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A duality for isotropic median algebras

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Abstract. We establish categorical dualities between varieties of isotropic median algebras and suitable categories of operational and relational topological structures. We follow a general duality theory of B.A. Davey and H. Werner. The duality results are used to describe free isotropic median algebras. If the number of free generators is less than five, the description is detailed.

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1. Preliminaries.

In [2] B.A. Davey and H. Werner developed a general scheme for constructing dualities for certain classes of algebras. Let us recall their setting.

We assume we have an algebra $\overline{P}$ and a topological structure $\widetilde{P}$ (i.e. a set endowed with a topology, relations and operations) on the same underlying set P such that all operations of $\overline{P}$ are continuous with respect to the topology of $\widetilde{P}$ and moreover all relations and operations of the structure $\widetilde{P}$ are algebraic over $\overline{P}$; i.e. they are subalgebras of appropriate powers of $\overline{P}$. (In fact, Davey and Werner consider also partial operations, but we will not need them.) Let $\mathcal{A} := \boldsymbol{ISP}(\overline{P})$ be the prevariety generated by $\overline{P}$ and let $\mathcal{X} := \boldsymbol{IS_cP}(\widetilde{P})$ be the class of all structures embeddable as a closed substructure into a power of $\widetilde{P}$. Then for each $\overline{A} \in \mathcal{A}$ the set $D(\overline{A})$ of all homomorphisms $\overline{A} \longrightarrow \overline{P}$ is a natural member of $\mathcal{X}$, a substructure of $\widetilde{P}^A$. Similarly, for each $\widetilde{X} \in \mathcal{X}$ the set $E(\widetilde{X})$ of all $\mathcal{X}$-morphisms (i.e. continuous structure-preserving maps) $\widetilde{X} \longrightarrow \widetilde{P}$ forms a subalgebra of $\overline{P}^X$ and thus belongs to $\mathcal{A}$. We have thereby defined two contravariant (hom-)functors

$$D: \mathcal{A} \longrightarrow \mathcal{X}, \qquad E: \mathcal{X} \longrightarrow \mathcal{A},$$

which are adjoint to each other. Moreover, for each $\overline{A} \in \mathcal{A}$, the evaluation map $e_{\overline{A}}: \overline{A} \longrightarrow ED(\overline{A})$ defined by

$$e_{\overline{A}}(a)\,(f) = f(a) \text{ for every } a \in A,\ f \in D(\overline{A})$$

is an embedding. Similarly, for each $\widetilde{X} \in \mathcal{X}$, the evaluation map $\varepsilon_{\widetilde{X}}: \widetilde{X} \longrightarrow DE(\widetilde{X})$ is also an embedding. We call the pair (D, E) a duality if $e_{\overline{A}}$ is an isomorphism for each $\overline{A} \in \mathcal{A}$. We speak about a full duality if in addition each $\varepsilon_{\widetilde{X}}$ is an

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isomorphism. In this case D and E are equivalent to categorical antiisomorphisms between the categories $\mathcal{A}$ and $\mathcal{X}$ which are inverse to each other.

We are interested in a special case when the set P is finite, the topology of $\widetilde{P}$ is discrete and the algebra $\overline{P}$ has a near-unanimity term, i.e. (k+1)-ary term p such that $\overline{P}$ satisfies the equations

$$p(x, y, \dots, y) = p(y, x, y, \dots, y) = \dots = p(y, \dots, y, x) = y.$$

Let k be a positive integer, R a subalgebra of $\overline{P}^k$. Let I be an arbitrary set and $X \subseteq P^I$. We say that a function $f: X \longrightarrow P$ preserves R if $[f(u_1), \dots, f(u_k)] \in R$ holds whenever $u_1, \dots, u_k \in X$, $u_j = (x_{ij})_{i \in I}$, are such that $[x_{i1}, \dots, x_{ik}] \in R$ for every $i \in I$. We say that the structure on $\widetilde{P}$ generates R if, for every $\widetilde{X} \in \mathcal{X}$, each $\mathcal{X}$-morphism $\varphi: \widetilde{X} \longrightarrow \widetilde{P}$ preserves R.

We say that an object $\widetilde{Z} \in \mathcal{X}$ is injective in $\mathcal{X}$ if for any $\mathcal{X}$-embedding $\alpha: \widetilde{X} \longrightarrow \widetilde{Y}$ and any $\mathcal{X}$-morphism $\varphi: \widetilde{X} \longrightarrow \widetilde{Z}$ there exists an $\mathcal{X}$-morphism $\psi: \widetilde{Y} \longrightarrow \widetilde{Z}$ with $\varphi = \psi\alpha$.

1.1. Theorem ([2, Theorem 1.19]). *Let $\overline{P}$ be a finite algebra having a $(k+1)$-ary near unanimity term function. If the structure on $\widetilde{P}$ generates all subalgebras of $\overline{P}^k$, then (D, E) is a duality between $\mathcal{A}$ and $\mathcal{X}$. Moreover, $\widetilde{P}$ is injective in $\mathcal{X}$.*

1.2. Theorem ([2, Theorem 1.20]). *Assume that the operations in the type of $\widetilde{P}$ are at most unary and that (D, E) is a duality between $\mathcal{A}$ and $\mathcal{X}$. Then (D, E) is a full duality provided the following condition holds:*

(E) *if $\widetilde{X}$ is a proper substructure of some finite $\widetilde{Y} \in \mathcal{X}$, then there exist two different $\mathcal{X}$-morphisms $\varphi, \psi: \widetilde{Y} \longrightarrow \widetilde{P}$ such that $\varphi \restriction \widetilde{X} = \psi \restriction \widetilde{X}$.*

We want to apply these theorems to the varieties of isotropic median algebras. Basic facts about median algebras and isotropic median algebras can be found in [5] and [3]. The similarity type of these algebras consists of a single ternary operation m called the median. The variety $\mathcal{T}$ of all isotropic median algebras is generated by the algebra $\overline{T_\omega}$ with a countable infinite underlying set and the median operation defined by the rule

$$m(x, y, z) = \begin{cases} x & \text{if } y \neq z, \\ y & \text{if } y = z. \end{cases}$$

A function satisfying the above rule is known in the literature under the name of dual discriminator (see [4]). It is not hard to prove (see [5]) that the subvarieties of $\mathcal{T}$ form a chain

$$\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3 \subseteq \dots \mathcal{T},$$

where $\mathcal{T}_n$ is the variety generated by the algebra $\overline{T_n}$ with an n-element underlying set and the dual discriminator as the median operation. In fact, $\mathcal{T}_n = \boldsymbol{ISP}(\overline{T_n})$, since the only subdirectly irreducible algebras in $\mathcal{T}_n$ are $\overline{T_k}$'s with $k \leq n$ (see [5]). The algebras in $\mathcal{T}_2$ are called symmetric median algebras and are closely connected with distributive lattices (see [1]).

2. Dualities and full dualities.

In this section we construct for any $n \geq 2$ a full duality between $\mathcal{T}_n$ and an appropriate category of topological structures. Let us remark that a full duality for $\mathcal{T}_2$ has already been established by H. Werner (see [6, appendix]), but our duality will be slightly different.

Let us denote $T_n = \{1, 2, \ldots, n\}$, $\overline{T_n} = \langle T_n, m\rangle$, where m is the dual discriminator defined on T_n. To find a dual category to $\mathcal{T}_n = \boldsymbol{ISP}(\overline{T_n})$ we have to construct a suitable topological structure $\widetilde{T_n}$. We set $\widetilde{T_n} = \langle T_n; S_n, \mathbf{1}, \boldsymbol{H}, \boldsymbol{Z}, \boldsymbol{\tau}\rangle$, where
S_n is the set of all permutations on T_n (viewed as unary operations);
$\mathbf{1}$ is the nullary operation (constant) equal to $1 \in T_n$;
$\boldsymbol{H}$ is the binary relation on T_n defined by

$$[x, y] \in \boldsymbol{H} \text{ iff } x = 1 \text{ or } y = 1;$$

$\boldsymbol{Z} = \{2, 3, \ldots, n\}$ is a unary relation (subspace) on T_n;
$\boldsymbol{\tau}$ is the discrete topology on T_n.
We use the boldface letters for $\mathbf{1}$, $\boldsymbol{H}$, $\boldsymbol{\tau}$ (also for elements of S_n) to indicate that they also play the role of operational and relational symbols. These operations and relations extend pointwise to an arbitrary power $(T_n)^I$ (the topology on $(T_n)^I$ is the usual product topology) and we can restrict them to an arbitrary closed subset of $(T_n)^I$ that is closed under all operations from S_n and contains $\mathbf{1}$. Hence, every $\widetilde{X} \in \mathcal{X}_n := \boldsymbol{IS_cP}(\widetilde{T_n})$ is a set X endowed with a set S_n of $n!$ unary operations, one nullary operation $\mathbf{1}$, one binary relation $\boldsymbol{H}$, one unary relation $\boldsymbol{Z}$ and (compact Hausdorff) topology $\boldsymbol{\tau}$.

It is easy to see that any $\boldsymbol{\pi} \in S_n$ is a subalgebra of $(\overline{T_n})^2$, $\mathbf{1}$ and $\boldsymbol{Z}$ are subalgebras of $\overline{T_n}$ and $\boldsymbol{H}$ is a subalgebra of $(\overline{T_n})^2$. Thus, we have the adjoint pair (D, E) of functors

$$D: \mathcal{T}_n \longrightarrow \mathcal{X}_n, \qquad E: \mathcal{X}_n \longrightarrow \mathcal{T}_n,$$

defined as in the first section (with $\overline{T_n}$ and $\widetilde{T_n}$ playing the roles of $\overline{P}$ and $\widetilde{P}$, respectively). The median operation m of $\overline{T_n}$ is clearly a 3-ary near unanimity term. Theorem 1.1 suggests that we examine the subalgebras of $(\overline{T_n})^2$. Actually, this has been done by H. Werner [6]. (He used the term p-rectangular sets.)

2.1. Lemma ([6, p.798]). *Let S be a subalgebra of $\overline{T_n} \times \overline{T_n}$ and let us denote $Q = \{x \in \overline{T_n} \mid [x, y] \in S$ for some $y \in \overline{T_n}\}$, $R = \{y \in \overline{T_n} \mid [x, y] \in S$ for some $x \in \overline{T_n}\}$. Then one of the following cases occurs:*

(1) *$S = Q \times R$;*
(2) *S is a bijection between Q and R;*
(3) *$S = (\{x\} \times R) \cup (Q \times \{y\})$ for some $x \in Q$, $y \in R$.*

For $i, j \in T_n$ and $\boldsymbol{\pi} \in S_n$ we denote

$K^i = (T_n \setminus \{i\}) \times T_n$;
$K_i = T_n \times (T_n \setminus \{i\})$;
$L_{ij} = (\{i\} \times T_n) \cup (T_n \times \{j\})$;
$M_{\boldsymbol{\pi}} = \boldsymbol{\pi}$.

It is easy to see that, for every i, j and $\boldsymbol{\pi}$, the sets K^i, K_i, L_{ij} and $M_{\boldsymbol{\pi}}$ are subalgebras of $(\overline{T_n})^2$. The following assertion is now a consequence of 2.1.

2.2. Corollary. *Every subalgebra of $(\overline{T_n})^2$ is an intersection of algebras of the types K^i, K_i, L_{ij} and $M_{\boldsymbol{\pi}}$.*

Now we can prove the duality result.

2.3. Theorem. (D, E) *is a duality between $\mathcal{T}_n$ and $\mathcal{X}_n$.*

Proof: Since the median operation on $\overline{T_n}$ is a 3-ary near unanimity term, we have to show that the structure on $\widetilde{T_n}$ generates all subalgebras of $\overline{T_n} \times \overline{T_n}$. Let $\widetilde{X} \in \mathcal{X}_n$ be a substructure of $(\widetilde{T_n})^I$ and let $\varphi : \widetilde{X} \longrightarrow \widetilde{T_n}$ be an $\mathcal{X}_n$-morphism. We claim that φ preserves all subalgebras of the types K^i, K_i, L_{ij} and $M_{\boldsymbol{\pi}}$.

To see that φ preserves K^i, let $x, y \in \widetilde{X}$, $x = (x_k)_{k\in I}$, $y = (y_k)_{k\in I}$ be such that $[x_k, y_k] \in K^i$ for each $k \in I$. Hence, $x_k \neq i$ for each $k \in I$. Choose $\boldsymbol{\pi} \in S_n$ such that $\boldsymbol{\pi}(i) = 1$. Then $\boldsymbol{\pi}(x) \in \boldsymbol{Z}$. Since φ is an $\mathcal{X}_n$-morphism, we obtain $\varphi(\boldsymbol{\pi}(x)) \in \boldsymbol{Z}$, hence $\boldsymbol{\pi}(\varphi(x)) = \varphi(\boldsymbol{\pi}(x)) \in \boldsymbol{Z}$ and $\boldsymbol{\pi}(\varphi(x)) \neq 1$. This implies that $\varphi(x) \neq i$ and therefore $[\varphi(x), \varphi(y)] \in K^i$. The proof for K_i is similar.

To see that φ preserves L_{ij}, let $x, y \in \widetilde{X}$, $x = (x_k)_{k\in I}$, $y = (y_k)_{k\in I}$ be such that $[x_k, y_k] \in L_{ij}$ for each $k \in I$. Choose $\boldsymbol{\pi}, \boldsymbol{\rho} \in S_n$ such that $\boldsymbol{\pi}(i) = 1$ and $\boldsymbol{\rho}(j) = 1$. Then $[\boldsymbol{\pi}(x), \boldsymbol{\rho}(y)] \in \boldsymbol{H}$, hence $[\varphi(\boldsymbol{\pi}(x)), \varphi(\boldsymbol{\rho}(y))] = [\boldsymbol{\pi}(\varphi(x)), \boldsymbol{\rho}(\varphi(y))] \in \boldsymbol{H}$. This means that $\boldsymbol{\pi}(\varphi(x)) = 1$ or $\boldsymbol{\rho}(\varphi(y)) = 1$, hence $\varphi(x) = i$ or $\varphi(y) = j$. We obtain $[\varphi(x), \varphi(y)] \in L_{ij}$.

Finally, φ preserves any $M_{\boldsymbol{\pi}}$, since $\boldsymbol{\pi}$ is in the type of $\mathcal{X}_n$ as a basic operation.

It is easy to see that the property of being preserved by φ is closed under intersections. According to 2.2, φ preserves all subalgebras of $(\overline{T_n})^2$. By 1.1, (D, E) is a duality between $\mathcal{T}_n$ and $\mathcal{X}_n$. $\square$

2.4. Lemma. *Let $\widetilde{X}$ be a substructure of $(\widetilde{T_n})^I$, $x \in (\widetilde{T_n})^I \setminus \widetilde{X}$. Let $a, b \in \widetilde{X}$ and $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in S_n$ be such that $[a, \boldsymbol{\rho}_1(x)] \in \boldsymbol{H}$, $[b, \boldsymbol{\rho}_2(x)] \in \boldsymbol{H}$. Denote $J = \{i \in I \mid a_i = 1\}$, $K = \{i \in I \mid b_i = 1\}$. If $J \cap K = \emptyset$ then $J \cup K = I$ and $\boldsymbol{\rho}_1^{-1}(1) \neq \boldsymbol{\rho}_2^{-1}(1)$.*

Proof: Suppose that $J \cap K = \emptyset$. For $j \in I \setminus J$ and $k \in I \setminus K$ we have $\boldsymbol{\rho}_1(x_j) = 1 = \boldsymbol{\rho}_2(x_k)$, hence $x_j = \boldsymbol{\rho}_1^{-1}(1)$, $x_k = \boldsymbol{\rho}_2^{-1}(1)$. If $\boldsymbol{\rho}_1^{-1}(1) = \boldsymbol{\rho}_2^{-1}(1)$, then $x_j = \boldsymbol{\rho}_1^{-1}(1)$ for every $j \in I$, hence $x = \boldsymbol{\rho}_1^{-1}(\mathbf{1})$. Since $\widetilde{X}$ is a substructure of $(\widetilde{T_n})^I$, we have $\mathbf{1} \in \widetilde{X}$ and $\boldsymbol{\rho}_1^{-1}(\mathbf{1}) \in \widetilde{X}$, a contradiction. We obtain that $\boldsymbol{\rho}_1^{-1}(1) \neq \boldsymbol{\rho}_2^{-1}(1)$. This implies that $J \cup K = I$ because for $i \in I \setminus (J \cup K)$ we would have $\boldsymbol{\rho}_1^{-1}(1) = x_i = \boldsymbol{\rho}_2^{-1}(1)$. $\square$

2.5. Lemma. *Let $\widetilde{X}$ be a finite substructure of $(\widetilde{T_n})^I$, $x \in (\widetilde{T_n})^I \setminus \widetilde{X}$. Then there is an $\mathcal{X}_n$-morphism $\varphi : \widetilde{X} \longrightarrow \widetilde{T_n}$ such that $[a, \boldsymbol{\rho}(x)] \notin \boldsymbol{H}$ whenever $\boldsymbol{\rho} \in S_n$, $a \in \widetilde{X}$, $\varphi(a) \neq 1$.*

Proof: Denote $A = \{a \in \widetilde{X} \mid [\boldsymbol{\pi}(a), \boldsymbol{\rho}(x)] \in \boldsymbol{H}$ for some $\boldsymbol{\pi}, \boldsymbol{\rho} \in S_n\}$. The set A is closed under all $\boldsymbol{\pi} \in S_n$ and contains $\mathbf{1} = (1)_{i\in I}$. Hence, A is an underlying set of the substructure $\widetilde{A}$ of $\widetilde{X}$. Let us define a mapping $\varphi_0 : \widetilde{A} \longrightarrow \widetilde{T_n}$ by the rule

$$\varphi_0(a) = \boldsymbol{\pi}^{-1}(1), \text{ where } \boldsymbol{\pi} \in S_n \text{ is such that } [\boldsymbol{\pi}(a), \boldsymbol{\rho}(x)] \in \boldsymbol{H} \text{ for some } \boldsymbol{\rho} \in S_n.$$

First we prove the correctness of the definition. Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in S_n$ be such that $[\boldsymbol{\pi}_1(a), \boldsymbol{\rho}_1(x)] \in \boldsymbol{H}$, $[\boldsymbol{\pi}_2(a), \boldsymbol{\rho}_2(x)] \in \boldsymbol{H}$. Let us denote $J = \{i \in I \,|\, \boldsymbol{\pi}_1(a_i) = 1\}$, $K = \{i \in I \,|\, \boldsymbol{\pi}_2(a_i) = 1\}$. We have to show that $\boldsymbol{\pi}_1^{-1}(1) = \boldsymbol{\pi}_2^{-1}(1)$. This is clear if $J \cap K \neq \emptyset$, because for $i \in J \cap K$ we have $\boldsymbol{\pi}_1^{-1}(1) = a_i = \boldsymbol{\pi}_2^{-1}(1)$. Assume now that $J \cap K = \emptyset$. Then, by 2.4, $J \cup K = I$ and $\boldsymbol{\rho}_1^{-1}(1) \neq \boldsymbol{\rho}_2^{-1}(1)$. For any $j \in J = I \setminus K$ we have $\boldsymbol{\pi}_2(a_j) \neq 1$ and, since $[\boldsymbol{\pi}_2(a), \boldsymbol{\rho}_2(x)] \in \boldsymbol{H}$, $\boldsymbol{\rho}_2(x_j) = 1$. Similarly, $\boldsymbol{\rho}_1(x_k) = 1$ holds for any $k \in K$. If $\boldsymbol{\pi}_1^{-1}(1) \neq \boldsymbol{\pi}_2^{-1}(1)$, then $x = \boldsymbol{\sigma}(a)$, where $\boldsymbol{\sigma}$ is an arbitrary permutation from S_n with $\boldsymbol{\sigma}(\boldsymbol{\pi}_1^{-1}(1)) = \boldsymbol{\rho}_2^{-1}(1)$, $\boldsymbol{\sigma}(\boldsymbol{\pi}_2^{-1}(1)) = \boldsymbol{\rho}_1^{-1}(1)$. This is impossible, because $x \notin A$ and A is closed under all $\boldsymbol{\sigma} \in S_n$, hence $\boldsymbol{\pi}_1^{-1}(1) = \boldsymbol{\pi}_2^{-1}(1)$.

Now we show that φ_0 is a $\mathcal{X}_n$-morphism. It is easy to se that φ_0 preserves $\mathbf{1}$ and all $\boldsymbol{\pi} \in S_n$. Let $a \in A$, $a \in \boldsymbol{Z}$. Then $[\boldsymbol{\pi}(a), \boldsymbol{\rho}(x)] \in \boldsymbol{H}$ for some $\boldsymbol{\pi}, \boldsymbol{\rho} \in S_n$. Since $\boldsymbol{\rho}(x) \neq \mathbf{1}$ (otherwise $x = \boldsymbol{\rho}^{-1}(\mathbf{1}) \in A$), there must be $i \in I$ with $\boldsymbol{\pi}(a_i) = 1$. Since $a \in \boldsymbol{Z}$, we have $a_i \neq 1$, hence $\varphi_0(a) = \boldsymbol{\pi}^{-1}(1) \neq 1$, which means that $\varphi_0(a) \in \boldsymbol{Z}$.

To show that φ_0 preserves $\boldsymbol{H}$, let $a, b \in A$, $[a, b] \in \boldsymbol{H}$. Then $[\boldsymbol{\pi}_1(a), \boldsymbol{\rho}_1(x)] \in \boldsymbol{H}$, $[\boldsymbol{\pi}_2(b), \boldsymbol{\rho}_2(x)] \in \boldsymbol{H}$ for some $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \in S_n$. Denote $J = \{i \in I \,|\, \boldsymbol{\pi}_1(a_i) = 1\}$, $K = \{i \in I \,|\, \boldsymbol{\pi}_2(b_i) = 1\}$. If there exists $i \in J \cap K$, then $1 \in \{a_i, b_i\} = \{\boldsymbol{\pi}_1^{-1}(1), \boldsymbol{\pi}_2^{-1}(1)\} = \{\varphi_0(a), \varphi_0(b)\}$, hence $[\varphi_0(a), \varphi_0(b)] \in \boldsymbol{H}$. Assume now that $J \cap K = \emptyset$. Then, by 2.4, $J \cup K = I$ and $\boldsymbol{\rho}_1^{-1}(1) \neq \boldsymbol{\rho}_2^{-1}(1)$. For any $j \in J$, $k \in K$ we have $\boldsymbol{\rho}_1(x_k) = \boldsymbol{\rho}_2(x_j) = 1$. We claim that there are $i, k \in K$ with $\boldsymbol{\pi}_1(a_i) \neq \boldsymbol{\pi}_1(a_k)$. Indeed, if $\boldsymbol{\pi}_1(a_k) = q \in \{2, 3, \dots, n\}$ holds for every $k \in K$, then $x = \boldsymbol{\sigma}(a)$, where $\boldsymbol{\sigma} \in S_n$ is such that $\boldsymbol{\sigma}(\boldsymbol{\pi}_1^{-1}(q)) = \boldsymbol{\rho}_1^{-1}(1)$ and $\boldsymbol{\sigma}(\boldsymbol{\pi}_1^{-1}(1)) = \boldsymbol{\rho}_2^{-1}(1)$. This is impossible because $x \notin A$ and $\boldsymbol{\sigma}(a) \in A$. Thus, we have $i, k \in K$ with $\boldsymbol{\pi}_1(a_i) \neq \boldsymbol{\pi}_1(a_k)$, or equivalently, $a_i \neq a_k$. Without loss of generality, $a_i \neq 1$. From $[a, b] \in \boldsymbol{H}$ we obtain that $b_i = 1$. Since $i \in K$, we have $\boldsymbol{\pi}_2(b_i) = 1$, hence $\varphi_0(b) = \boldsymbol{\pi}_2^{-1}(1) = 1$. This implies that $[\varphi_0(a), \varphi_0(b)] \in \boldsymbol{H}$.

Since the structure $\widetilde{A}$ is finite, its topology is discrete and the mapping φ_0 is continuous. Thus, we have shown that φ_0 is an $\mathcal{X}_n$-morphism. According to 1.1, $\widetilde{T_n}$ is injective in $\mathcal{X}_n$ and therefore φ_0 extends to an $\mathcal{X}_n$-morphism $\varphi :\ \widetilde{X} \longrightarrow \widetilde{T_n}$.

It remains to show that φ has the required property. Let $\boldsymbol{\rho} \in S_n$, $a \in X$ and $[a, \boldsymbol{\rho}(x)] \in \boldsymbol{H}$. Then clearly $a \in A$ and $\varphi(a) = \varphi_0(a) = \boldsymbol{\iota}^{-1}(1) = 1$, where $\boldsymbol{\iota}$ denotes the identical permutation. $\square$

2.6. Theorem. *(D, E) is a full duality between $\mathcal{T}_n$ and $\mathcal{X}_n$.*

Proof: According to 2.1, we have to prove that the condition (E) is fulfilled. Let $\widetilde{X}$ be a proper substructure of some finite $\widetilde{Y} \in \mathcal{X}_n$. Let $x = (x_i)_{i \in I} \in \widetilde{Y} \setminus \widetilde{X}$ and let $\varphi :\ \widetilde{X} \longrightarrow \widetilde{T_n}$ be the $\mathcal{X}_n$-morphism constructed in 2.5. The set $W = X \cup \{\boldsymbol{\pi}(x) \,|\, \boldsymbol{\pi} \in S_n\}$ is the underlying set of the substructure $\widetilde{W}$ of $\widetilde{Y}$. Since $\mathbf{1} \notin \{\boldsymbol{\pi}(x) \,|\, \boldsymbol{\pi} \in S_n\}$, there are $i, j \in I$ with $x_i \neq x_j$. We define the mappings $\varphi_1, \varphi_2 :\ W \longrightarrow T_n$ by

$\varphi_1(a) = \varphi_2(a) = \varphi(a)$ for $a \in \widetilde{X}$;

$\varphi_1(\boldsymbol{\pi}(x)) = \boldsymbol{\pi}(x_i)$, $\varphi_2(\boldsymbol{\pi}(x)) = \boldsymbol{\pi}(x_j)$ for $\boldsymbol{\pi} \in S_n$.

It is easy to see that both φ_1 and φ_2 are $\mathcal{X}_n$- morphisms $\widetilde{W} \longrightarrow \widetilde{T_n}$. Because of the injectivity of $\widetilde{T_n}$, they extend to $\mathcal{X}_n$-morphisms ψ_1 and $\psi_2 :\ \widetilde{Y} \longrightarrow \widetilde{T_n}$, respectively.

We obtain that $\psi_1 \restriction \widetilde{X} = \psi_2 \restriction \widetilde{X} = \varphi$ and $\psi_1(x) \neq \psi_2(x)$. $\square$

Thus, we have constructed a full duality for each of the varieties $\mathcal{T}_n$ $(n = 2, 3, \dots)$. Let us remark that a full duality for the variety $\mathcal{T}_2$ (symmetric median algebras) was established in [6], but our duality (for $n = 2$) is different. This is caused by a different choice of a relational structure on $\widetilde{T_2}$. Instead of the binary relation $\boldsymbol{H}$, Werner considers the partial ordering relation $\leq = \{[1,1],[1,2],[2,2]\}$. Of course, our and Werner's dual categories are concretely isomorphic to each other. Similarly, for $n > 2$ the relation $\boldsymbol{H}$ on $\widetilde{T_n}$ can be replaced by any of the relations $\boldsymbol{H}_{ij} = \{[x,y] \in \widetilde{T_n} \mid x = i \text{ or } y = j\}$ $(i, j \in T_n)$, but none of them is a partial ordering.

3. Free isotropic median algebras.

A duality theory is a powerful tool for investigation of free algebras. In fact, this is often the reason why dualities for certain varieties are constructed (see [7], for example).

It is well-known that the free algebra in the variety generated by an algebra $\overline{P}$ with a set S of free generators is isomorphic to the algebra of all term functions $\overline{P}^S \longrightarrow \overline{P}$ (with the projections as free generators and the operations defined pointwise.). Let D and E be the functors defined in Section 1. Since the structure of $\widetilde{P}$ is algebraic over $\overline{P}$, any term function $\overline{P}^S \longrightarrow \overline{P}$ must be an $\mathcal{X}$-morphism $\widetilde{P}^S \longrightarrow \widetilde{P}$. Conversely, if (D, E) is a duality, then 1.8 of [2] says that any $\mathcal{X}$-morphism $\widetilde{P}^S \longrightarrow \widetilde{P}$ is a term function $\overline{P}^S \longrightarrow \overline{P}$. (Although 1.8 of [2] asserts this only for finite S, the proof contains the easy claim that any continuous function $\widetilde{P}^S \longrightarrow \widetilde{P}$ depends only on a finite number of variables.) Hence, we have the following result.

3.1. Corollary. *For any set I and any integer $n > 1$, the $\mathcal{T}_n$- free algebra over the set I of free generators, denoted by $F(\mathcal{T}_n, I)$, is isomorphic to the algebra of all $\mathcal{X}_n$-morphisms $\widetilde{T_n}^I \longrightarrow \widetilde{T_n}$.*

As for the variety $\mathcal{T}$, if the set I is finite, $|I| = n$, then $F(\mathcal{T}, I)$ is isomorphic to $F(\mathcal{T}_n, I)$. The reason for this is that all subalgebras of $\overline{T_\omega}$ generated by at most n elements belong to $\mathcal{T}_n$. Similarly, if $k \geq n = |I|$, then $F(\mathcal{T}_n, I) \cong F(\mathcal{T}_k, I)$.

If the set I is finite, then any mapping $\widetilde{T_n}^I \longrightarrow \widetilde{T_n}$ is continuous and we can rewrite 3.1 as follows.

3.2. Theorem. *Let k and n be integers, $k \geq 2$, $n \geq 1$. Then the algebra $F(\mathcal{T}_k, n)$ is isomorphic to the algebra of all functions $f : \{1, \dots, k\}^n \longrightarrow \{1, \dots, k\}$ satisfying the following conditions:*

(i) *$f(\boldsymbol{\pi}(x_1), \dots, \boldsymbol{\pi}(x_n)) = \boldsymbol{\pi}(f(x_1, \dots, x_n))$ for every $\boldsymbol{\pi} \in S_k$ and $x_1, \dots, x_n \in \{1, \dots, k\}$;*

(ii) *$f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}$ for every $x_1, \dots, x_n \in \{1, \dots, k\}$;*

(iii) *$1 \in \{f(x_1, \dots, x_n), f(y_1, \dots, y_n)\}$ for every $x_1, \dots, x_n, y_1, \dots, y_n \in \{1, \dots, k\}$ such that $1 \in \{x_1, y_1\} \cap \dots \cap \{x_n, y_n\}$.*

The condition (iii) can be replaced by a stronger condition in which the element $1 \in T_n$ does not play a special role:

3.3. Lemma. *Let $k \geq 2$, $n \geq 1$ and let $f : \{1, \ldots, k\}^n \longrightarrow \{1, \ldots, k\}$ satisfy* (i)-(iii). *Then it also satisfies*

(iv) *if $x_1, \ldots, x_n, y_1, \ldots, y_n, i, j \in \{1, \ldots, k\}$ satisfy $\{q \mid x_q = i$ or $y_q = j\} = \{1, \ldots, k\}$, then $f(x_1, \ldots, x_n) = i$ or $f(y_1, \ldots, y_n) = j$.*

PROOF: Let $x_1, \ldots, x_n, y_1, \ldots, y_n, i, j$ satisfy the assumption of (iv). Choose $\boldsymbol{\pi}, \boldsymbol{\rho} \in S_n$ such that $\boldsymbol{\pi}(i) = \boldsymbol{\rho}(j) = 1$. Then $1 \in \{\boldsymbol{\pi}(x_1), \boldsymbol{\rho}(y_1)\} \cap \cdots \cap \{\boldsymbol{\pi}(x_n), \boldsymbol{\rho}(y_n)\}$ and, by (iii), $f(\boldsymbol{\pi}(x_1), \ldots, \boldsymbol{\pi}(x_n)) = 1$ or $f(\boldsymbol{\rho}(y_1), \ldots, \boldsymbol{\rho}(y_n)) = 1$. In the first case we have (by (i)) $\boldsymbol{\pi}(f(x_1, \ldots, x_n)) = 1$, hence $f(x_1, \ldots, x_n) = i$. The second case leads to $f(y_1, \ldots, y_n) = j$. □

It is easy to see that, for any $n \geq 2$, $F(\mathcal{T}_n, 1)$ is a one-element algebra and $F(\mathcal{T}_n, 2)$ is isomorphic to $\overline{T_2}$. The algebra $F(\mathcal{T}_2, 3)$ has four elements and is isomorphic to the subalgebra of $(\overline{T_2})^3$ consisting of the elements $[1, 1, 2]$, $[1, 2, 1]$, $[2, 1, 1]$ and $[1, 1, 1]$ (see [1], where also the 12-element algebra $F(\mathcal{T}_2, 4)$ is described). The algebra $F(\mathcal{T}_n, 3)$ for $n \geq 3$ is isomorphic to the subalgebra $\{[1, 1, 2, 1], [1, 2, 1, 2], [2, 1, 1, 3], [1, 1, 1, 1], [1, 1, 1, 2], [1, 1, 1, 3]\}$ of $(\overline{T_2})^3 \times \overline{T_3}$ (see [5]; this 6-element algebra is in fact free in a much larger class of the so called taut median algebras). Both $F(\mathcal{T}_2, 3)$ and $F(\mathcal{T}_3, 3)$ can be easily visualized. In the fig. 1 and the fig. 2, they are represented as subsets of the lattices L_1 and L_2 respectively, with the median operation defined by the rule

$$m(x, y, z) = (x \vee (y \wedge z)) \wedge (y \vee z).$$

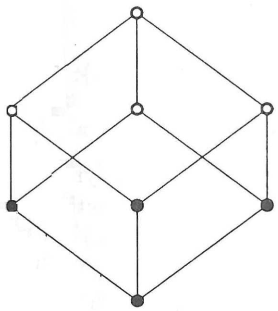

fig.1

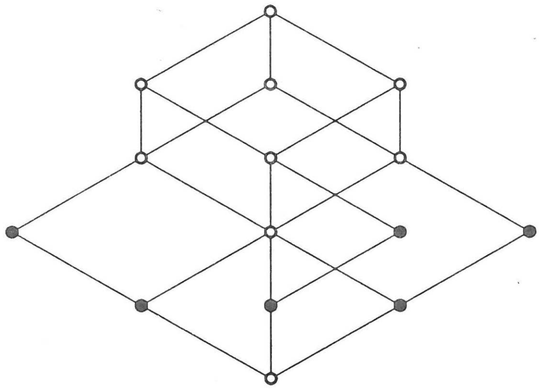

fig.2

In this section we give a detailed description of $F(\mathcal{T}_n, 4)$ for $n \geq 3$. Denote by F the set of all functions $(T_4)^4 \longrightarrow T_4$ satisfying (i)-(iii) of 3.2 (with n=k=4). According to the condition (i), every $f \in F$ is determined uniquely by its values in the following 15 elements of its domain:

$$\begin{array}{lll} a_1 = [1,1,1,1], & a_6 \ = [1,1,2,2], & a_{11} = [1,2,3,2], \\ a_2 = [1,1,1,2], & a_7 \ = [1,1,2,3], & a_{12} = [1,2,2,1], \\ a_3 = [1,1,2,1], & a_8 \ = [1,2,3,3], & a_{13} = [1,2,3,1], \\ a_4 = [1,2,1,1], & a_9 \ = [1,2,1,2], & a_{14} = [1,2,2,3], \\ a_5 = [2,1,1,1], & a_{10} = [1,2,1,3], & a_{15} = [1,2,3,4]. \end{array}$$

3.4. Lemma. *Let $f \in F$ be such that $f(a_i) \neq 1$ for some $i \in \{1,2,3,4,5\}$. Then f is one of the four projections.*

PROOF: The case $f(a_1) \neq 1$ is impossible because of (ii). If $[x,y,z,t]$ is an arbitrary element of $(T_4)^4$, then, by 3.3, $f(a_2) = 1$ or $f(x,y,z,t) = t$. Thus, if $f(a_2) \neq 1$, then f must be a projection on the fourth coordinate. The remaining cases are similar. □

Let us denote $A = \{f \restriction \{a_6, a_7, a_8\} \mid f \in F\}$, $B = \{f \restriction \{a_9, a_{10}, a_{11}\} \mid f \in F\}$, $C = \{f \restriction \{a_{12}, a_{13}, a_{14}\} \mid f \in F\}$. Thus, A contains the restrictions of all $f \in F$ to $\{a_6, a_7, a_8\}$. It is clear that A, B and C are isotropic median algebras with respect to the median operation defined pointwise (they are homomorhic images of $F(\mathcal{T}_4, 4)$). Obviously,these algebras are isomorphic to each other. Now we give their description.

The Lemma 3.4 and (ii) imply the following facts:

(1) if $f \in F$, then $f(a_6) \in \{1,2\}$, $f(a_7) \in \{1,2,3\}$, $f(a_8) \in \{1,2,3\}$:
(2) if $f \in F$ and $f(a_6) = 1$, then $f(a_7) = 1$;
(3) if $f \in F$ and $f(a_6) = 2$, then $f(a_8) = 3$.

It is easy to see that there are 6 functions $\{a_6, a_7, a_8\} \longrightarrow T_4$ satisfying (1), (2) and (3); namely the restrictions of the four projections and the functions g and h defined by $g(a_6) = g(a_7) = h(a_7) = 1$, $h(a_6) = 2$, $g(a_8) = h(a_8) = 3$. The function g is the restriction of $\overline{g} \in F$ defined by $\overline{g}(x,y,z,t) = m(z,x,y)$. Similarly, $h = \overline{h} \restriction \{a_6, a_7, a_8\}$ for $\overline{h} \in F$ given by $\overline{h}(x,y,z,t) = m(x,z,t)$. Thus, we have proved that A is the 6-element subalgebra of $(T_4)^3$ (in fact, of $(T_3)^3$) consisting of the elements $[1,1,1]$, $[1,1,2]$, $[2,2,3]$, $[2,3,3]$, $[1,1,3]$ and $[2,1,3]$. It can be easily visualized. Let L be the lattice depicted in the fig. 3. The six middle layer elements with the operation m given by $m(x,y,z) = (x \wedge (y \vee z)) \vee (y \wedge z)$ form an algebra isomorphic to A.

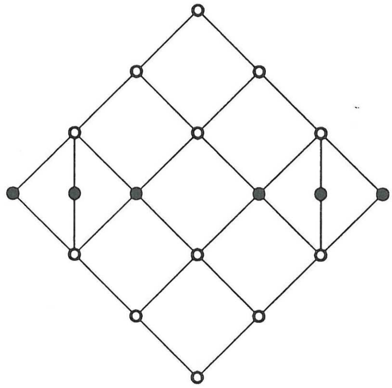

fig. 3

3.5. Lemma. *Let $g_1 \in A$, $g_2 \in B$, $g_3 \in C$, $k \in \{1,2,3,4\}$. Then there is a function $f \in F$ satisfying $f \restriction \{a_6, a_7, a_8\} = g_1$, $f \restriction \{a_9, a_{10}, a_{11}\} = g_2$, $f \restriction \{a_{12}, a_{13}, a_{14}\} = g_3$, $f(a_{15}) = k$ and $f(a_1) = f(a_2) = f(a_3) = f(a_4) = f(a_5) = 1$.*

Proof: Let $f_0 : \{a_1, \dots, a_{15}\} \longrightarrow T_4$ be the unique function satisfying the conditions of our assertion. (Except that it does not belong to F, of course.) We define the function f by the rule

$$f(x,y,z,t) = \boldsymbol{\pi}^{-1}(f_0(\boldsymbol{\pi}(x), \boldsymbol{\pi}(y), \boldsymbol{\pi}(z), \boldsymbol{\pi}(t))),$$

where $\boldsymbol{\pi} \in S_4$ is such that $[\boldsymbol{\pi}(x), \boldsymbol{\pi}(y), \boldsymbol{\pi}(z), \boldsymbol{\pi}(t)] \in \{a_1, \dots, a_{15}\}$. This definition is correct. Indeed, $\boldsymbol{\pi}_1(x,y,z,t) = a_i$ and $\boldsymbol{\pi}_2(x,y,z,t) = a_j$ is only possible if $a_i = a_j$, $\boldsymbol{\pi}_1(x) = \boldsymbol{\pi}_2(x)$, $\boldsymbol{\pi}_1(y) = \boldsymbol{\pi}_2(y)$, $\boldsymbol{\pi}_1(z) = \boldsymbol{\pi}_2(z)$ and $\boldsymbol{\pi}_1(t) = \boldsymbol{\pi}_2(t)$. Since f_0 satisfies (ii), we have $f_0(a_i) \in \{\boldsymbol{\pi}_1(x), \boldsymbol{\pi}_1(y), \boldsymbol{\pi}_1(z), \boldsymbol{\pi}_1(t)\}$, hence $\boldsymbol{\pi}_1^{-1}(f_0(a_i)) = \boldsymbol{\pi}_2^{-1}(f_0(a_i))$.

It is clear that $f \restriction \{a_1, \dots, a_{15}\} = f_0$, because one can take the identity permutation for $\boldsymbol{\pi}$. It remains to show that $f \in F$. The validity of (i) and (ii) is clear. To show (iii), let $\overline{u_i} = [x_i, y_i, z_i, t_i] \in (T_4)^4$ (for $i = 1, 2$) be such that $1 \in \{x_1, x_2\} \cap \{y_1, y_2\} \cap \{z_1, z_2\} \cap \{t_1, t_2\}$. If the element 1 occurs at least three times in the quadruple $\overline{u_1}$ then $\boldsymbol{\pi}(\overline{u_1}) \in \{a_1, \dots, a_5\}$ for some $\boldsymbol{\pi} \in S_4$ with $\boldsymbol{\pi}(1) = 1$, hence $f(\overline{u_1}) = 1$. Similarly if 1 occurs at least three times in $\overline{u_2}$, then $f(\overline{u_2}) = 1$. The remaining case is that the element 1 occurs exactly twice in $\overline{u_1}$ and exactly twice in $\overline{u_2}$. Without loss of generality, $x_1 = y_1 = z_2 = t_2 = 1$ and $z_1, t_1, x_2, y_2 \neq 1$. If $\boldsymbol{\pi} \in S_4$ is such that $\boldsymbol{\pi}(\overline{u_i}) \in \{a_1, \dots, a_{15}\}$, then $\boldsymbol{\pi}(\overline{u_i}) \in \{a_6, a_7, a_8\}$. The function g_1 is a restriction of some $g \in F$ and we have $f(\overline{u_i}) = \boldsymbol{\pi}^{-1}(f_0(\boldsymbol{\pi}(\overline{u_i}))) = \boldsymbol{\pi}^{-1}(g(\boldsymbol{\pi}(\overline{u_i}))) = \boldsymbol{\pi}^{-1}(\boldsymbol{\pi}(g(\overline{u_i}))) = g(\overline{u_i})$. Since g fulfils (iii), we obtain that $1 \in \{f(\overline{u_1}), f(\overline{u_2})\}$. $\square$

3.6. Theorem. *Let p_1, p_2, p_3, p_4 denote the four projections $(T_4)^4 \longrightarrow T_4$. Then $F \setminus \{p_1, p_2, p_3, p_4\}$ is a subalgebra of $F(\mathcal{T}_4, 4)$ isomorphic to $A \times B \times C \times T_4$.*

Proof: By 3.4 one can see easily that $F_0 = F \setminus \{p_1, p_2, p_3, p_4\}$ is indeed a subalgebra of $F(\mathcal{T}_4, 4)$. We define a map $h: F_0 \longrightarrow A \times B \times C \times T_4$ by the rule

$$h(f) = [f \restriction \{a_6, a_7, a_8\}, f \restriction \{a_9, a_{10}, a_{11}\}, f \restriction \{a_{12}, a_{13}, a_{14}\}, f(a_{15})].$$

Since the median operation in $F(\mathcal{T}_4, 4)$ is defined pointwise, the map h is a homomorphism. Since any $f \in F$ is determined by its values on the set $\{a_1, \dots, a_{15}\}$, the map h is one-to-one. Finally, 3.5 says that it is also surjective. □

3.7. Corollary. *$F(\mathcal{T}_4, 4)$ has 868 elements.*

The Theorem 3.6 yields an explicit description of $F(\mathcal{T}_4, 4)$. Namely, $F(\mathcal{T}_4, 4)$ is isomorphic to $\{p_1, p_2, p_3, p_4\} \cup A \times B \times C \times T_4$ with the median operation defined by

$$m(x, y, z) = \begin{cases} x & \text{if } x = z \text{ or } x = y, \\ y & \text{if } y = z, \\ m(q(x), q(y), q(z)) & \text{computed in } A \times B \times C \times T_4 \text{ if } x \neq y \neq z \neq x, \end{cases}$$

where $q(x) = x$ for $x \in A \times B \times C \times T_4$ and $q(p_i) = [p_i \restriction \{a_6, a_7, a_8\}, p_i \restriction \{a_9, a_{10}, a_{11}\}, p_i \restriction \{a_{12}, a_{13}, a_{14}\}, p_i(a_{15})]$.

By a similar way one can prove that $F(\mathcal{T}_3, 4) \setminus \{p_1, p_2, p_3, p_4\}$ is isomorphic to $A \times B \times C$ and hence that $F(\mathcal{T}_3, 4)$ has 220 elements. The only difference is that the element a_{15} is not in $(T_3)^4$ and should be omitted in considerations, which makes the proofs a little bit simpler. If we investigate $F(\mathcal{T}_2, 4)$, we have to omit the elements a_7, a_8, a_{10}, a_{11}, a_{13}, a_{14} and a_{15}. In this case one finds that $F(\mathcal{T}_2, 4) \setminus \{p_1, p_2, p_3, p_4\}$ is isomorphic to $(T_2)^3$.

Free isotropic median algebras with more than four free generators have a much more complicated structure. However, combinatorial considerations and 3.2 allow to compute their cardinalities. One can show that $|F(\mathcal{T}_2, 5)| = 81$, $|F(\mathcal{T}_3, 5)| = 36207977$, $|F(\mathcal{T}_4, 5)| = 19583346143237$ and $|F(\mathcal{T}_n, 5)| = 97916730716165$ for $n > 4$.

References

[1] Bandelt H.J., Hedlíková J., *Median algebras*, Discrete Math. **45** (1983), 1–30.

[2] Davey B.A., Werner H., *Dualities and equivalences for varieties of algebras*, Coll. Math. Soc. János Bolyai 33 (Contributions to lattice theory), North-Holland, 1983,pp. 101–276.

[3] Draškovičová H., *Contributions to General Algebra 7*, Holder-Pichler-Tempsky, Wien, 1991, pp. 119–125.

[4] Fried E., Pixley A.F., *The dual discriminator function in universal algebra*, Acta Sci. Math. (Szeged) **41** (1979), 83–100.

[5] Isbell J.R., *Median algebra*, Trans. Amer. Math. Soc. **260** (1980), 319–362.

[6] Werner H., *A duality for weakly associative lattices*, Coll. Math. Soc. János Bolyai 28 (Finite algebra and multiple-valued logic), North-Holland, 1982, pp. 781–808.

[7] ———, *A duality for the lattice variety generated by M_3*, Coll. Math. Soc. János Bolyai 43 (Lectures in universal algebra), North-Holland, 1986, pp. 561-572.

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