Order continuous linear functionals on non-locally convex Orlicz spaces

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Abstract. The space of all order continuous linear functionals on an Orlicz space L^φ defined by an arbitrary (not necessarily convex) Orlicz function φ is described.

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1. Introduction and preliminaries.

In the theory of duality of function spaces an investigation of the order continuous dual is of importance. In this paper we examine order continuous dual $(L^\varphi)^\sim_n$ of an Orlicz space L^φ defined by an arbitrary (not necessarily convex) finite valued Orlicz function φ over a σ-finite measure space. By making use of Kalton's and Drewnowski's results concerning the Mackey topology of E^φ (= the ideal of L^φ of all elements with order continuous F-norm) we describe the Köthe dual $(L^\varphi)^x$ of L^φ. Thus in view of the Riesz isomorphism between $(L^\varphi)^x$ and $(L^\varphi)^\sim_n$ we can establish the general form of order continuous linear functionals on L^φ. Moreover, considering L^φ (equipped with its usual integral modular m_φ) from the viewpoint of Nakano's theory of modular spaces [16] one can define on the topological dual $(L^\varphi)^*$ of L^φ the conjugate convex semimodular $\overline{m}_\varphi$, and next by means of $\overline{m}_\varphi$, we can define two modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $|||\cdot|||_{\overline{m}_\varphi}$. In this paper we obtain a description of the semimodular $\overline{m}_\varphi$ and the modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $|||\cdot|||_{\overline{m}_\varphi}$ restricted to $(L^\varphi)^\sim_n$.

We generalize the well-known results concerning the duality of Orlicz spaces obtained by W.A. Luxemburg and A.C. Zaanen [10], J. Musielak and W. Orlicz [13], W. Orlicz [20] and B. Gramsch [5] (see Remarks 3.2, 3.3 and Remark 4.1).

For the terminology concerning Riesz spaces we refer to [1], [24].

Let (Ω, Σ, μ) be a σ-finite and atomless measure space, and let L^0 denote the set of equivalence classes of all real valued measurable functions defined and finite a.e. on Ω. Then L^0 is a super Dedekind complete Riesz space under the ordering $x \le y$, whenever $x(t) \le y(t)$ a.e. on Ω. For a subset A of Ω, $\mathcal{X}_A$ stands for the characteristic function of A.

Now we recall some notation and terminology concerning Orlicz spaces (see [8], [9], [23], [24] for more details).

By an Orlicz function we mean a function $\varphi : [0,\infty) \to [0,\infty]$ that is non-decreasing, left continuous at 0 with $\varphi(0) = 0$, non identically equal to 0.

A convex Orlicz function is usually called a Young function.

An Orlicz function φ determines a functional $m_\varphi : L^0 \to [0,\infty]$ by the formula

$$m_\varphi(x) = \int_\Omega \varphi(|x(t)|)\, d\mu.$$

The Orlicz space determined by φ is the ideal of L^0 defined as follows:

$$L^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \ \text{ for some } \ \lambda > 0\}.$$

The functional m_φ restricted to L^φ is an orthogonally additive semimodular (see [13], [15], [16]). The space L^φ can be equipped with the complete metrizable topology $\mathcal{T}_\varphi$ of the Riesz F-norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \le \lambda\}.$$

Moreover, if φ is a Young function, then two norms (equivalent to $\|\cdot\|_\varphi$) on L^φ can be defined by

$$\|x\|_\varphi = \inf_{\lambda>0}\{\frac{1}{\lambda}(1 + m_\varphi(\lambda x))\},$$
$$|||x|||_\varphi = \inf\{\lambda > 0 : m_\varphi(x/\lambda) \le 1\}$$

and $|||x|||_\varphi \le \|x\|_\varphi \le 2|||x|||_\varphi$.

Let

$$E^\varphi = \{x \in L^0 : m_\varphi(\lambda x) < \infty \ \text{ for all } \ \lambda > 0\}.$$

It is well known that E^φ coincides with the ideal of L^φ of all elements with order continuous F-norm $\|\cdot\|_\varphi$, that is,

$$E^\varphi = \{x \in L^\varphi : |x| \ge u_n \downarrow 0 \ \text{ in } \ L^\varphi \ \text{ implies } \ \|u_n\|_\varphi \downarrow 0\}.$$

Since $\operatorname{supp} E^\varphi = \Omega$, there exists a sequence (Ω_n) of μ-measurable subsets of Ω, such that $\Omega_n \uparrow$, $\bigcup_{n=1}^\infty \Omega_n = \Omega$ and $\mathcal{X}_{\Omega_n} \in E^\varphi$ (see [24, Theorem 86.2]).

Throughout the paper, for a given $x \in L^\varphi$, we will denote by $x^{(n)}$ $(n = 1, 2, \dots)$ the functions defined on Ω as follows:

$$x^{(n)}(t) = \begin{cases} x(t) & \text{if } |x(t)| \le n \text{ and } t \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

By $(L^\varphi)^*$ we will denote the dual of L^φ with respect to $\mathcal{T}_\varphi$. Since $\mathcal{T}_\varphi$ is a complete, metrizable locally solid topology, we have (see [1, Theorem 16.9]):

$$(L^\varphi)^* = (L^\varphi)^\sim,$$

where $(L^\varphi)^\sim$ stands, as usual, for the space of all order bounded linear functionals on L^φ.

A linear functional f on L^φ is said to be order continuous whenever $x_\sigma \xrightarrow{0} 0$ in L^φ implies $f(x_\sigma) \to 0$ for a net (x_σ) in L^φ. Since the measure space is σ-finite, a linear functional f is order continuous iff f is σ-order continuous (i.e., $x_n \xrightarrow{0} 0$ in L^φ implies $f(x_n) \to 0$ for a sequence (x_n)). As usual, let $(L^\varphi)_n^\sim$ stand for the space of all order continuous linear functionals on L^φ. It is known that $(L^\varphi)_n^\sim \subset (L^\varphi)^\sim$ and $(L^\varphi)_n^\sim$ is a band of $(L^\varphi)^\sim$ (see [22, Proposition 5.22]). Moreover, it is known that $(L^\varphi)_n^\sim = (L^\varphi)^\sim$ whenever φ satisfies the so-called Δ_2-condition, that is

$$\limsup \frac{\varphi(2u)}{\varphi(u)} < \infty \quad \text{as} \quad u \to 0 \quad \text{and} \quad u \to \infty.$$

In view of [20] the dual $(L^\varphi)^*$ is a Banach space under the norm

$$P_{m_\varphi}(f) = \sup\{|f(x)| : x \in L^\varphi,\ m_\varphi(x) \le 1\},$$

which is called, due to Nakano [16], a polar of the semimodular m_φ.

In general, given a linear topological space (X,ξ), by $(X,\xi)^*$ we will denote its topological dual.

2. The convex minorant of an Orlicz function.

Throughout the remainder of the paper we will assume that an Orlicz function φ takes only finite values.

For an Orlicz function φ satisfying the condition

$$(+) \qquad \liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$$

let

$$\varphi^*(v) = \sup\{uv - \varphi(u) : u \ge 0\} \quad \text{for} \quad v \ge 0.$$

Then φ^* is a Young function, complementary to φ in the sense of Young. The function

$$\overline{\varphi}(u) = (\varphi^*)^*(u) \quad \text{for} \quad u \ge 0$$

is called a convex minorant of φ, because it is the largest Young function that is smaller than φ on $[0,\infty)$.

In this section we give more details about φ^* and $\overline{\varphi}$ that will be useful in the sections 3 and 4. To the end of this section we will assume that the condition (+) is satisfied.

We start with the following

Lemma 2.1. (i) *If* $\liminf_{u\to 0} \frac{\varphi(u)}{u} = 0$, *then* φ^* *vanishes only at zero.*

(ii) *If* $\liminf_{u\to 0} \frac{\varphi(u)}{u} > 0$, *then* φ^* *vanishes in some neighbourhood of zero.*

Proof: (i) For every $u > 0$ and $v > 0$, there exists $0 < u_1 < u$ such that $\varphi(u_1) < u_1 v$. Hence $\varphi^*(v) \ge u_1 v - \varphi(u_1) > 0$.

(ii) There exist $u_1 > 0$ and $v_1 > 0$ such that $\varphi(u) > uv_1$ for all $u \geq u_1$, and there exist $u_2 > 0$ and $v_2 > 0$ such that $\varphi(u) \geq uv_2$ for all $0 < u \leq u_2$. We can assume that $u_2 < u_1$ and let us take $v_3 > 0$ such that $1/v_3 = \sup\{u/\varphi(u) : u_2 \leq u \leq u_1\}$. Then putting $v' = \max(v_1, v_2, v_3)$ we have $uv' \leq \varphi(u)$ for all $u \geq 0$. Hence $\varphi^*(v') = 0$. Since the function φ^* is convex and left continuous, there exists a number $v_0 > 0$ such that $\varphi^*(v_0) = 0$ for $0 \leq v \leq v_0$ and $\varphi^*(v) > 0$ for $v \geq v_0$. □

In case of φ satisfying the condition $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = \infty$, the functions φ^* and $\overline{\varphi}$ were examined by Z. Birnbaum and W. Orlicz [2], W. Orlicz [20] and W. Matuszewska, W. Orlicz [12], and the main properties of φ^* and $\overline{\varphi}$ can be summarized in the following

Lemma 2.2. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. *Then the following hold:*

(i) *For every $v > 0$ there exists the least number $u_v > 0$ such that*

$$\varphi^*(v) + \varphi(u_v) = u_v v.$$

(ii) *The set $\{u_v : v \in A \subset [0, \infty)\}$ is bounded if the set A is bounded.*

(iii) $u_v \to 0$ *as* $v \to 0$.

(iv) $\overline{\varphi}(u_v) = \varphi(u_v)$ *for* $v > 0$.

(v) $\lim_{v\to 0} \frac{\varphi^*(v)}{v} = 0$.

(vi) φ^* *takes only finite values and* $\lim_{v\to\infty} \frac{\varphi^*(v)}{v} = \infty$.

Now we are going to extend the results of the previous lemma to the case of Orlicz function φ satisfying the condition $\liminf_{u\to\infty} \frac{\varphi(u)}{u} < \infty$.

Lemma 2.3. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = a < \infty$. *Then the following statements hold:*

(i) *For every $0 < v < a$ there exists the least number $u_v > 0$ such that*

$$\varphi^*(v) + \varphi(u_v) = u_v v.$$

(ii) *The set $\{u_v : 0 < v < a\}$ is bounded.*

(iii) $u_v \to 0$ *as* $v \to 0$ $(0 < v < a)$.

(iv) $\overline{\varphi}(u_v) = \varphi(u_v)$ *for* $0 < v < a$.

(v) $\lim_{v\to 0} \frac{\varphi^*(v)}{v} = 0$.

(vi) φ^* *jumps to infinity, more precisely:* $\varphi^*(v) < \infty$ *for* $v \leq a$, $\varphi^*(v) = \infty$ *for* $v > a$.

Proof: (i) For every $0 < v < a$ there exists $c_v > 0$ such that $\varphi(u)/u > v$ for $u > c_v$. Hence $\varphi^*(v) = \sup\{uv - \varphi(u) : 0 \leq u \leq c_v\}$ and for every $0 < v < a$ there exists the least number $u_v > 0$ such that $\varphi^*(v) = u_v v - \varphi(u_v)$.

(ii) Assume that the set $\{u_v : 0 < v < a\}$ is not bounded. Then there would exist a sequence (v_n) such that $0 < v_n < a$ and $u_{v_n} > \max(n, c_{v_n})$. Hence $\varphi(u_{v_n}) > u_{v_n} v_n$, so $\varphi^*(v_n) = u_{v_n} v_n - \varphi(u_{v_n}) < 0$. This contradiction establishes the boundedness of our set.

(iii) Let $v_n \to 0$ $(0 < v_n < a)$ and assume by way of contradiction that $u_{v_n} \not\to 0$. Then there would exist a number $\alpha > 0$ and an increasing sequence (k_n) of natural numbers such that $u_{v_{k_n}} > \alpha$. On the other hand, in view of (ii) there exists $\beta > 0$ such that $u_{v_{k_n}} \le \beta$. Choose an index n_0 such that $v_{k_n} < \varphi(\alpha)/\beta$ for $n \ge n_0$. Then, by (i), for $n \ge n_0$ we have

$$\varphi^*(v_{k_n}) = u_{v_{k_n}}(v_{k_n} - (\varphi(u_{v_{k_n}})/u_{v_{k_n}})) \le u_{v_{k_n}}(v_{k_n} - \frac{\varphi(\alpha)}{\beta}) < 0.$$

This contradiction establishes that $v_n \to 0$ implies $u_{v_n} \to 0$.

(iv) In view of (i), for $0 < v < a$, the equality $\varphi(u_v) + \varphi^*(v) = u_v v$ holds. On the other hand, from the definition of $\overline{\varphi}$ we get $u_v v \le \overline{\varphi}(u_v) + \varphi^*(v)$. Hence $\varphi(u_v) \le \overline{\varphi}(u_v)$, so $\overline{\varphi}(u_v) = \varphi(u_v)$ because $\overline{\varphi}(u) \le \varphi(u)$ for $u \ge 0$.

(v) Let $v_n \to 0$. Without loss of generality we can assume that $v_n < a$. Thus, by (i) and (ii) we get

$$0 \le \varphi^*(v_n)/v_n = u_{v_n} - (\varphi(u_{v_n})/v_n) \le u_{v_n} \to 0.$$

(vi) Let $v > a$. Then $v > a + \varepsilon$ for some $\varepsilon > 0$, and since $\liminf_{u\to\infty} \varphi(u)/u < a + \varepsilon$, there exists a sequence (u_n) such that $0 < u_n \uparrow \infty$ and $\varphi(u_n) < (a+\varepsilon)u_n$. Hence

$$\begin{aligned}\varphi^*(v) &= \sup\{u(v - (a+\varepsilon)) + u(a+\varepsilon) - \varphi(u) : u \ge 0\}\\ &\ge u_n(v - (a+\varepsilon)) + u_n(a+\varepsilon) - \varphi(u_n) \to \infty,\end{aligned}$$

so $\varphi^*(v) = \infty$ for $v > a$. Moreover, it follows from (i) that $\varphi^*(v) < \infty$ for $v < a$, and, by the left-hand continuity of φ^* we get $\varphi^*(a) < \infty$. □

The following lemma will be of importance in the proof of Theorem 4.2.

Lemma 2.4. (i) *If* $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \infty$, *then for every measurable bounded function* $y \ge 0$ *there exists a measurable bounded function* $z \ge 0$ *such that*

$$\varphi(z(t)) + \varphi^*(y(t)) = z(t)y(t) \quad \textit{for all} \quad t \in \Omega.$$

(ii) *If* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = a < \infty$, *then for every measurable function* y *such that* $0 \le y(t) \le a$ *for* $t \in \Omega$, *there exists a measurable bounded function* $z \ge 0$ *such that*

$$\varphi(z(t)) + \varphi^*(y(t)) = z(t)y(t) \quad \textit{for all} \quad t \in \Omega.$$

Proof: (i) By Lemma 2.2 for every $v > 0$ there is $u_v > 0$ such that

$$\overline{\varphi}(u_v) + \varphi^*(v) = u_v v. \tag{+}$$

It is well known that φ^* is of the form $\varphi^*(v) = \int_0^v q(s)\,ds$, where the function $q : [0,\infty) \to [0,\infty)$ is non-decreasing with $q(0) = 0$ and left continuous (see [9, p. 37]). By Theorem 1 of [9, Ch. II, § 1], the equality (+) holds if $u_v = q(v)$ for $v > 0$. Putting $z(t) = q(y(t))$ for $t \in \Omega$, by Lemma 2.2 we get that $z \ge 0$ is a measurable bounded function on Ω and $\varphi(z(t)) + \varphi^*(y(t)) = z(t)y(t)$ for $t \in \Omega$.

(ii) Proceeding as in (i) and making use of Lemma 2.3 we get (ii). □

3. Order continuous linear functionals on L^φ.

In this section we will find the general form of order continuous linear functionals on an Orlicz space L^φ.

Let us recall that the Köthe dual X^x of a function space $X \subset L^0$ (with supp $X = \Omega$) is defined as follows:

$$X^x = \{y \in L^0 : \int_\Omega |x(t)y(t)|\, d\mu < \infty \ \text{ for all } \ x \in X\}.$$

It is well known that $(L^\varphi)^x = L^{\varphi^*}$, whenever φ is a Young function (see [9], [24]). It was originally proved by Z. Birnbaum and W. Orlicz [2].

Putting

$$f_y(x) = \int_\Omega x(t)y(t)\, d\mu \ \text{ for all } \ x \in X,$$

we have the following important equality

$$X_n^\sim = \{f_y : y \in X^x\} \tag{3.1}$$

where the mapping $X^x \ni y \to f_y \in X_n^\sim$ is a Riesz isomorphism (see [7, Ch. 6, § 1, Theorem 1]).

Next, let us recall that the Mackey topology of a linear topological space (X, ξ) is the finest locally convex topology τ on X that produces the same continuous linear functionals as the original topology ξ.

The following result due to N.J. Kalton [6] and L. Drewnowski [4, Corollary 1, Corollary 2] will be of importance in this section.

Theorem 3.1. (i) *There exists a nonzero continuous linear functional on $(E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})$ iff $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$.*

(ii) *The Mackey topology τ_{E^φ} of $(E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})$ coincides with the seminormed topology $\mathcal{T}_{\overline{\varphi}|_{E^\varphi}}$, i.e., $\tau_{E^\varphi} = \mathcal{T}_{\overline{\varphi}|_{E^\varphi}}$.*

Now we are ready to give a description of the Köthe dual $(L^\varphi)^x$.

Theorem 3.2. (i) *If $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = 0$, then $(L^\varphi)^x = (E^\varphi)^x = \{0\}$.*

(ii) *If $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$, then $(L^\varphi)^x = (E^\varphi)^x = (E^{\overline{\varphi}})^x = L^{\varphi^*}$.*

Proof: (i) We have $(E^\varphi)_n^\sim \subset (E^\varphi)^\sim = (E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})^*$ (see [1, Proposition 16.9]), so by Theorem 3.1, $(E^\varphi)_n^\sim = \{0\}$. Hence, by (3.1), $(E^\varphi)^x = \{0\}$, and since $(L^\varphi)^x \subset (E^\varphi)^x$, the proof is completed.

(ii) First, we shall show that $(L^\varphi)^x = (E^\varphi)^x = (E^{\overline{\varphi}})^x$. It suffices to show that $(E^\varphi)^x \subset (L^\varphi)^x$ and $(E^\varphi)^x \subset (E^{\overline{\varphi}})^x$. Indeed, let $y \in (E^\varphi)^x$, i.e., $\int_\Omega |x(t)y(t)|\, d\mu < \infty$ for all $x \in E^\varphi$. Putting

$$g_y(z) = \int_\Omega z(t)y(t)\, d\mu \ \text{ for } \ z \in E^\varphi,$$

we get $g_y \in (E^\varphi)^\sim_n$ by (3.1). But according to Theorem 3.1 we have $(E^\varphi)^\sim_n \subset (E^\varphi)^\sim = (E^\varphi, \mathcal{T}_{\varphi|_{E^\varphi}})^* = (E^\varphi, \mathcal{T}_{\overline{\varphi}|_{E^\varphi}})^*$, so we can put

$$\|g_y\|_{\overline{\varphi}} = \sup\Big\{\Big|\int_\Omega z(t)y(t)\,d\mu\Big| : z \in E^\varphi,\ |||z|||_{\overline{\varphi}} \le 1\Big\}.$$

To prove that $y \in (L^\varphi)^x$ (resp. $y \in (E^{\overline{\varphi}})^x$), let now $x \in L^\varphi$ (resp. $x \in E^{\overline{\varphi}}$), $x \neq 0$. Then $|x^{(n)}(t)y(t)| \uparrow_n |x(t)y(t)|$ on Ω, so by applying Fatou's lemma we get

$$\begin{aligned}\frac{1}{|||x|||_{\overline{\varphi}}}\int_\Omega |x(t)y(t)|\,d\mu &\le \frac{1}{|||x|||_{\overline{\varphi}}}\sup_n \int_\Omega (|x^{(n)}(t)|\operatorname{sign} y(t))y(t)\,d\mu\\ &\le \sup\Big\{\Big|\int_\Omega z(t)y(t)\,d\mu\Big| : z \in E^\varphi,\ |||z|||_{\overline{\varphi}} \le 1\Big\} = \|g_y\|_{\overline{\varphi}}.\end{aligned}$$

Hence $y \in (L^\varphi)^x$ (resp. $y \in (E^{\overline{\varphi}})^x$), so $(L^\varphi)^x = (E^\varphi)^x = (E^{\overline{\varphi}})^x$.

Since the topology $\mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}}$ is locally convex and satisfies the Lebesgue property, we have $(E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^* \subset (E^{\overline{\varphi}})^\sim_n \subset (E^{\overline{\varphi}})^\sim$ (see [1, Theorem 9.1]). On the other hand $(E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^* = (E^{\overline{\varphi}})^\sim$, because $\mathcal{T}_{\overline{\varphi}|_{E^\varphi}}$ is metrizable and complete (see [1, Theorem 16.9]). Thus $(E^{\overline{\varphi}})^\sim_n = (E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^*$. By the mapping $(y \mapsto g_y)$ the space $(E^{\overline{\varphi}})^x$ can be identified with $(E^{\overline{\varphi}})^\sim_n$ (see (3.1)) and the space $L^{\overline{\varphi}^*}$ with $(E^{\overline{\varphi}}, \mathcal{T}_{\overline{\varphi}|_{E^{\overline{\varphi}}}})^*$ (see [10, Ch. II, § 3, Theorem 2]). Thus $(E^{\overline{\varphi}})^x = L^{\overline{\varphi}^*}$, so $(E^{\overline{\varphi}})^x = L^{\varphi^*}$, because $\overline{\varphi}^* = \varphi^*$. □

As an application of Theorem 3.2 and the equality (3.1) we obtain a condition for the existence of non-zero order continuous linear functionals on L^φ.

Theorem 3.3. *The following statements are equivalent:*

(i) $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$.
(ii) $(L^\varphi)^\sim_n \neq \{0\}$.

Finally, by making use of Theorem 3.2 and (3.1) we can establish the general form of order continuous linear functionals on L^φ.

Theorem 3.4. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$. *Then for a linear functional* f *on* L^φ *the following statements are equivalent:*

(i) f *is order continuous, i.e.,* $f \in (L^\varphi)^\sim_n$.
(ii) *There exists a unique* $y \in L^{\varphi^*}$ *such that*

$$f(x) = f_y(x) = \int_\Omega x(t)y(t)\,d\mu \quad \text{for } x \in L^\varphi.$$

Moreover, the map $L^{\varphi^*} \ni y \mapsto f_y \in (L^\varphi)^\sim_n$ *is a Riesz isomorphism.*

Remark 3.1. The equality $(L^\varphi)^x = L^{\varphi^*}$ from Theorem 3.2 has been recently proved in a different way by L. Maligranda and W. Wnuk ([11, Theorem 2]).

Remark 3.2. In the theory of Orlicz spaces the class of modular continuous linear functionals is considered. Let us recall that a linear functional f on L^φ is called modular continuous if $m_\varphi(x_n) \to 0$ implies $f(x_n) \to 0$ for a sequence (x_n) in L^φ [21]. In some special cases the class of all modular continuous linear functional on L^φ was investigated by J. Musielak and W. Orlicz [13], W. Orlicz [20] and J. Musielak and A. Waszak [14]. On the other hand, from [18, Theorem 5.4] it follows that the class of all modular continuous linear functionals on L^φ coincides with $(L^\varphi)^\sim_n$, and one can see that the well known results concerning modular continuous linear functionals on L^φ follow immediately from Theorems 3.3 and 3.4.

Remark 3.3. B. Gramsch [5] examined the topological dual of L^φ when φ is a concave Orlicz function. Gramsch's result contains the classical result of M.M. Day [3] on the triviality of the duals of L^p for $0 < p < 1$. But for φ being concave, the topological dual $(L^\varphi)^*$ coincides with $(L^\varphi)^\sim_n$ and the Gramsch's result follows easily from Theorems 3.3 and 3.4.

Remark 3.4. The order continuous dual of Orlicz sequence spaces l^φ (without local convexity) was described by the present author [19].

4. The conjugate semimodular and modular norms on $(L^\varphi)^\sim_n$.

In view of [16], the conjugate $\overline{m}_\varphi$ of the semimodular m_φ can be defined on the algebraic dual $\widetilde{L^\varphi}$ of L^φ as follows:

$$\overline{m}_\varphi(f) = \sup\{|f(x)| - m_\varphi(x) : x \in L^\varphi\}.$$

According to [17, Theorem 3.1] we have the following

Theorem 4.1. (i) $(L^\varphi)^* = \{f \in \widetilde{L^\varphi} : \overline{m}_\varphi(\lambda f) < \infty$ *for some* $\lambda > 0\}$.

(ii) *The conjugate* $\overline{m}_\varphi$ *restricted to* $(L^\varphi)^*$ *is a convex orthogonally additive semimodular. Moreover, if* $f \geq 0$, *then*

$$\overline{m}_\varphi(f) = \sup\{f(x) - m_\varphi(x) : 0 \leq x \in L^\varphi,\ m_\varphi(x) < \infty\}.$$

By means of the conjugate semimodular $\overline{m}_\varphi$, one can define on the dual $(L^\varphi)^*$ two Riesz norms (see [21]):

$$\|f\|_{\overline{m}_\varphi} = \inf_{\lambda>0}\Big\{\frac{1}{\lambda}(1 + \overline{m}_\varphi(\lambda f))\Big\},$$
$$|||f|||_{\overline{m}_\varphi} = \inf\Big\{\lambda > 0 : \overline{m}_\varphi\Big(\frac{f}{\lambda}\Big)\Big\}.$$

In view of the general fact (see [21, 1.51]), for any $f \in (L^\varphi)^*$,

$$|||f|||_{\overline{m}_\varphi} \leq \|f\|_{\overline{m}_\varphi} \leq 2|||f|||_{\overline{m}_\varphi} \ \text{ and } \ |||f|||_{\overline{m}_\varphi} \leq 1 \ \text{ iff } \ \overline{m}_\varphi(f) \leq 1.$$

Since $((L^\varphi)^*, P_{m_\varphi})$ is a Banach lattice, in view of the above inequalities and by applying the Open Mapping Theorem, the dual $(L^\varphi)^*$ endowed with the modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $\|\!|\cdot|\!\|_{\overline{m}_\varphi}$ is also a Banach lattice. Further, since $(L^\varphi)^\sim_n$ is a band of $(L^\varphi)^*$ (see [1, Theorem 3.7]), $(L^\varphi)^\sim_n$ is closed with respect to P_{m_φ} (resp. $\|\cdot\|_{\overline{m}_\varphi}$, $\|\!|\cdot|\!\|_{\overline{m}_\varphi}$) restricted to $(L^\varphi)^\sim_n$ by [1, Theorem 5.6]. Thus $(L^\varphi)^\sim_n$ is a Banach lattice with respect to the norms P_{m_φ} (resp. $\|\cdot\|_{\overline{m}_\varphi}, \|\!|\cdot|\!\|_{\overline{m}_\varphi}$) restricted to $(L^\varphi)^\sim_n$.

Remark 4.1. In 1956, W.A. Luxemburg and A.C. Zaanen [10, Theorem 1] showed that if φ is a Young function, then for any $y \in L^{\varphi^*}$,

$$m_{\varphi^*}(y) = \sup\{\Big|\int_\Omega x(t)y(t)\,d\mu\Big| - m_\varphi(x) : x \in L^\varphi\}.$$

Now we will extend the above equality over an arbitrary finite valued Orlicz function. Moreover, using this equality we will obtain a description of the modular norms $\|\cdot\|_{\overline{m}_\varphi}$ and $\|\!|\cdot|\!\|_{\overline{m}_\varphi}$ and the polar P_{m_φ} restricted to $(L^\varphi)^\sim_n$. The details follow.

Theorem 4.2. *Let* $\liminf_{u\to\infty} \frac{\varphi(u)}{u} > 0$. *Then for every* $y \in L^{\varphi^*}$ *the following equalities hold:*

(i) $\overline{m}_\varphi(f_y) = m_{\varphi^*}(y)$.

(ii) $\|f_y\|_{\overline{m}_\varphi} = \|y\|_{\varphi^*} = \sup\{\big|\int_\Omega x(t)y(t)\,d\mu\big| : x \in E^\varphi,\ m_{\overline{\varphi}}(x) \le 1\}$.

(iii) $\|\!|f_y|\!\|_{\overline{m}_\varphi} = \|\!|y|\!\|_{\varphi^*}$.

(iv) $P_{m_\varphi}(f_y) = \sup\{\big|\int_\Omega x(t)y(t)\,d\mu\big| : x \in E^\varphi,\ m_\varphi(x) \le 1\}$.

Proof: (i) From the definition of φ^* it easily follows that

$$\overline{m}_\varphi(f_y) \le m_{\varphi^*}(y).$$

Now we shall show that $\overline{m}_\varphi(f_y) \ge m_{\varphi^*}(y)$.

I. Assume first that $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = a < \infty$. Then, by Theorem 2.2, $\varphi^*(v) < \infty$ for $0 \le v \le a$ and $\varphi^*(v) = \infty$ for $v > a$. Hence the inclusion $L^{\varphi^*} \subset L^\infty$ holds and we can consider two subcases:

1°. $\|y\|_\infty \le a$ ($\|\cdot\|_\infty$ — the norm in L^∞), i.e., $|y(t)| \le a$ a.e. on Ω. Let $y_n(t) = y^{(n)}(t)$ for $t \in \Omega$ ($n = 1, 2, \dots$). Then by Lemma 2.4, there exists a sequence (z_n) of bounded, measurable functions such that $z_n \ge 0$, supp $z_n \subset \Omega_n$ and

$$\varphi(z_n(t)) + \varphi^*(|y_n(t)|) = |z_n(t)y_n(t)|$$

for $n = 1, 2, \dots$ and $t \in \Omega$. Putting $x_n(t) = (\operatorname{sign} y_n(t))z_n(t)$ for $n = 1, 2, \dots$, we have $x_n \in L^\varphi$. Since $\varphi^*(|y_n(t)|) \uparrow_n \varphi^*(|y(t)|)$ for $t \in \Omega$, by applying Fatou's lemma we get

$$\begin{aligned} m_{\varphi^*}(y) &\le \sup_n \int_\Omega \varphi^*(|y_n(t)|)\,d\mu \\ &= \sup_n\Big\{\int_\Omega |z_n(t)y_n(t)|\,d\mu - \int_\Omega \varphi(z_n(t))\,d\mu\Big\} \\ &= \sup\Big\{\Big|\int_\Omega x_n(t)y(t)\,d\mu\Big| - m_\varphi(x_n)\Big\} \le \overline{m}_\varphi(f_y). \end{aligned}$$

Thus the equality $\overline{m}_\varphi(f_y) = m_{\varphi^*}(y)$ holds.

2°. $\|y\|_\infty > a$. Then $m_{\varphi^*}(y) = \infty$. Let us take $0 < \lambda < 1$ and $0 < \delta < a$ such that $\|\lambda y\|_\infty = a$ and $\lambda(a+\delta)/(a-\delta) < 1$. Let $F = \{t \in \Omega : |\lambda y(t)| > a - \delta\}$ and choose a measurable subset E of F such that $0 < \mu(E) < \infty$.

There exists a sequence (u_n) of positive numbers such that $u_n \uparrow \infty$ and $\varphi(u_n) < (a+\delta)u_n$.

Putting $x_n = u_n \cdot \mathcal{X}_E$ $(n = 1, 2, \ldots)$ we can easily show that

$$\int_\Omega \varphi(|x_n(t)|)\, d\mu \le \frac{\lambda(a+\delta)}{a-\delta} \int_\Omega x_n(t)|y(t)|\, d\mu.$$

Hence

$$\begin{aligned}
\overline{m}_\varphi(f_y) &\ge \int_\Omega x_n(t)|y(t)|\, d\mu - \int_\Omega \varphi(|x_n(t)|)\, d\mu \\
&\ge \Big(1 - \frac{\lambda(a+\delta)}{a-\delta}\Big) \int_\Omega x_n(t)|y(t)|\, d\mu \\
&\ge \Big(1 - \frac{\lambda(a+\delta)}{a-\delta}\Big) \frac{a-\delta}{\lambda} u_n \mu(E).
\end{aligned}$$

Thus $\overline{m}_\varphi(f_y) = \infty$ and $\overline{m}_\varphi(f_y) = m_{\varphi^*}(y)$.

II. Next assume that $\liminf_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. Then in view of Lemma 2.4 the same proof as in 1° works.

(ii) Since $\lambda f_y = f_{\lambda y}$, by making use of (i) and (1.1) we get

$$\begin{aligned}
\|f\|_{\overline{m}_\varphi} &= \inf_{\lambda>0} \Big\{\frac{1}{\lambda}(1 + \overline{m}_\varphi(\lambda f_y))\Big\} \\
&= \inf \Big\{\frac{1}{\lambda}(1 + m_{\varphi^*}(\lambda y))\Big\} = \|y\|_{\varphi^*}.
\end{aligned}$$

It is well known that (see [10]) that

$$\|y\|_{\varphi^*} = \sup\{|\int_\Omega z(t)y(t)\, d\mu| : z \in L^{\overline{\varphi}},\ m_{\overline{\varphi}}(z) \le 1\}.$$

Let $z \in L^{\overline{\varphi}}$ with $m_{\overline{\varphi}}(z) \le 1$. Putting $x_n(t) = |z^{(n)}(t)| \operatorname{sign} y(t)$ for $t \in \Omega$ $(n = 1, 2, \ldots)$, we have that $x_n \in E^\varphi$, $m_{\overline{\varphi}}(x_n) \le 1$ and $|z^{(n)}(t)y(t)| \uparrow_n |z(t)y(t)|$ for $t \in \Omega$. Hence by applying Fatou's lemma we easily get

$$|\int_\Omega z(t)y(t)\, d\mu| \le \sup_n |\int_\Omega x_n(t)y(t)\, d\mu|.$$

Thus $\|y\|_{\varphi^*} = \sup\{|\int_\Omega x(t)y(t)\, d\mu| : x \in E^\varphi,\ m_{\overline{\varphi}}(x) \le 1\}$.

(iii) Using (i) and (1.2) we get

$$\begin{aligned}
|||f_y|||_{\overline{m}_\varphi} &= \inf\{\lambda > 0 : \overline{m}_\varphi(f_y/\lambda) \le 1\} \\
&= \inf\{\lambda > 0 : m_{\varphi^*}(y/\lambda) \le 1\} = |||y|||_{\varphi^*}.
\end{aligned}$$

(iv) Similarly as in (ii). □

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