact. Since βX is compact, there exists a cofinite inverse system of compact polyhedra $\boldsymbol{X} = (X_a, p_{aa'}, A)$ and a collection of mappings $q_a : \beta X \to X_a$, $a \in A$, such that $\boldsymbol{q} = (q_a)$ is an inverse limit of βX. Note that $\boldsymbol{q}$ is a resolution and therefore satisfies the conditions (R1) and (R2) (see [10, I, 6.1, Theorem 1]). We now define mappings $p_a : X \to X_a$, $a \in A$, by putting $p_a = q_a e$. The desired implication will be established if we prove the following lemma.

Lemma 1. *$\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$ is a resolution of X.*

In order to prove Lemma 1 we need the following simple fact.

Lemma 2. *A pseudocompact subspace Q of a polyhedron P is a compact space.*

PROOF OF LEMMA 2: Let K be a simplicial complex whose geometric realization $|K|$ is P. Denote by $|K|_{CW}$ and $|K|_m$ the spaces obtained by endowing K with the CW-topology and the metric topology, respectively. Let $j : |K|_{CW} \to |K|_m$ denote the identity mapping, which is known to be continuous. Since the continuous image of a pseudocompact space is pseudocompact, we see that $j(Q)$ is a pseudocompact subspace of the metric space $|K|_m$. Therefore, $j(Q)$ is compact (see [2, Theorems 3.10.21 and 5.1.20]) and thus a closed subset of $|K|_m$. This implies that $Q = j^{-1}j(Q)$ is a closed subset of $P = |K|_{CW}$. Since P is paracompact, Q is also paracompact. However, paracompact pseudocompact spaces are compact [loc. cit.] and we conclude that indeed, Q is a compact space. $\square$

PROOF OF LEMMA 1: Clearly, $p_{aa'}p_{a'} = p_a$, for $a \leq a'$. Therefore, it remains to verify the conditions (R1) and (R2).

Verification of (R1)*:* Let P be a polyhedron, $\mathcal{V}$ an open covering of P and $f : X \to P$ a mapping. Clearly, $f(X)$ is pseudocompact. By Lemma 1, it follows that $f(X)$ is even compact. Therefore, there exists a compact subpolyhedron $Q \subseteq P$ such that $f(X) \subseteq Q$. If we view f as mapping $f : X \to Q$, it admits an extension $\tilde{f} : \beta X \to Q$, $\tilde{f}e = f$. Applying (R1) for $\boldsymbol{q}$ to $\tilde{f}$ and $\mathcal{V} \mid Q$, we obtain an $a \in A$ and a mapping $g : X_a \to Q \subseteq P$, such that $(gq_a, \tilde{f}) \prec \mathcal{V} \mid Q$. Therefore, we also have the desired relation $(gp_a, f) \prec \mathcal{V}$.

Verification of (R2)*:* Let P be a polyhedron and $\mathcal{V}$ an open covering of P. Choose a covering $\mathcal{V}'$ of P, by (R2) applied to $\boldsymbol{q}$. Let $\mathcal{W}$ be a star-refinement of $\mathcal{V}$, i.e. $\operatorname{st}(\mathcal{W})$ refines $\mathcal{V}'$, where $\operatorname{st}(\mathcal{W})$ denotes the covering formed by all the stars $\operatorname{st}(W, \mathcal{W})$, $W \in \mathcal{W}$. We claim that $\mathcal{W}$ has the properties required by (R2) for $\boldsymbol{p}$, i.e. that whenever $a \in A$ and $g, g' : X_a \to P$ are mappings, for which

$$(gp_a, g'p_a) \prec \mathcal{W}, \tag{3}$$

then there exists an $a' \geq a$ such that

$$(gp_{aa'}, g'p_{aa'}) \prec \mathcal{V}. \tag{4}$$

It suffices to show that (3) implies

$$(gq_a, g'q_a) \prec \mathcal{V}', \tag{5}$$

because (4) follows from (5), by the choice of $\mathcal{V}'$. To verify (5), consider any point $y \in \beta X$. Choose members W_1, W_2 of $\mathcal{W}$ so that

$$gq_a(y) \in W_1,\ g'q_a(y) \in W_2. \tag{6}$$

Choose an open neighborhood U of y in βX so small that

$$gq_a(U) \subseteq W_1,\ g'q_a(U) \subseteq W_2. \tag{7}$$

Since $e(X)$ is dense in βX, there exists a point $x \in X$ such that $e(x) \in U$. By (3), there exists a $W \in \mathcal{W}$ such that

$$gp_a(x), g'p_a(x) \in W. \tag{8}$$

Since $e(x), y \in U$, (7) implies

$$gq_a(y), gp_a(x) = gq_a e(x) \in W_1,\ g'q_a(y), g'p_a(x) = g'q_a e(x) \in W_2. \tag{9}$$

Now, (8) and (9) yield

$$gq_a(y), g'q_a(y) \in \mathrm{st}\,(W, \mathcal{W}). \tag{10}$$

Since $\mathcal{W}$ is a star-refinement of $\mathcal{V}'$, there exists a $V' \in \mathcal{V}'$ such that $\mathrm{st}\,(W, \mathcal{W}) \subseteq V'$ and therefore, $gq_a(y), g'q_a(y) \in V' \in \mathcal{V}'$, which proves (5). □

(iv) ⇒ (v) is obvious.

(v) ⇒ (iii). Let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a resolution, which consists of compact Hausdorff spaces X_a. We must show that each mapping $f : X \to \mathbb{R}$ is bounded. Let $\mathcal{V}$ be the open covering of $\mathbb{R}$, which consists of all intervals of length 1. By (R1), there exist an $a \in A$ and a mapping $g : X_a \to \mathbb{R}$ such that

$$(f, gp_a) \prec \mathcal{V}. \tag{11}$$

Since $g(X_a)$ is compact, it is contained in a segment $[b, c] \subseteq \mathbb{R}$. However, $gp_a(X) \subseteq g(X)_a$. Therefore, also $gp_a(X) \subseteq [b, c]$. Now (11) shows that

$$f(X) \subseteq [b-1, c+1]. \tag{12}$$

4. Strong shape.

If $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are cofinite polyhedral resolutions, then there is a functorial one-to-one correspondence between strong shape morphisms $F : X \to Y$ and morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ of the category $\boldsymbol{CPHTop}$ of coherent prohomotopy. Here $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a coherent mapping and $[\boldsymbol{f}]$ is its homotopy class. Therefore, F is an isomorphism of $\boldsymbol{SSh}$ if and only if $[\boldsymbol{f}]$ is an isomorphism of $\boldsymbol{CPHTop}$. For a given mapping $f : X \to Y$, there exists a unique morphism $[\boldsymbol{f}] : \boldsymbol{X} \to \boldsymbol{Y}$ of $\boldsymbol{CPHTop}$ such that

$$[\boldsymbol{f}][\boldsymbol{p}] = [\boldsymbol{q}][f]. \tag{13}$$

By definition, $S_1([f])$ is given by $[\boldsymbol{f}]$. Therefore, in order to prove that a mapping $f : X \to Y$ is a strong shape equivalence, it suffices to find cofinite polyhedral resolutions $\boldsymbol{p}, \boldsymbol{q}$ and an isomorphism $[\boldsymbol{f}]$ of $\boldsymbol{CPHTop}$ such that (13) holds. Also recall that there is a functor $C : \boldsymbol{pro\text{-}Top} \to \boldsymbol{CPHTop}$. It takes mappings of systems into morphisms of $\boldsymbol{CPHTop}$. Now, it is clear that the following assertion holds.

Lemma 3. *Let $f : X \to Y$ be a mapping and let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be cofinite polyhedral resolutions. If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is an isomorphism of $\boldsymbol{pro\text{-}Top}$ and*

$$\boldsymbol{f}\boldsymbol{p} = \boldsymbol{q}f, \tag{14}$$

then f is a strong shape equivalence.

(iv) $\Rightarrow$ (i). Let X admit a cofinal resolution $\boldsymbol{p} = (p_a) : X \to \boldsymbol{X}$, where each X_a, $a \in A$, is a compact polyhedron. We want to prove that $e : X \to \beta X$ is a strong shape equivalence by applying Lemma 3. Therefore, we first define a suitable polyhedral resolution for βX. It is of the form $\boldsymbol{q} : \beta X \to \boldsymbol{X}$, i.e. it uses the same inverse system $\boldsymbol{X}$. For each $a \in A$, we take for $q_a : \beta X \to X_a$, $a \in A$, the unique extension of $p_a : X \to X_a$ to βX, so that

$$p_a = q_a e, \ a \in A. \tag{15}$$

The extension q_a exists because X_a is compact. Uniqueness of q_a follows from the density of $e(X)$ in βX. This is also the reason why $p_{aa'}p_{a'} = p_a$ implies $p_{aa'}q_{a'} = q_a$, for $a \le a'$. Consequently, the mappings q_a, $a \in A$, form a mapping of systems $\boldsymbol{q} : \beta X \to \boldsymbol{X}$.

Lemma 4. *$\boldsymbol{q} : \beta X \to \boldsymbol{X}$ is a cofinite polyhedral resolution of βX.*

Once Lemma 4 is proved, we can apply Lemma 3 to the mapping $e : X \to \beta X$, to the polyhedral resolutions $\boldsymbol{p}$ and $\boldsymbol{q}$ and to the identity isomorphism $\boldsymbol{1} : \boldsymbol{X} \to \boldsymbol{X}$ of $\boldsymbol{pro\text{-}Top}$. In this case (14) becomes

$$\boldsymbol{1}\boldsymbol{p} = \boldsymbol{q}e. \tag{16}$$

It holds, because it reduces to (15).

Proof of Lemma 4: We need only to verify the conditions (R1) and (R2).

Verification of (R1)*:* Let P be a polyhedron, $\mathcal{V}$ an open covering of P and $f : \beta X \to P$ a mapping. Choose a covering $\mathcal{U}$ of P, which is a star-refinement of $\mathcal{V}$. By (R1) for $\boldsymbol{p}$, there exist an index $a \in A$ and a mapping $g : X_a \to P$, such that

$$(fe, gp_a) \prec \mathcal{U}. \tag{17}$$

We claim that

$$(f, gq_a) \prec \mathcal{V}. \tag{18}$$

Indeed, consider any point $y \in \beta X$. Choose members U_1, U_2 of $\mathcal{U}$ so that

$$f(y) \in U_1,\ \ gq_a(y) \in U_2. \tag{19}$$

Choose an open neighborhood W of y in βX so small that

$$f(W) \subseteq U_1,\ \ gq_a(W) \subseteq U_2. \tag{20}$$

Since $e(X)$ is dense in βX, there exists a point $x \in X$ such that $e(x) \in W$. By (17), there exists a $U \in \mathcal{U}$ such that

$$fe(x), gp_a(x) \in U. \tag{21}$$

Since $y, e(x) \in W$, (20) implies

$$f(y), fe(x) \in U_1,\ \ gq_a(y), gp_a(x) = gq_a e(x) \in U_2. \tag{22}$$

Now, (21) and (22) imply

$$f(y), gq_a(y) \in \mathrm{st}\,(U, \mathcal{U}). \tag{23}$$

Since $\mathcal{U}$ is a star-refinement of $\mathcal{V}$, there exists a $V \in \mathcal{V}$ such that $\mathrm{st}\,(U, \mathcal{U}) \subseteq V$ and therefore, $f(y), gq_a(y) \in V \in \mathcal{V}$, which proves (18).

Verification of (R2)*:* Let P be a polyhedron and $\mathcal{V}$ an open covering of P. We choose a covering $\mathcal{V}'$ of P, by (R2) applied to $\boldsymbol{p}$. We claim that this covering also fulfills the requirements of the condition (R2) for $\boldsymbol{q}$. Indeed, let $a \in A$ and let $g, g' : X_a \to P$ be mappings such that

$$(gq_a, g'q_a) \prec \mathcal{V}'. \tag{24}$$

Then also

$$(gp_a, g'p_a) \prec \mathcal{V}'. \tag{25}$$

Therefore, there exists an $a' \geq a$ such that

$$(gp_{aa'}, g'p_{aa'}) \prec \mathcal{V}, \tag{26}$$

which is the desired conclusion. □

Remark 1. In our theorem one cannot replace the Stone-Čech compactification $e : X \to \beta X$ by an arbitrary compactification $i : X \to \tilde{X}$. E.g., if $X = (0,1) \subseteq \mathbb{R}$ and $\tilde{X} = [0,1] \subseteq \mathbb{R}$, then the inclusion $i : X \to \tilde{X}$ is a homotopy equivalence and therefore, also a (strong) shape equivalence. However, X is not pseudocompact. K. Morita showed [11, Corollary 5.3] that every pseudocompact space has the shape of a compact space. This also follows from our theorem. The converse does not hold because $\mathbb{R}$ has compact shape, i.e. the shape of a point, but is not pseudocompact.

Remark 2. Two spaces X and Y can have the same strong shape, $\boldsymbol{SSh}(X) = \boldsymbol{SSh}(Y)$, but the strong shape of their Stone-Čech compactifications can be different, $\boldsymbol{SSh}(\beta X) \neq \boldsymbol{SSh}(\beta Y)$. E.g., if $X = \mathbb{R}$ and $Y = \{*\}$ is a point, then X and Y are of the same homotopy type and therefore have the same strong shape. On the other hand, already C.H. Dowker showed in [2] that the first Čech cohomology group with integer coefficients $\check{H}^1(\beta\mathbb{R}, \mathbb{Z})$ is an uncountable group (see also [5]). Since the Čech cohomology groups are (strong) shape invariants and $\check{H}^1(\{*\}, \mathbb{Z}) = 0$, it follows that $\boldsymbol{SSh}(\beta X) \neq \boldsymbol{SSh}(\beta Y)$. There exist spaces X, Y with $\boldsymbol{SSh}(X) \neq \boldsymbol{SSh}(Y)$, but $\boldsymbol{SSh}(\beta X) = \boldsymbol{SSh}(\beta Y)$. Such an example is given by $X = \mathbb{R}$ and $Y = \beta\mathbb{R}$.

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