Comment.Math.Univ.Carolin. 33,3 (1992)551–555

A note on splittable spaces

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Abstract. A space X is splittable over a space Y (or splits over Y) if for every $A \subset X$ there exists a continuous map $f : X \to Y$ with $f^{-1}fA = A$. We prove that any n-dimensional polyhedron splits over $\mathbf{R}^{2n}$ but not necessarily over $\mathbf{R}^{2n-2}$. It is established that if a metrizable compact X splits over $\mathbf{R}^n$, then $\dim X \leq n$. An example of n-dimensional compact space which does not split over $\mathbf{R}^{2n}$ is given.

Keywords: splittable, polyhedron, dimension

Classification: 54A25

The notion of splittability was introduced by A.V. Arhangel'skii [1]. A space X is splittable (or splits) over a space Y if for any $A \subset X$ there exists a continuous map $f : X \to Y$ such that $f^{-1}fA = A$. Many results were obtained by A.V. Arhangel'skii and D.B. Shakhmatov ([1]–[3]) on spaces splittable over $\mathbf{R}^\omega$. The author had also written a paper [4] on this topic.

Recently, A.V. Arhangel'skii had shown that a compact space X splits over $\mathbf{R}$ iff it embeds in $\mathbf{R}$. He also proved that any 1-dimensional polyhedron splits over $\mathbf{R}^2$ so that not every compact space splittable over $\mathbf{R^2}$ embeds in $\mathbf{R}^2$. We prove here that any n-dimensional polyhedron splits over $\mathbf{R}^{2n}$ but not necessarily over $\mathbf{R}^{2n-2}$. We establish also that there exists a compact space $X_n \subset \mathbf{R}^{2n+1}$ with $\dim X_n = n$ and not splittable over $\mathbf{R}^{2n}$. Another result is Corollary 8 answering Question 2 and 3 in [1].

Notations and terminology. All spaces under consideration are Tychonoff ones. Given two spaces X and Y, denote by $C(X,Y)$ the set of all continuous functions from X to Y. The space $\mathbf{R}$ is the real line with its usual topology and $2 = \{0,1\}$. If $x, y \in \mathbf{R}^n$ then $[x,y] = \{tx+(1-t)y : 0 \leq t \leq 1\}$ is the segment connecting x and y, $|x-y|$ is its length. It is always clear from the context whether $|\cdot|$ denotes cardinality of some set or length of a segment. Of course, $[x,y) = \{tx+(1-t)y : 0 < t \leq 1\}$ and $(x,y) = \{tx+(1-t)y : 0 < t < 1\}$. The simplex in $\mathbf{R}^n$ with vertices $a_0, \dots, a_k$ will be denoted by $[a_0, \dots, a_k]$, then$\langle T \rangle = \{t_0a_0 + \dots + t_ka_k : t_i > 0$ for all $i \in k+1$ and $\sum_{i=0}^k t_i = 1\}$. Let $x \in \mathbf{R}^n$ and $A \subset \mathbf{R}^n$. Then $\operatorname{con}(x,A) = \bigcup\{[x,a] : a \in A\}$.

Other notations are standard and can be found in [9].

1. Lemma. *Given any spaces X and Y and an infinite cardinal k with $d(X) \leq k$, (d is the density character), $|Y| \leq 2^k$, suppose that $\bigcup\{X_s : s \in 2^k\} \subset X$, $X_s \cap X_t = \emptyset$ if $s \neq t$ and no X_s can be continuously injected into Y. Then X does not split over Y.*

PROOF: Clearly, $|C(X,Y)| \leq 2^k$, so let $C(X,Y) = \{f_s : s < 2^k\}$. For any $s < 2^k$ there is an $x_s \in X$ such that $|f_s^{-1} f_s(x_s) \cap X_s| > 1$ because $f \restriction X$ is not an injection. Then the set $A = \{x_s : s < 2^k\}$ witnesses non-splittability of X over Y. □

2. Example. *For any natural n there exists an n-dimensional metrizable compact space X_n which does not split over $\mathbf{R}^{2n}$.*

PROOF: Take any metrizable compact space Y_n with $Y_n \not\hookrightarrow \mathbf{R}^{2n}$ and $\dim Y_n = n$. Then let $X_n = 2^\omega \times Y$. It is obvious that the family $\{\{s\} \times Y_n : s \in 2^\omega\}$ satisfies the conditions of Lemma 1 for $X = X_n$, $k = \omega$ and $Y = \mathbf{R}^{2n}$. Hence X_n does not split over $\mathbf{R}^{2n}$. □

3. Example. *There exists an n-dimensional compact polyhedron P_n which is not splittable over $\mathbf{R}^{2n-2}$.*

PROOF: There exists an $(n-1)$-dimensional compact polyhedron Y_n which does not embed in $\mathbf{R}^{2n-2}$. Then $P = Y_n \times [0,1]$ is what was required, because the family $\{Y_n \times \{t\} : t \in [0,1]\}$ satisfies the conditions of Lemma 1 for $X = P_n$, $k = \omega$ and $Y = \mathbf{R}^{2n-2}$. □

4. Theorem. *Let P be an n-dimensional compact polyhedron. Then P splits over $\mathbf{R}^{2n}$.*

PROOF: Denote by $a_1, \dots, a_k$ the vertices of P. Let $\{S_1, \dots, S_r\}$ be the set of all $(n-1)$-dimensional simplexes from P nd let $\mu = \{T_1, \dots, T_m\}$ be some set of its n-dimensional ones. Take any hyperplane $H \subset \mathbf{R}^{2n}$ and let $b_1, \dots, b_k$ be some points generally positioned in H. Define a polyhedron Q_{n-1} in the following way: the vertices of Q_{n-1} are $b_1, \dots, b_k$ and a simplex $[b_{i_1}, \dots, b_{i_l}]$, $l \leq n$ belongs to Q_{n-1} iff the simplex $[a_{i_1}, \dots, a_{i_l}]$ belongs to P. If P_{n-1} is the union of all $\leq (n-1)$-dimensional simplexes, then the simplicial map $f : P_{n-1} \to Q_{n-1}$ defined by $f(a_i) = b_i$ is a homeomorphism because H is isomorphic to $\mathbf{R}^{2n-1}$. Pick any $D \in \mathbf{R}^{2n} \setminus H$. □

5. Lemma. *There exist m sets $L_1, \dots, L_m$ and a continuous map*

$$g = g(f, D) : P_{n-1} \cup T_1 \cup \dots \cup T_m \to \mathbf{R}^{2n}$$

with the following properties:

(1) *L_i is a subset of $\mathbf{R}^{2n}_D$, where the last set is the component of $\mathbf{R}^{2n} \setminus H$ containing D, $i = 1, \dots, m$;*
(2) *L_i is homeomorphic to $\langle T_i \rangle$, $i = 1, \dots, m$;*
(3) *$L_i \cap L_j = \{D\}$ if $i \neq j$;*
(4) *$g \restriction P_{n-1} = f$;*
(5) *$g \restriction (P_{n-1} \cup T_i)$ is a homeomorphism onto $Q_{n-1} \cup L_i$.*

PROOF OF THE LEMMA: Let $R = \operatorname{con}(D, Q_{n-1})$, $R_i = \operatorname{con}(D, f(S_i))$, $i = 1, \dots, r$. The set R being compact, there is a sphere (in $\mathbf{R}^{2n}$) containing it. Pick any $E \in \mathbf{R}^{2n}_D$ outside this sphere and not belonging to any of $(2n-1)$-dimensional planes, spanned in $\mathbf{R}^{2n}$ by some $2n$ points from the set $\{b_1, \dots, b_k, D\}$.

We are going to construct a continuous function $q : R \to [0,1)$ such that

$$q^{-1}(0) = Q_{n-1} \cup \{D\}; \tag{6}$$

$$\text{if } x \in R \setminus (Q_{n-1} \cup \{D\}) \text{ and } y \in (x,E) \cap R, \text{ then } q(x) < \frac{|x-y|}{|y-E|}. \tag{7}$$

To do that let $W_i = \{x \in R : (x,E) \cap R_i \neq \emptyset\}$. For each $x \in R$ and each i, there is at most one point in $(x,E) \cap R_i$, for otherwise E would belong to the n-dimensional plane spanned by R_i. Put for $x \in R_i$:

$$r_i(x) = \frac{|x-y|}{|y-E|} \text{ where } y \in (x,E) \cap R_i, \text{ and } r_i(D) = 0.$$

Let us prove that $\operatorname{dom} r_i = W_i \cup \{D\}$ is a closed subset of R. It suffices to show that $\operatorname{dom} r_i \cap R_j$ is closed for each j. If $R_i \cap R_j \cap H \neq \emptyset$ then, owing to the choice of E, $\operatorname{dom} r_i \cap R_j = \{D\}$. Suppose $R_i \cap R_j \cap H = \emptyset$, hence $R_i \cap R_j = \{D\}$. Let $x \in R_j \setminus \operatorname{dom} r_i$. Then $(x,E) \cap R_i = \emptyset$ and $[x,E] \cap R_i = \emptyset$ as well. Since $[x,E]$ is compact and R_i is closed, the distance between these sets is positive, say ε, and whenever $|z-x| < \varepsilon$ then clearly $[z,E] \cap R_i = \emptyset$. Since r_i is continuous, there exists a continuous $q_i : R \to [0,1)$ with $q^{-1}(0) = Q_{n-1} \cup \{D\}$ and $q_i(x) < r_i(x)$ for all $x \in W_i \setminus (Q_{n-1} \cup \{D\})$. Now it suffices to put

$$q(x) = \min\{q_i(x) : i = 1, \ldots, r\}.$$

Let $M_i = \operatorname{con}(D, f(T_i \setminus \langle T_i \rangle))$, $i = 1, \ldots, m$. It is clear that M_i is homeomorphic to T_i. Define an injective continuous map $s_i : M_i \to \mathbf{R}^{2n}_D$ in the following way: if $x \in M_i$ then find the point $y \in [x,E)$ with

$$\frac{|x-y|}{|y-E|} = \frac{q(x)}{i+1}$$

and put $s_i(x) = y$.

Evidently, s_i is a homeomorphism. Let $L_i = s_i(M_i \setminus f(T_i \setminus \langle T_i \rangle))$. We are going to define the map $g = g(f,D)$ and verify (1)–(5). Take any homeomorphism $u_i : T_i \to M_i$ with $u_i \restriction (T_i \setminus \langle T_i \rangle) = f$. Then let $g(x)$ be equal to $f(x)$ if $x \in P_{n-1}$ and to $s_i(u_i(x))$ for $x \in T_i \setminus P_{n-1}$, $i = 1, \ldots, m$.

Only (3) needs to be verified.

Let $x \in M_i \setminus (\{D\} \cup Q_{n-1})$, $y \in M_j \setminus (\{D\} \cup Q_{n-1})$. If $g(x) = g(y)$ then x, y and E are linearly dependent. We may assume without loss of generality that $y \in [x,E]$. There are two possibilities: $x = y$, and $x \neq y$.

If $x = y$ then

$$\frac{|x-g(x)|}{|g(x)-E|} = \frac{q(x)}{i+1} \text{ and } \frac{|x-g(y)|}{|g(y)-E|} = \frac{q(y)}{j+1} = \frac{q(x)}{j+1} \neq \frac{q(x)}{i+1},$$

so that $g(x) \neq g(y)$, which is a contradiction.

If $x \neq y$ and $x \in R_{t_1}$, $y \in R_{t_2}$, $R_{t_1} \cap R_{t_2} \cap Q_{n-1} \neq \emptyset$ then it is impossible that $y \in [x, E]$ — a contradiction.

If $R_{t_1} \cap R_{t_2} \cap Q_{n-1} = \emptyset$ then

$$\frac{|x - g(x)|}{|g(x) - E|} < \frac{|x - y|}{|y - E|}$$

and therefore $g(x) \in [x, y)$ while $g(y) \in [y, E)$ and $g(x) \neq g(y)$ — a contradiction again and we established (3) together with our lemma. $\square$

We have all we need to split P over $\mathbf{R}^{2n}$. Let $A \subset P$. Pick a point $D_1 \in \mathbf{R}^{2n} \setminus (H \cup \mathbf{R}^{2n}_D)$. Let $T_1, \dots, T_m, T_{m+1}, \dots, T_{m_1}$ be all n-dimensional simplexes of P numerated in such a way that $A \cap \langle T_i \rangle \neq \emptyset$, $i = 1, \dots, m$, $(P \setminus A) \cap \langle T_i \rangle \neq \emptyset$, $i = m+1, \dots, m_1$. Using Lemma 5 find the sets $L_1, \dots, L_{m_1}$ and maps $g = g(f, D)$ and $g_1 = g(f, D_1)$ such that

$$L_i \subset \mathbf{R}^{2n}_D,\ i = 1, \dots, m,\ L_i \subset \mathbf{R}^{2n}_D,\ i = m+1, \dots, m_1; \tag{8}$$

$$L_i \text{ is homeomorphic to } \langle T_i \rangle,\ i = 1, \dots, m_1; \tag{9}$$

$$L_i \cap L_j = \{D\},\ i \neq j,\ j \in 1, \dots, m; \tag{10}$$

$$L_i \cap L_j = \{D\},\ i = j,\ i, j \in m+1, \dots, m_1; \tag{11}$$

$$g \restriction P_{n-1} = g_1 \restriction P_{n-1} = f; \tag{12}$$

$$g \restriction (P_{n-1} \cup T_i) \text{ is a homeomorphism onto } Q_{n-1} \cup L_i,\ i = 1, \dots, m; \tag{13}$$

$$g \restriction (P_{n-1} \cup T_i) \text{ is a homeomorphism onto } Q_{n-1} \cup L_i,\ i = m+1, \dots, m_1; \tag{14}$$

Pick some points $c_1, \dots, c_{m_1}$ with $c_i \in A \cap \langle T_i \rangle$, $i = 1, \dots, m$, $c_i \in (P \setminus A) \cap \langle T_i \rangle$, $i = m+1, \dots, m_1$ and the points $d_1, \dots, d_{m_1}$ with $g(d_i) = D$, $i = 1, \dots, m$, $g_1(d_i) = D_1$, $i = m+1, \dots, m_1$ (observe that automatically $d_i \in \langle T_i \rangle$ for each i). Let $G = g \cup g_1$. Then G is a continuous map, $G : P \to \mathbf{R}^{2n}$. There exists a homeomorphism $h : P \to P$ with $h \restriction P_{n-1} = \mathrm{id}_{P_{n-1}}$ and $h(c_i) = d_i$, $i = 1, \dots, m_1$. The map $F = G \circ h$ separates A from $P \setminus A$, because $|F^{-1}(x)| = 1$ if $x \notin \{D, D_1\}$ and $F^{-1}(D) = \{c_1, \dots, c_m\} \subset A$, $F^{-1}(D_1) = \{c_{m+1}, \dots, c_{m_1}\} \subset P \setminus A$, and our theorem is proved. $\square$

6. Proposition. *Let X be a compact space and $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and any compact $K \subset X_i$ is scattered $(i = 1, 2)$. Assume that X splits over a space Y with $\dim Y \leq n$. Then $\dim X \leq n$.*

Proof: Take a continuous $f : X \to Y$ with $f^{-1}f(X_1) = X_1$. If $y \in Y$ then $f^{-1}(y)$ is a compact subset of some X_i $(i = 1, 2)$ and is thus scattered. Hence $\dim f^{-1}(y) = 0$ for every $y \in f(X)$. But $\dim X \leq \dim f(X) + \dim f \leq n$ [10] and the proof is over. $\square$

7. Corollary. *If a compact space X is splittable over $\mathbf{R}^n$, then $\dim X \leq n$.*

Proof: The space X must be metrizable [3]. It is widely known (see e.g. [5]) that metrizable compact spaces satisfy the assumptions of Proposition 6, so our proof is over. $\square$

This corollary answers Questions 2 and 3 in [1].

8. Corollary ($ACP^{\#}$). *If X is a compact space splittable over a space Y then $\dim X \leq \dim Y$. (The definition of $ACP^{\#}$ can be found in* [5]*).*

9. Corollary. *If X is a metrizable compact space splittable over a space Y then $\dim X \leq \dim Y$.*

10. Example. *Compactness is essential in* 7–9*, for there exist infinite-dimensional second countable spaces which can be injectively mapped in* $\mathbf{R}$ [6].

11. Proposition. *If X is an infinite extremally disconnected compact space splittable over a space Y then $\beta\omega \hookrightarrow Y$.*

PROOF: It is true in ZFC (see [7]) that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ and every compact $K \subset X$ is finite $(i = 1, 2)$. Pick a continuous map $f : X \to Y$ with $f^{-1}f(X_i) = X_i$. The space $\beta\omega$ embeds in X and $f \restriction \beta\omega$ has finite point-inverses, so that $\beta\omega \hookrightarrow f(\beta\omega)$ [8] and our proposition is proved.

12. Corollary. *If $\beta\omega$ splits over a space Y then $\beta\omega$ embeds in Y.*

□

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(*Received* May 6, 1991, *revised* April 21, 1992)