Ramsey-like properties for bi-Lipschitz mappings of finite metric spaces

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Abstract. Let (X,ρ), (Y,σ) be metric spaces and $f: X \to Y$ an injective mapping. We put $\|f\|_{Lip} = \sup\{\sigma(f(x),f(y))/\rho(x,y);\ x,y \in X,\ x \neq y\}$, and $\operatorname{dist}(f) = \|f\|_{Lip}.\|f^{-1}\|_{Lip}$ (the *distortion* of the mapping f). Some Ramsey-type questions for mappings of finite metric spaces with bounded distortion are studied; e.g., the following theorem is proved: Let X be a finite metric space, and let $\varepsilon > 0$, K be given numbers. Then there exists a finite metric space Y, such that for every mapping $f: Y \to Z$ (Z arbitrary metric space) with $\operatorname{dist}(f) < K$ one can find a mapping $g: X \to Y$, such that both the mappings g and $f|_{g(X)}$ have distortion at most $(1+\varepsilon)$. If X is isometrically embeddable into a ℓ_p space (for some $p \in [1,\infty]$), then also Y can be chosen with this property.

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1. Notation.

If X is a set and κ a cardinal number (usually a natural number), then $[X]^\kappa$ (resp. $[X]^{<\kappa}$) denotes the set of all subsets of X of cardinality κ (resp. less than κ).

Let (X,ρ), (Y,σ) be metric spaces and $f: X \to Y$ an injective mapping. We put

$$\|f\|_{Lip} = \sup\{\frac{\sigma(f(x),f(y))}{\rho(x,y)};\ x,y \in X, x \neq y\}\,,$$

(the *Lipschitz norm of* f) and

$$\operatorname{dist}(f) = \|f\|_{Lip} \cdot \|f^{-1}\|_{Lip}$$

(the *distortion* of the mapping f). A mapping with distortion at most C is also called a C*-isomorphism* and the metric spaces X and $f(X)$ are called C-isomorphic. Note that 1-isomorphism need not be an isometry, but rather a *homothetic* mapping, which expands every distance by the same factor. A mapping for which $\sigma(f(x),f(y)) \geq \rho(x,y)$ is called *non-contracting*. For such a mapping, $\operatorname{dist}(f) \leq \|f\|_{Lip}$. Finally, for metric spaces X and Z, we define

$$\operatorname{dist}(X, \subseteq Z) = \inf\{\operatorname{dist}(f);\ f: X \to Z \text{ an injective mapping}\}.$$

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If (X, ρ) is a metric space and $a > 0$ a real number, we define a metric space aX as a metric space on the same set as X, with a metric σ given by $\sigma(x, y) = a\rho(x, y)$.

If X and Y are metric spaces, we define the metric space $X \dot{\cup} Y$ (the *disjoint union* of X and Y); the point set of this metric space will be the disjoint union of point sets of X and Y, the metric on X and Y is preserved and it is arbitrarily extended to a metric on the whole set (one easily sees that this is always possible).

All Banach spaces are considered real. The spaces ℓ_p $(1 \le p \le \infty)$ are the spaces of real sequences with the appropriate ℓ_p-norm, $L_p = L_p(0, 1)$. The symbol ℓ_p^n will denote the space of n-tuples of real numbers with ℓ_p-norm. For Banach spaces E and F of the same dimension, the *Banach-Mazur distance* of E and F is the quantity

$$d(E, F) = \inf\{\|T\| \cdot \|T^{-1}\|;\ T : E \to F \text{ an invertible linear operator}\}.$$

The Banach-Mazur distance basically says how similar the unit balls of E and F can be made by a suitable affine transformation.

The symbol C_1^n will denote the n-dimensional *Hamming cube*, which is the set $\{0, 1\}^n$ with the metric of subspace of ℓ_1^n.

The metric spaces $\{1, \dots, n\}$ and $\mathbb{Z}$ (the set of integers) are considered with the metric of subspace of the real numbers.

2. Motivation and background.

There are many theorems saying that a sufficiently big structure of a given type must contain very regular substructures. Such a type of theorems in combinatorics belong to so-called Ramsey Theory (see [GRS80], [Ne88]). We will consider such a type of theorems for metric spaces, in the following sense: Suppose that we are given a finite metric space X. We want to find a bigger metric space Y, such that if we deform it by any mapping with a bounded distortion, then we can find a copy of X in it which is deformed only very slightly. A class of metric spaces in which we can find such a Y for every X will be called *BD-Ramsey* (BD stands for "bounded distortion").

We will establish the BD-Ramsey property for the class of all finite metric spaces (using combinatorial tools) and for the classes of all finite subspaces of the spaces ℓ_p (using a result on differentiability, a Ramsey-type result for Banach spaces and a compactness argument).

Ramsey-type statements are intensively studied for Banach spaces, in the framework of so-called Local Theory. A monograph on this subject is [MiS86], to which we refer for proofs and references concerning the Local Theory results mentioned in the sequel. A basic result in this area is the Dvoretzky theorem, which says that for any $\varepsilon > 0, n > 0$ there exists N such that any N-dimensional Banach space contains an n-dimensional subspace E whose Banach-Mazur distance from ℓ_2^n is $\le 1 + \varepsilon$ (see [MiS86] for a modern proof). Also, nearly tight asymptotic bounds are known for the minimum necessary value of such N.

As for subspaces nearly isomorphic to ℓ_p for p other than 2, there is a theorem of Krivine and its strong generalization by Maurey and Pisier. We will use the following version of Krivine's theorem:

Theorem 2.1. *Let $1 \le p \le \infty$. For every n, $\varepsilon > 0$, $C > c > 0$ there exists N, such that if ψ is any norm on ℓ_p^N with $c \le \psi(x) \le C$ for every x with $\|x\|_p = 1$, then there exists a linear n-dimensional subspace $F \subseteq \ell_p^N$ and a constant $a > 0$ in such a way that $(1-\varepsilon)a\|x\|_p < \psi(x) < (1+\varepsilon)a\|x\|_p$ for all $x \in F$.* □

Let us remark that this version is a reformulation of the one given in [MiS86], which emphasizes the aspects of this result we will need. The above version is obtained from the one in [MiS86] by straightforward considerations.

If the norm ψ has some special properties, one can say more about subspaces almost isomorphic to ℓ_p^n. In particular, the bounds on the dimension of subspaces almost isomorphic to ℓ_p^n in spaces ℓ_q^N have been intensively studied (e.g. [AM83], [Sche81], [JS82]).

As for works with more "metric" aspects, the classical papers of Enflo [Enf69a], [Enf69b], [Enf70] should be mentioned; here a minimum distortion necessary for embedding certain finite metric spaces into certain Banach spaces is established (this is used as a tool for other things in these papers). Metric analogs of Banach-space notions are investigated in [BMW86]. This work also extends the Enflo's results and it is very close in spirit to questions investigated in the present paper; among others, it establishes the BD-Ramsey property for the class of finite subspaces of ℓ_1. Other Ramsey-type questions for metric spaces are investigated in [BFM86]; more about this in Section 7.

The plan of this paper is the following: Section 3 has an auxiliary character; it gives some easy connections between an embeddability of a given (infinite) metric space and the embeddability of all its finite subspaces ("compactness" conditions). The definitions and easy examples concerning the BD-Ramsey property are given in Section 4. Sections 5 and 6 contain some theorems on the BD-Ramsey property. Section 7 applies a combinatorial approach to finding special subspaces in metric spaces.

3. Embedding of finite and infinite spaces.

In this section we show some results on the connection between the embeddability of a metric space Y into a given metric space Z and the embeddability of all finite subspaces of Y into Z. Let us begin by some definitions:

Definition 3.1. Let $\mathcal{Z}$ be a class of metric spaces. The class $\mathcal{Z}$ is called *homothetically closed*, if for every $Z \in \mathcal{Z}$ and for every number $a > 0$ there exists $Z' \in \mathcal{Z}$, such that aZ is isometric to a subspace of Z',

closed on finite subspaces, if for every finite metric space (X, ρ) the following implication holds: If for every $\varepsilon > 0$ there exists $(Z, \sigma) \in \mathcal{Z}$ and a mapping $f : X \to Z$ with

$$(1-\varepsilon)\rho(x,y) < \sigma(f(x), f(y)) < (1+\varepsilon)\rho(x,y)$$

(such a mapping is naturally called a $(1+\varepsilon)$-*isometry*), then X can be isometrically embedded into some $Z \in \mathcal{Z}$.

We say that *embeddability into $\mathcal{Z}$ is determined by finite subspaces for weight κ* (κ a cardinal), if for every metric space of weight $\le \kappa$ the following implication holds: If every metric space $F \in [X]^{<\omega}$ can be isometrically embedded into some $Z \in \mathcal{Z}$, then also X can be isometrically embedded into some $Z \in \mathcal{Z}$.

The appendix "for weight κ" will be omitted if $\mathcal{Z}$ has the property for every κ.

If $\mathcal{Z}$ consists of a single metric space, we will speak directly about this metric space instead of the class.

The notion of "homothetically closed" appears naturally in connection with Ramsey-type questions. The further two notions are (rather special) properties of the class $\mathcal{Z}$, allowing to use a compactness argument for this class.

Example 3.2. A simplest example of a metric space, which is not closed on finite subspaces, are the rationals. But even a (complete) Banach space does not necessarily have this property: Let us choose a sequence of numbers $p_n > 1$ tending to 1. Let us consider a direct ℓ_2-sum of the spaces $L_{p_n}, n \in \omega$ (which is a space of sequences $(x_1, x_2, \dots)$, where $x_i \in L_{p_i}$ and $(\|x_1\|_{p_1}, \|x_2\|_{p_2}, \dots) \in \ell_2$; the norm of such a sequence is defined as the ℓ_2-norm of the corresponding sequence of norms — see e.g. [Lin66]). This metric space contains subspaces arbitrarily near to the 2-dimensional Hamming cube C_1^2, but it does not contain this cube 1-isomorphically. Similar direct sums were used as the examples of uniformly homeomorphic non-isomorphic Banach spaces — see the survey paper [Ben85].

A simple sufficient condition for the closedness on finite subspaces is the following (the proof is simple and we omit it):

Lemma 3.3. *Let Z be a metric space such that for every $n \geq 1$ and every $r > 0$ there exists a compact subset $K_{n,r} \subseteq Z$ such that every n-point subspace of Z with diameter at most r can be isometrically embedded into $K_{n,r}$. Then Z is closed on finite subspaces.* $\square$

By this criterion, all finite-dimensional Banach spaces are closed on finite subspaces. The spaces ℓ_p and L_p are also closed on finite subspaces (this can be seen from the fact that each their n-point subspace can be realized in $\ell_p^{n(n-1)/2}$ — see [Fi88]).

Let us remark that if a class of metric spaces is defined by a system of non-strict inequalities, which must hold on subsets of a certain constant size (e.g. the four-point condition defining tree metric spaces) then it is closed on finite subspaces.

It is not difficult to verify that the embeddability into any finite dimensional Banach space is determined by finite subspaces (using a simple compactness argument). Bretagnolle et al. [BDK66] prove that for every $p \in [1, \infty]$, the embeddability into the class of metric spaces

$$\{L_p(P);\ P \text{ a measure space}\}$$

is determined by finite subspaces (the proof is quite complicated). By the universality of $L_p(0,1)$ for *separable* Banach spaces of the form $L_p(P)$ (see e.g. [LT73]), embeddability into $L_p(0,1)$ is determined by finite subspaces for weight ω (a direct proof of this fact for the separable Hilbert space is easy).

On the other hand, the space L_p ($p \neq 2, \infty$) cannot be isometrically embedded into ℓ_p, so e.g. embeddability into ℓ_p is not determined by finite subspaces, not even for weight ω.

The following is an analog of Rado's lemma (used in compactness arguments in combinatorics):

Lemma 3.4. *Let (X, ρ) be a metric space, $K \geq 1$ a number. For every $F \in [X]^{<\omega}$, let a metric ρ_F on F be given, so that*

$$K^{-1}\rho(x,y) \leq \rho_F(x,y) \leq K\rho(x,y)$$

for every $x, y \in F$. Then there exists a metric σ on X with the following property: For every $F \in [X]^{<\omega}$ and for every $\varepsilon > 0$ there exists $G \in [X]^{<\omega}$, $F \subseteq G$, such that the values of σ and ρ_G differ by at most ε for every pair of points $x, y \in F$. In particular, $K^{-1}\rho(x,y) \leq \sigma(x,y) \leq K\rho(x,y)$ for every $x, y \in X$.

Proof: Let us consider the topological product P of intervals $[1/K, K]$, indexed by all pairs $\{x, y\} \in [X]^2$. P is a compact topological space, whose points can be identified with symmetric real functions on $X \times X$, which are zero on the diagonal. For every $F \in [X]^{<\omega}$ let us consider the set

$$C_F = \{\sigma \in P;\ \forall \varepsilon > 0\ \exists G \in [X]^{<\omega} : F \subseteq G\ \&\ \forall \{x,y\} \in [F]^2 : |\sigma(x,y) - \rho_G(x,y)| < \varepsilon\} = \overline{\{\sigma \in P;\ \exists G \in [X]^{<\omega} : F \subseteq G\ \&\ \forall \{x,y\} \in [F]^2 : \sigma(x,y) = \rho_G(x,y)\}}.$$

Each C_F is nonempty and closed (this is seen from the second formula for C_F). Furthermore

$$C_{F_1 \cup F_2 \cup \ldots \cup F_n} \subseteq C_{F_1} \cap C_{F_2} \cap \ldots \cap C_{F_n} \qquad (F_1, \ldots, F_n \in [X]^{<\omega}),$$

and thus the sets C_F form a centered system of closed sets. It is easily verified that any function σ from the (nonempty) intersection of all C_F satisfies the claim of the lemma. $\square$

4. BD-Ramsey property - definitions and examples.

Definition 4.1. Let X be a metric space (we will be interested mainly in the case when X is finite). Let Y be a metric space, $\mathcal{Z}$ a class of metric spaces, and $\varepsilon > 0, K > 1$.

We say that Y is *(K, ε)-Ramsey for X relative to $\mathcal{Z}$* if for any $Z \in \mathcal{Z}$ and any embedding $f : Y \to Z$ with $\mathrm{dist}(f) < K$, there exists a mapping $g : X \to Y$, such that $\mathrm{dist}(g) \leq 1 + \varepsilon$, $\mathrm{dist}(f|_{g(X)}) \leq 1 + \varepsilon$.

If we leave out "relative to $\mathcal{Z}$" we mean relative to the class of all metric spaces.

We say that a class $\mathcal{Y}$ is *BD-Ramsey for X*, if for every $\varepsilon > 0$ and every $K > 1$ there exists $Y \in \mathcal{Y}$, which is (K, ε)-Ramsey for X relative to $\mathcal{Z}$. Finally a class $\mathcal{Y}$ is called *BD-Ramsey* if it is BD-Ramsey for every $X \in \mathcal{Y}$ (all the previous could be again taken relative to a class $\mathcal{Z}$, and instead of a class consisting of a single metric space we will refer directly to this metric space).

Remarks: If $\mathcal{Z}' \subseteq \mathcal{Z}$, then the BD-Ramsey property relative to $\mathcal{Z}$ implies the BD-Ramsey property relative to $\mathcal{Z}'$.

A metric space Y can be BD-Ramsey for (any) X relative to $\mathcal{Z}$ for a trivial reason — there is no embedding of Y into a metric space of $\mathcal{Z}$. Such cases we will not discuss in the sequel.

Another interpretation of the definition of a BD-Ramsey class is the following: If $\mathcal{Y}$ contains an "uniform" counterexample on the isometric embeddability into $\mathcal{Z}$ (the distortion needed for embedding into each $Z \in \mathcal{Z}$ is at least $1+\varepsilon$, $\varepsilon > 0$ fixed), then $\mathcal{Y}$ also contains spaces whose embedding into $\mathcal{Z}$ requires arbitrarily large distortions.

Definition 4.2. We say that a class $\mathcal{Y}$ is *representable* in $\mathcal{Y}'$, if for every $\varepsilon > 0$ and every $Y \in \mathcal{Y}$ there exists $Y' \in \mathcal{Y}'$, such that Y' contains a $(1+\varepsilon)$-isomorphic copy of Y. The classes $\mathcal{Y}$ and $\mathcal{Y}'$ are *equivalent* if they are representable in each other.

Obviously, if $\mathcal{Z}$ is homothetically closed and $\mathcal{Y}'$ and $\mathcal{Y}$ are equivalent classes, then $\mathcal{Y}$ is BD-Ramsey relative to $\mathcal{Z}$ iff $\mathcal{Y}'$ is.

Example 4.3. Some groups of mutually equivalent classes:

(i) $[\mathbb{R}]^{<\omega}$, $[\mathbb{Z}]^{<\omega}$, $\{\{1,2,\dots,n\};\ n \in \omega\}$.

(ii) The class of all finite metric spaces and the class of all (unweighted) graphs (as metric spaces).

(iii) $[\ell_1]^{<\omega}$ and the class of Hamming cubes $\{C_1^n;\ n \in \omega\}$ (here one notes that a n-dimensional Hamming cube contains the metric space $\{1,2,\dots,n\} \subseteq \mathbb{R}$ isometrically).

(iv) $[\ell_p]^{<\omega}$ and the class of all metric spaces $\{1,2,\dots,n\}^n$ with ℓ_p-metric ($1 \leq p \leq \infty$).

The following "compactness principle" often allows to infer the BD-Ramsey property of a class of finite metric spaces from the BD-Ramsey property of an infinite metric space.

Lemma 4.4. *Let X be a finite metric space, and let $\mathcal{Z}$ be class of metric spaces which is homothetically closed, closed on finite subspaces, and such that embeddability into $\mathcal{Z}$ is determined by finite subspaces for weight κ. Then if a metric space Y of weight $\leq \kappa$ is BD-Ramsey for X relative to $\mathcal{Z}$, then the class $[Y]^{<\omega}$ is also BD-Ramsey for X relative to $\mathcal{Z}$.*

PROOF: For contradiction, assume that $[Y]^{<\omega}$ is not BD-Ramsey for X under the assumptions of the lemma. Then for some fixed $\varepsilon_0 > 0$, K_0 the following is true: For every $F \in [Y]^{<\omega}$ there exists an embedding $f_F : F \to Z$ ($(Z,\sigma) \in \mathcal{Z}$), satisfying

(a) $\rho(x,y) < \sigma(f(x), f(y)) < K_0\rho(x,y)$ for every $x, y \in X$ and
(b) on no subspace $X' \subseteq F$, which is $(1+\varepsilon_0)$-isomorphic to X, f behaves like an $(1+\varepsilon_0)$-isomorphism.

Each f_F defines a metric ρ_F on F. By Lemma 3.4, there exists a metric σ on Y, such that on every $F \in [Y]^{<\omega}$ it can be approximated by some $\rho_G|_F$, where $F \subseteq G \in [Y]^{<\omega}$. This and the assumptions of the lemma imply that there exists an isometry $g : (Y, \sigma) \to Z \in \mathcal{Z}$. Let h denote the identity map from (Y, ρ) to (Y, σ). The mapping $f = g \circ h$ has distortion $\leq K_0$ and it is easily seen that this gives a contradiction with the BD-Ramsey property of Y for X. $\square$

The assumptions of the preceding lemma are satisfied e.g. for $\mathcal{Z}$ being the class of all metric spaces or the class of all subspaces of some L_p (in this case with $\kappa = \omega$).

We will now mention several easy negative examples connected to BD-Ramsey property. The following example shows that it would not be reasonable to require an isometric embedding of X into $f(Y)$ in the definition of BD-Ramsey property:

Consider the metric space $Z = \ell_\infty \cap \mathrm{Q}^\omega$, i.e. the space ℓ_∞ where we take only points with rational coordinates. Any finite metric space can be embedded into this space with the deformation arbitrarily close to 1, but (when it contains an irrational distance) not isometrically. Not even the completeness of the metric space Z saves the situation — it suffices to consider the union of all finite metric spaces with rational metrics. The important property here is the closedness on finite subspaces.

Another simple example shows why we require only an almost isomorphic copy of X in Y and not an almost isometric copy: Let ρ denote the usual metric on the space ℓ_∞, and let us consider a function $\sigma = \tau \circ \rho$, where $\tau : [0, \infty) \to (0, \infty]$ is a function defined by $\tau(x) = \min(x, (x + 1)/2)$. The function τ is a minimum of two additive functions, and from this it is seen that it is subadditive, hence $\tau \circ \rho$ is a metric. Let Z denote the metric space with the same point set as ℓ_∞ and with metric σ. The identity map from ℓ_∞ to Z has distortion at most 2, so $\mathrm{dist}(X, \subseteq Z) \leq 2$ for every finite metric space X. At the same time, for the metric space $U = \{0, 1, 2\}$ we have $\mathrm{dist}(U, \subseteq Z) > 1$ (if $f : U \to Z$ were an isometry, it would have to be $\rho(f(0), f(1)) = \rho(f(1), f(2)) = \tau^{-1}(1) = 1$, and at the same time $\rho(f(0), f(2)) = \tau^{-1}(2) > 2$ — a contradiction).

Finally let us remark that a class of metric spaces of the form $[\ell_p^n]^{<\omega}$ for some n, p is never BD-Ramsey, not even relative to L_p. Indeed, for $p \neq 2$ we may (for instance) use the fact that the identity mapping of ℓ_p^n into ℓ_2^n has a bounded distortion (for n, p fixed), but there exists a finite subset F of ℓ_p^2 which cannot be isometrically embedded into ℓ_2, so $\mathrm{dist}(F, \subseteq \ell_2^n) > 1 + \varepsilon$ for some fixed $\varepsilon > 0$. At the same time, ℓ_2^n is isometrically embeddable into L_p. For $p = 2$, we may consider the mapping (in the coordinate form) $(x_1, x_2, \ldots, x_n) \mapsto (2x_1, x_2, \ldots, x_n)$. This mapping obviously does not map the vertices of any regular n-dimensional simplex in ℓ_2^n with distortion 1.

5. Proofs of the BD-Ramsey property using differentiability.

In this section we will prove the BD-Ramsey property, using theorems about differentiability of nice (in our case Lipschitz) maps. The basic instance of such a theorem is perhaps the result of Lebesgue that any absolutely continuous real function has a derivative almost everywhere. Rademacher proved this result for real Lipschitz functions on finite-dimensional Euclidean spaces. Many authors have considered this problem in more general settings (see [Ben85] for an account); the most general results so far were obtained by D. Preiss [Pre88].

We start by proving the BD-Ramsey property of $\mathbb{R}$ for every its finite subspace relative to ℓ_2 (this result will be generalized later). Any Lipschitz map $f : \mathbb{R} \to \ell_2$ has a derivative at some point x_0 (almost everywhere, in fact). This means that on a small neighborhood of x_0, f can be approximated by a linear mapping:

$$\|f(x_0 + h) - f(x_0) - h \cdot z\| = o(|h|)$$

($z \in \ell_2$ is some vector, and if f has a bounded distortion, we have $z \neq 0$).

Now let the numbers n and $\varepsilon > 0$ be given. Let us put $\nu = \varepsilon/2n$ and choose a number $\delta > 0$ so small that when $|h| < \delta$, then

$$\|f(x_0 + h) - f(x_0) - h.z\| \leq \nu|h|.$$

If we now consider the metric space $X = \{x_0, x_0 + \delta/n, \ldots, x_0 + (n-1)\delta/n\}$, it is easily seen that the mapping $f|_X$ has distortion at most $1 + \varepsilon$. This shows that $\mathbb{R}$ is BD-Ramsey for every its finite subspace relative to ℓ_2, and by compactness (Lemma 4.4) $[\mathbb{R}]^{<\omega}$ is also a BD-Ramsey class relative to ℓ_2.

There are two obstacles to a generalization of the above argument. If the domain space is more complex than $\mathbb{R}$ (e.g. a higher-dimensional Banach space), then a linear map on it need not be a homothety and some additional "Ramsey-type" result for Banach spaces is needed. Such results will be the theorem of Krivine.

The second problem is connected to the space into which the mappings go. If it does not have Radon-Nikodým property (see e.g. [DU77] for definition and discussion), then there exist Lipschitz maps with differential at no point. However, the famous example of such a mapping (from $\mathbb{R}$ to L_1, see [Ar76]) is even an isometry, so "metrically" it behaves quite nicely. This motivates the following definition:

Definition 5.1. Let X be a Banach space (with a norm $\|\cdot\|$) and (Z, σ) a metric space. We say that a mapping $f : X \to Z$ has a *metric differential* at a point $x_0 \in X$, if there exists a pseudonorm ψ on X, such that

$$\sigma(f(x_0 + h), f(x_0 + k)) = \psi(h - k) + o(\|h\| + \|k\|) \quad (h, k \in X).$$

B. Kirchheim proved a theorem (in a quite different context), which in our terminology can be reformulated as follows:

Theorem 5.2 [Kir88]. *Let Z be any metric space and let $f : \ell_2^n \to Z$ be a Lipschitz mapping. Then f has a metric differential almost everywhere.* □

Obviously in the above theorem, we can take any norm on ℓ_2^n such that the identity map has a bounded distortion with respect to the Euclidean norm (e.g. the ℓ_p-norm). The theorem essentially says that a Lipschitz deformation of the Euclidean metric is almost everywhere locally very simple. The proof is not simple; it uses a theorem on the existence of points of density of a measurable set and further tools from measure theory on metric spaces.

As a first easy consequence we obtain (by an argument analogous to the one given on the beginning of this section, but using the metric differential).

Theorem 5.3. *The class of all finite subspaces of* $\mathbb{R}$ *is BD-Ramsey.* □

The main result following from the Kirchheim theorem is the following:

Theorem 5.4. *For every* $p \in [1, \infty]$, *the class* $[L_p]^{<\omega}$ *is BD-Ramsey* (*for* $p = \infty$, *we mean just the class of all finite metric spaces*).

PROOF: By $\|\cdot\|$ we will denote the norm on L_p and its subspaces. Let $f : L_p \to Z$ be a mapping with bounded distortion and let X be a finite subspace of L_p. Let us choose a subspace $E = \ell_p^N$ of L_p, where N is sufficiently large. We will again consider a point x_0, where $f|_E$ has a metric differential, and let ψ be the pseudonorm appearing in the definition of the metric differential. Since the distortion of f is bounded, the value $\psi(x)$ for unit vectors $x \in E$ is bounded by constants both from above and from below (the constants are independent of N) — in particular, ψ is a norm.

Now we can use Krivine's theorem (Theorem 2.1). When we consider ℓ_p^N where N is sufficiently large, then there exists an n-dimensional linear subspace F, on which the norm ψ is almost a constant multiple of the ℓ_p norm. If we now consider a homothetic copy of the space X, embedded into the space F into a small neighborhood of the point x_0, also f behaves almost like a homothety on it. Hence the space L_p is BD-Ramsey for every its finite subspace. The proof is finished by a compactness argument using Lemma 4.4. □

For $p = \infty$ we give an alternative proof of this result in the next section, using quite different methods.

6. Combinatorial proofs of the BD-Ramsey property.

In this section we shall use finite (combinatorial) methods to get alternative proofs of BD-Ramsey property in some cases. These methods also allow to obtain quantitative bounds on the size of the Ramsey space for a given metric space (but we will not do this). Perhaps a simplest example of this approach is the following:

Theorem 6.1. *The class* $\{D_n;\ n \in \omega\}$ (*where* D_n *denotes an* n-*point metric space where every two points have distance* 1) *is BD-Ramsey.*

PROOF: Let $f : D_n \to (Z, \sigma)$ be a mapping with $\mathrm{dist}(f) \le K$. Without loss of generality we may assume that f is non-contracting and $\|f\|_{Lip} \le K$. Color every pair $\{x, y\} \in [D_n]^2$ by the color $\lceil \sigma(f(x), f(y))/\varepsilon \rceil \in \{0, 1, \dots, \lceil K/\varepsilon \rceil\}$. When $n = n(K, \varepsilon, k)$ is large enough, by Ramsey's theorem there exists a k-point subspace X (copy of D_k) in D_n, for which all pairs $\{x, y\} \in [X]^2$ have the same color, which means that $f|_X$ is a $(1 + \varepsilon)$-isomorphism. □

Now we will give a combinatorial **proof of Theorem 5.3**. By Example 4.3 (i) it suffices to show that $[\mathbb{Z}]^{<\omega}$ is a BD-Ramsey class for every $X = \{1, 2, \dots, n\}$. The metric space Y will be chosen of the form $Y = \{1, 2, \dots, N\}$, where $N = N(n, K, \varepsilon)$ is large enough.

Let $f : Y \to (Z, \sigma)$ be a non-contracting mapping with $\|f\|_{Lip} \le K$. For $d = 1, 2, \dots, N$ we define numbers

$$K(d) = \max\{\frac{\sigma(f(x), f(y))}{|x - y|};\ x, y \in Y,\ |x - y| = d\}.$$

Since $\|f\|_{Lip} \leq K$, we have $K(d) \in [1, K]$ for every d.

A basic observation is that for every natural number m and for every d it is $K(md) \leq K(d)$ (from the triangle inequality in the metric space Z).

Now let t be a large enough integer and M a large enough power of 2 (depending on n, K, ε). Let us consider the nonincreasing sequence of values $K(1), K(M), K(M^2), \dots, K(M^{t-1})$. By the pigeonhole principle, there must be some consecutive pair of these values, say $K(M^i)$ and $K(M^{i+1})$, differing by less than $\nu = K/t$.

Let $d_1 = M^i$, $d_2 = M^{i+1}$ and $K_1 = K(d_1), K_2 = K(d_2)$. Then $d_2/d_1 = M$ and $K_2 \leq K_1 \leq K_2 + \nu$. Let the points $x_0 < y_0$ have distance d_2 and their f-images distance $K_2 d_2$.

Let d be a divisor of d_2 and a multiple of d_1, so that $d_2/d \geq n$ and simultaneously $d > \nu d_2/\varepsilon$. We show that on the subspace $X' = \{x_0, x_0 + d, x_0 + 2d, \dots, y_0\} \subseteq Y$ the mapping f is very near to a homothety with ratio K_1.

Let $x = x_0 + rd$, $y = x_0 + sd \in X'$ $(r < s)$. We have

$$\sigma(f(x), f(y)) \leq K(|x - y|)|x - y| \leq K_1|x - y|.$$

Let us assume that $\sigma(f(x), f(y)) < (K_1 - \varepsilon)|x - y|$. Then

$$\begin{aligned} K_2 d_2 = \sigma(f(x_0), f(y_0)) &\leq \sigma(f(x_0), f(x)) + \sigma(f(x), f(y)) + \sigma(f(y), f(y_0)) \leq \\ &\leq K_1(|x - x_0| + |y_0 - y|) + (K_1 - \varepsilon)|x - y| \leq K_2 d_2 + \nu d_2 - \varepsilon|x - y| < K_2 d_2, \end{aligned}$$

since $\nu d_2 < \varepsilon d \leq \varepsilon |x - y|$ — a contradiction. □

Let us remark that the size of the Ramsey space for $\{1, \dots, n\}$, which we could calculate from the above proof, is nearly the best possible in general.

The main result of this section is an alternative proof of 5.4 for $p = \infty$, i.e. of the following:

Theorem 6.2. *The class of all finite metric spaces is BD-Ramsey.*

For the proof of this theorem we use the following strong tool from Ramsey theory:

Lemma 6.3 [NR79]**.** *For every graph metric space X and any natural number r there exists a graph metric space Y with the following property: Whenever we color all pairs of points of Y by r colors, then there exists a subspace X' isometric to X in Y, such that the color of a pair of points of X' depends on the distance of these points only.* □

This result has been announced in [NR79], and its proof has never been published. Today it can be quite easily proved using so-called induced Ramsey theorems for set systems (see [Ne88]). These theorems are based on so-called partite constructions — the metric space Y arises by gluing some copies of X along edges of the same length. Since an amalgamation of e.g. Euclidean metric spaces can in general give rise to non-Euclidean metric spaces, this method cannot be used to prove BD-Ramsey property e.g. for $[\ell_2]^{<\omega}$. At present there are probably no known analogs of the previous lemma, where the "Ramsey" object Y would satisfy some additional requirement of an algebraic nature.

Proof of Theorem 6.2: Let X be a given finite metric space (we may take it to be a graph metric, by Example 4.3 (ii)) and let ε, K be given numbers. Let us put

$$\Delta = \{\rho(x,y);\ x \neq y,\ x,y \in X\}$$

and let us consider Δ as a metric subspace of $\mathbb{R}$. By Theorem 5.3 there exists $\Delta' \subseteq \mathbb{R}$, such that for any embedding of Δ' into some metric space with $\operatorname{dist}(f) \leq K$, there exists an embedding $g : \Delta \to \Delta'$, such that both g and $f|_{g(\Delta)}$ are $(1+\varepsilon)$-isomorphisms. Moreover, we may obviously require that $\|g\|_{Lip} \in A$, where A is some finite set of numbers (not depending on f). Let us put

$$X' = \Delta' \dot\cup \dot{\bigcup_{a\in A}} aX.$$

Let us put $r = \lceil K/\varepsilon \rceil$ and let Y be a metric space constructed for X' (and r) according to Lemma 6.3.

Let f be a non-contracting mapping of Y into some metric space (Z,σ), satisfying $\|f\|_{Lip} \leq K$. For every pair $\{x,y\} \in [Y]^2$ we define the color of this pair as the number

$$\left\lfloor \frac{\sigma(f(x),f(y))}{\varepsilon\rho(x,y)} \right\rfloor \in \{0,1,\dots,r-1\}.$$

Now there exists an isometric copy X'_0 of the space X' in Y, where the color of pairs of points depends on their distance only. Since X' contains Δ', we get that there exists a $(1+\varepsilon)$-isomorphic embedding $g : \Delta \to X'_0$, such that $f|_{g(\Delta)}$ is a $(1+\varepsilon)$-isomorphic mapping and moreover $\|g\|_{Lip} = a \in A$. But this means that all the distances occurring in the metric space aX are deformed in the same ratio (upto a factor $(1+\varepsilon)^2$) by the mapping $f|_{X'_0}$. This means that there exists a copy of aX, contained in X'_0, which is mapped by f with distortion $\leq (1+\varepsilon)^3$. $\square$

Another class, for which one can show the BD-Ramsey property by essentially combinatorial means, is $[\ell_1]^{<\omega}$. This property easily follows from a more general result of [BMW86].

7. Further applications of the combinatorial approach.

The combinatorial approach yields also some theorems about special subspaces of metric spaces. We will mention two examples of such results here and sketch one proof; details can be found in [Ma89].

A nice example is a result of [BFM86], a certain (not very deep) analogy of the Dvoretzky theorem for metric spaces.

Theorem 7.1. *For every $\varepsilon > 0$ there exists a positive constant $C(\varepsilon)$ (one may choose $C(\varepsilon) = \Omega(\frac{\varepsilon}{\log \varepsilon^{-1}})$), such that for every n-point metric space (X,ρ) there exists a subset $Y \subseteq X$, $|Y| \geq C(\varepsilon)\log n$, such that* $\operatorname{dist}(Y, \subseteq \ell_2) \leq 1+\varepsilon$. $\square$

One can prove this theorem analogously to 6.1; we give only a sketch of the proof. A pair $\{x,y\} \in X$ is colored by the color $\lceil \log_{1+\varepsilon/2} \rho(x,y) \rceil \bmod r$, where $r = \lceil (\log \frac{1}{\varepsilon})/\varepsilon \rceil$. If Y is a k-point homogeneous subset, then any two distances in Y have ratio roughly ε^q (q an integer), upto a factor $1+\varepsilon$. One easily shows (by induction on the number of points) that such a metric space can be almost isomorphically embedded into ℓ_2. As for quantitative results, from the known bounds on Ramsey numbers (see [GRS80]) one gets the same order of magnitude for the size of the subspace Y as in 7.1, only with this method the value of $C(\varepsilon)$ will be larger by a factor $\log \varepsilon^{-1}$.

Actually one can find even much more special subspaces in any sufficiently large metric space. We will state the result in infinite form, without quantitative bounds, and we omit the proof.

Theorem 7.2. *For every $\varepsilon > 0, K > 0$ the following holds: Every infinite metric space X contains either a $(1+\varepsilon)$-isomorphic copy of D_ω, or a $(1+\varepsilon)$-isomorphic copy of a metric space of the form $\{x_0, x_1, \dots\} \subseteq (0,\infty)$, where $x_i/x_{i-1} \geq K$ for every $i = 1, 2, \dots$.* □

References

[AM83] Alon N., Milman J., *Embeddings of l_∞^k in finite dimensional Banach spaces*, Israel J. Math. **45** (1983), 265–280.

[Ar76] Aronszajn N., *Differentiability of Lipschitz mappings between Fréchet spaces*, Studia Math. **57** (1976), 147–190.

[BDK66] Bretagnolle J., Dacunha-Castelle D., Krivine J.L., *Lois stables et espaces L^p*, Ann. Inst. H. Poincaré, Sect. B 2(1966) pp. 231–259.

[Ben85] Benyamini Y., *The uniform classification of Banach spaces*, Longhorn Notes, The Univ. of Texas at Austin, Functional analysis seminar 1984-85, pp. 15–38.

[BFM86] Bourgain J., Figiel T., Milman V., *On Hilbertian subspaces of finite metric spaces*, Israel J. Math. **55** (1986), 147–152.

[BMW86] Bourgain J., Milman V., Wolfson H., *On type of metric spaces*, Trans. Am. Math. Soc. **294** (1986), 295–317.

[DU77] Diestel J., Uhl J.J., Jr., *Vector measures*, Math. Surveys 15, AMS, Providence, 1977.

[Enf69a] Enflo P., *On a problem of Smirnov*, Ark. Mat. **8** (1969), 107–109.

[Enf69b] ———, *On the nonexistence of uniform homeomorphisms between L_p-spaces*, Ark. Mat. **8** (1969), 103–105.

[Enf70] ———, *Uniform structures and square roots in topological groups II*, Israel J. Math. **8** (1970), 253–272.

[Fi88] Fichet B., *L_p spaces in data analysis*, in: Classification and related methods of data analysis, H.H. Bock ed., North Holland, 1988, pp. 439–444.

[GRS80] Graham R.L., Rothschild B.L., Spencer J.H., *Ramsey theory*, J.Wiley & sons, 1980.

[JS82] Johnson W., Schechtman G., *Embedding l_p^m into l_1^n*, Acta Math. **149** (1982), 71–85.

[Kir88] Kirchheim B., *Geometry of measures* (in Czech), thesis, Charles University, Prague, 1988.

[Lin66] Lindenstrauss J., *On nonlinear projections in Banach spaces*, Michigan Math. J. **11** (1966), 268–287.

[LT73] Lindenstrauss J., Tzafriri L., *Classical Banach spaces*, Lecture Notes in Mathematics 338, Springer-Verlag, 1973.

[Ma89] Matoušek J., *Lipschitz distance of metric spaces* (in Czech), CSc. degree thesis, Charles University, 1989.

[MiS86] Milman V.D., Schechtman G., *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Mathematics 1200, Springer-Verlag, 1986.

[Neš91] Nešetřil J., *Ramsey Theory*, Chapter for Handbook of Combinatorics, North-Holland, to appear.

[NR79] Nešetřil J., Rödl V., *Partition theory and its applications*, in: Surveys in Combinatorics, (B. Bollobás ed.), Cambridge Univ. Press, Cambridge-London, 1979 pages 96–156.

[Pre88] Preiss D., *Differentiability of Lipschitz functions on Banach spaces*, Journal of Functional Analysis **91** (1990), 312–345.

[Sche81] Schechtman G., *Random embeddings of Euclidean spaces in sequence spaces*, Israel J. Math. **40** (1981), 187–192.

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