Strong sequences and the weight of regular spaces

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Abstract. It will be shown that if in a family of sets there exists a strong sequence of the length $(\kappa^{\lambda})^{+}$ then this family contains a subfamily consisting of λ^{+} pairwise disjoint sets. The method of strong sequences will be used for estimating the weight of regular spaces.

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In his paper [1], B.A. Efimov introduced the method of strong sequences in generalized Cantor discontinua. Among others he has proved that in generalized Cantor discontinua strong sequences do not exist. As a corollary, Marczewski's theorem on cellularity of dyadic space and Shanin's theorem on calibers in dyadic spaces are obtained.

The aim of this paper is to give the answer to the question "What follows from the existence of strong sequences?" It will be shown that if in a family of sets there exists a strong sequence of the length $(\kappa^{\lambda})^{+}$ then this family contains a subfamily consisting of λ^{+} pairwise disjoint sets. The method of strong sequences will be used for estimating the weight of regular spaces. A new cardinal invariant
$\lambda(X) = \inf\{\sup\{\chi(x, X) : x \in M\} : M$ dense in $X\} + \omega$ is introduced.

It was proved in [3] that the wight of a regular space is not greater than $\pi_{\chi}(X)^{c(X)}$ (the π-character of space to the power of cellularity of space). It will be proved that the weight of regular space is not greater than $\lambda(X)^{c(X)}$ (in [1] Efimov proved that if X is dyadic space, than the weight of X is equal to $\lambda(X)$). It is easy to see that these two powers are equal. But it will be shown (Examples 2 and 3) that no inequality can be established between $\lambda(X)$ and the π-character of X.

Let X be a set and $\mathbf{B} \subseteq \mathbf{P}(X)$ be a family of non-empty subsets of X closed with respect to the finite intersections. Let S be a finite subfamily contained in $\mathbf{B}$. A pair (S, H) where $H \subseteq \mathbf{B}$ will be called <u>connected</u> if $S \cup H$ is centered. A sequence (S_{ϕ}, H_{ϕ}); $\phi < \alpha$, consisting of connected pairs is called <u>a strong sequence</u> if $S_{\lambda} \cup H_{\phi}$ is not centered whenever $\lambda > \phi$.

Let T be an infinite set. Denote a Cantor cube by

$$D^{T} := \{p : p : T \to \{0, 1\}\}.$$

For $v \subseteq T$, $i : v \to \{0, 1\}$ we shall use the following notation

$$H_{v}^{i} := \{p \in D^{T} : p \mid v = i\}.$$

Efimov investigated in his paper [1] strong sequences in the subbase $\{H^i_{\{\alpha\}} : \alpha \in T\}$ in D^T.

Efimov proved the following

Theorem. *Let κ be a regular, uncountable cardinal number. In the space D^T there is not a strong sequence $(\{H^i_\alpha : \alpha \in v_\zeta\}, \{H^l_\beta : \beta \in w_\zeta\})$; $\zeta < \kappa$ such that $\operatorname{card} w_\zeta < \kappa$ and $\operatorname{card} v_\zeta < \omega$ for each $\zeta < \kappa$.*

Let X be a topological space.
Denote by $w(X) = \min\{\operatorname{card} B : B$ a base for $X\} + \omega$, the weight of the space X. A pairwise disjoint collection of non-empty open sets in X is called cellular family. The cellularity of X is defined as follows:

$$c(X) = \sup\{\operatorname{card} \mathcal{V} : \mathcal{V} \text{ a cellular family in } X\} + \omega.$$

Let $\mathcal{V}$ be a collection of non-empty open sets in X, let $p \in X$. Then $\mathcal{V}$ is a local π-base for p if for each open neighborhood U of p, one has $V \subset U$ for some $V \in \mathcal{V}$. If in addition one has $p \in V$ for all $V \in \mathcal{V}$, then $\mathcal{V}$ is a local base for p.

Denote by

$$\begin{aligned}
\pi w(X) &= \min\{\operatorname{card} \mathcal{V} : \mathcal{V} \text{ a } \pi\text{-base for } X\} + \omega \\
\chi(p, X) &= \min\{\operatorname{card} \mathcal{V} : \mathcal{V} \text{ is a local base for } p\} \\
\pi\chi(p, X) &= \min\{\operatorname{card} \mathcal{V} : \mathcal{V} \text{ is a local } \pi\text{-base for } p\} \\
\chi(M, X) &= \sup\{\chi(p, X) : p \in M \subset X\} \\
\lambda(X) &= \min\{\chi(M, X) : M \text{ is a dense subset of } X\} + \omega \\
\pi\chi(X) &= \sup\{\pi\chi(p, X) : p \in X\}.
\end{aligned}$$

Define the density of X as follows:

$$d(X) = \min\{\operatorname{card} S : S \subset X \text{ and } \operatorname{cl} S = X\}.$$

Example 1. It is easy to give an example of a strong sequence of length greater than a cellularity of family of sets. For this purpose let us take a regular space such that $c(X) < d(X)$ and $\chi(X) < d(X)$. Let M be a dense subset of X such that $\operatorname{card} M = d(X)$. Let $B(x)$ be a base in a point $x \in M$ such that $\operatorname{card} B(x) \leq \chi(X)$. Let us start from an arbitrary point $x_1 \in M$ and $U \in B(x_1)$. Then we take a pair $(\{U\}, B(x_1))$ as a first pair of a strong sequence. Suppose that for $\beta < \gamma < \max(c(X), \chi(X))^+$ the strong sequence has been defined. Let us take the set $\{x_\beta : \beta < \gamma\}$. The set $(X \setminus \operatorname{cl}\{x_\beta : \beta < \gamma\}) \cap M$ is non-empty and hence we can take a point $x_\beta \in (X \setminus \operatorname{cl}\{x_\beta : \beta < \gamma\}) \cap M$. Since X is a regular space, there exists $U_\gamma \in B(x_\gamma)$ such that $\operatorname{cl} U_\gamma \cap \operatorname{cl}\{x_\beta : \beta < \gamma\} = \emptyset$. Hence for each $\beta < \gamma$ $\{U_\gamma\} \cup B(x_\beta)$ is not centered. Hence the strong sequence has been defined.

Theorem 1. *If for* $\mathbf{B} \subseteq \mathbf{P}(X)$ *there exists a strong sequence* $\mathbf{S} = (S_\zeta, H_\zeta)$*;* $\zeta < (\kappa^\lambda)^+$ *such that* $\operatorname{card}(H_\zeta) \leq \kappa$ *for each* $\zeta < (\kappa^\lambda)^+$ *then the family* $\mathbf{B}$ *contains a subfamily of cardinality* λ^+ *consisting of pairwise disjoint sets.*

Proof: Denote $\alpha = \kappa^\lambda$, let $\{(S_\mu, H_\mu) : \mu < \alpha^+\}$ be a strong sequence in $\mathbf{B}$. We may and shall assume that every H_μ is closed under finite intersections. We construct an increasing chain $\{I_\zeta : \zeta < \lambda^+\}$ of subsets of α^+ by transfinite induction as follows:

$$I_0 = \{0\};\ \text{if } \zeta \text{ is a limit ordinal, then } I_\zeta = \bigcup\{I_\nu : \nu < \zeta\}.$$

The induction assumption is $|I_\zeta| \leq \alpha$. Suppose $\zeta = \eta + 1$, $\zeta < \lambda^+$ and I_η known, $|I_\eta| \leq \alpha$. Call a pair (K, f) admissible, if $K \subseteq I_\eta$, $|K| \leq \lambda$, $f \in \prod\{H_\eta : \eta \in K\}$ and there is some $\varphi > \sup I_\eta$ such that for every $\mu \in K$, $\bigcap S_\varphi \cap f(\mu) = \emptyset$. If $|K| \leq \lambda$, then $|\prod\{H_\mu : \mu \in K\}| \leq \alpha$, because $|H_\mu| \leq \kappa$ for all $\mu < \alpha^+$. Since $|I_\eta| \leq \kappa^\lambda$, it contains not more than κ^λ subsets of size $\leq \lambda$. Therefore there are at most $\kappa^\lambda \kappa^\lambda = \kappa^\lambda$ admissible pairs.

For every admissible pair (K, f) select one $\varphi(K, f) > \sup I_\eta$ witnessing the fact that (K, f) is admissible and let $I_\zeta = I_\eta \cup \{\varphi(K, f) : (K, f) \text{ admissible}\}$. This completes the inductive definitions. We have already verified that $|I_\zeta| \leq \alpha$ again holds.

Let $I = \bigcup\{I_\zeta : \zeta \leq \lambda^+\}$. Since $|I| \leq \lambda^+ \kappa^\lambda = \kappa^\lambda$, I is not cofinal in α^+. Thus there is some $\sup I < \varphi < \alpha^+$. For this φ, $S_\varphi \cup H_\mu$ is not centered whenever $\mu \in I$.

Choose $\mu(0) \in I_1 \setminus I_0$ and $F_0 \in H_{\mu(0)}$ with $F_0 \cap \bigcap S_\varphi = \emptyset$. Proceeding further, suppose $\mu(\eta) \in I_{\eta+1} \setminus I_\eta$ and $F_\eta \in H_{\mu(\eta)}$ are known and $F_\eta \cap \bigcap S_\varphi = \emptyset$ for all $\eta < \zeta$, $\zeta < \lambda^+$. Then the pair (K, f), where $K = \{\mu(\eta) : \eta < \zeta\}$ and $f(\mu(\eta)) = F_\eta$ is admissible, so it remains to define $\mu(\zeta) = \varphi(K, f)$ in accordance with the inductive definition of $I_{\zeta+1}$ and let $F_\zeta \in H_{\mu(\zeta)}$ be such that $F_\zeta \cap \bigcap S_\varphi = \emptyset$.

By our choice of I_ζ, $\mu(\zeta)$ and F_ζ, it is obvious now that $\{\bigcap S_{\mu(\zeta)} \cap F_\zeta : \zeta < \lambda^+\}$ is the required disjoint family. □

A pairwise disjoint collection of non-empty sets in a family A is called a cellular family. The cellularity of A is defined as follows:

$$c(A) = \sup\{\operatorname{card} V : \mathcal{V} \text{ a cellular family in } A\} + \omega.$$

Theorem 2. *Let* $\mathbf{A}$ *be a family of non-empty sets closed with respect to the finite non-empty intersections. For each cardinal number* κ *such that* $(\kappa^{c(\mathbf{A})})^+ \leq \operatorname{card} \mathbf{A}$ *there exists a centered subfamily* $\mathbf{B} \subseteq \mathbf{A}$ *such that* $\operatorname{card} \mathbf{B} > \kappa$*.*

Proof: Suppose that if $\mathbf{B} \subseteq \mathbf{A}$ and $\mathbf{B}$ is centered, then $\operatorname{card} \mathbf{B} \leq \kappa$. Let us take an arbitrary centered subfamily $\mathbf{H}_1 \subseteq \mathbf{A}$. Let $S_1 \subseteq \mathbf{H}_1$ be a finite subfamily and $(S_1, \mathbf{H}_1)$ be a first pair of a strong sequence. Suppose that for $\gamma < \beta < (\kappa^{c(\mathbf{A})})^+$ the strong sequence $(S_\gamma, \mathbf{H}_\gamma)$ $\gamma < \beta$ has been defined. Since $(\kappa^{c(\mathbf{A})})^+$ is regular and

$\operatorname{card}\mathbf{H}_\gamma \le \kappa$ for each γ, hence $\operatorname{card}\{\bigcup \mathbf{H}_\gamma : \gamma < \beta\} < (\kappa^{c(\mathbf{A})})^+ < \operatorname{card}\mathbf{A}$. From this it follows that there exists $S \in \mathbf{A}$ such that $\mathbf{H}_\gamma \cup \{S\}$ is not a centered family for each $\gamma < \beta$. Let us take a maximal centered family $\mathbf{H}_\beta \subseteq \mathbf{A}$ such that $S \in \mathbf{H}_\beta$. Hence the strong sequence has been defined. Hence, by Theorem 1, there are $c(\mathbf{A})^+$ pairwise disjoint sets in $\mathbf{A}$. A contradiction. □

Lemma. *If X is a regular space, then $d(X) \le \lambda(X)^{c(X)}$.*

Proof: Suppose that $d(X) > \lambda(X)^{c(X)}$. Let us choose a dense subset M such that $\chi(M, X) = \lambda(X)$. Let us choose an arbitrary point $x \in M$ and let $\mathcal{B}(x)$ be a base in the point x of cardinality not greater than $\lambda(X)$. Let $(U_x, \mathcal{B}(x))$ be the first pair of a strong sequence, where $U_x \in \mathcal{B}(x)$. Let us assume that for each $\zeta < \lambda < (\lambda(X)^{c(X)})^+$ a connected pair has been defined. So we have a strong sequence $(U_{x_\zeta}, \mathcal{B}(x_\xi))$ $\xi < \lambda$ where $x_\xi \in M$ and $\operatorname{card}\mathcal{B}(x_\xi) < \lambda(X)$ for each $\xi < \lambda$. Let $D_\lambda = \{x_\xi : \xi < \lambda\}$. From the assumption $X - \operatorname{cl} D_\lambda \ne \emptyset$. Let $B(x_\lambda)$ be a base in the point x_λ and $x_\lambda \in (X - \operatorname{cl} D_\lambda) \cap M$. Let $\operatorname{card} B(x_\lambda) < \lambda(X)$. There exists a set $U_{x_\lambda} \in \mathcal{B}(x_\lambda)$ and a neighborhood U of the set D_λ such that $U_{x_\lambda} \cap U = \emptyset$. From this it follows that for each $\xi < \lambda$ there exists $U_{x_\xi} \in \mathcal{B}(x_\xi)$ such that $U_{x_\xi} \cap U_{x_\lambda} = \emptyset$. Hence $\mathcal{B}(x_\xi) \cup \{U_{x_\lambda}\}$ is not a centered family for each $\xi < \lambda$. So we can extend strong sequence by adding the pair $\{U_{x_\lambda}, \mathcal{B}(x_\lambda)\}$. Hence we have a strong sequence of length $(\lambda(X)^{c(X)})^+$. This sequence fulfills the assumption of the theorem, so we have the family consisting of $c(X)^+$ pairwise disjoint open sets. □

In [2] Efimov proved that for any space X, $\operatorname{card} RO(X) \le \pi w(X)^{c(X)}$.

Theorem 3. *For X regular, $w(X) \le \lambda(X)^{c(X)}$.*

Proof: From the lemma, $d(X) \le \lambda(X)^{c(X)}$. Hence $\pi w(X) \le \lambda(X) d(X) \le \lambda(X)\lambda(X)^{c(X)} = \lambda(X)^{c(X)}$. Since for regular space $w(X) \le \operatorname{card} RO(X)$, hence, by Efimov's theorem, $w(X) \le \lambda(X)^{c(X)}$. □

(In [1] Efimov proved that if X is dyadic space, then $w(X) = \lambda(X)$.)

The reader of the Handbook of Set-Theoretic Topology can find the following

Theorem (R. Hodel [4]). *For X regular, $w(X) \le \pi_\chi(X)^{c(X)}$.*

Since $\lambda(X) \le w(X)$ and $\pi_\chi(X) \le w(X)$, hence

$$\lambda(X)^{c(X)} \le w(X)^{c(X)} \le \pi_\chi(X)^{c(X)^{c(X)}} = \pi_\chi(X)^{c(X)c(X)} = \pi_\chi(X)^{c(X)}$$

and

$$\pi_\chi(X)^{c(X)} \le w(X)^{c(X)} \le \lambda(X)^{c(X)^{c(X)}} = \lambda(X)^{c(X)c(X)} = \lambda(X)^{c(X)}.$$

From this it follows that $\pi_\chi(X)^{c(X)} = \lambda(X)^{c(X)}$. But the next two examples show that it can be $\lambda(X) < \pi_\chi(X)$ and also $\pi_\chi(X) < \lambda(X)$.

Example 2. Let $\omega_1 + 1$ denote the set of ordinal numbers no greater than ω_1. In the set $\omega_1 + 1$ let us consider the topology generated by order. This is a compact, Hausdorff, zero-dimensional space of weight ω_1. For each point $\alpha \in \omega_1$ we have $\chi(\alpha, \omega_1 + 1) = \omega$. Since the set ω_1 is dense in $\omega_1 + 1$, hence $\lambda(\omega_1 + 1) = \omega$. It is easy to see that $\pi_\chi(\omega_1, \omega_1 + 1) = \omega_1$. Hence $\pi_\chi(\omega_1 + 1) = \omega_1$. So $\lambda(\omega_1 + 1) < \pi_\chi(\omega_1 + 1)$.

Example 3. Let τ be an infinite cardinal number. Denote a Cantor cube by

$$D^\tau := \{p : p : \tau \to \{0, 1\}\}$$

and absolute of the Cantor cube by ρD^τ. B.A. Efimov [3] proved that

(a) for each cardinal number $\tau \geq \omega$ we have

$$\pi_\chi(x, \rho D^\tau) = \tau \text{ for each } x \in \rho D^\tau$$

and (b) if τ is a union of countably many fewer that τ cardinal numbers, then $\tau^+ \leq \chi(x, \rho D^\tau)$ for each point $x \in \rho D^\tau$.

From this theorem it follows that if we have a cardinal number as in the point (b), then $\pi_\chi(\rho D^\tau) < \lambda(\rho D^\tau)$.

References

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