A direct factor theorem for commutative group algebras

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Abstract. Suppose F is a field of characteristic $p \neq 0$ and H is a p-primary abelian A-group. It is shown that H is a direct factor of the group of units of the group algebra FH.

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1. Introduction.

Suppose F is a field of characteristic $p \neq 0$ and H is a multiplicative p-primary abelian group. If $\alpha = c_1 h_1 + \cdots + c_n h_n$ ($c_i \in F$, $h_i \in H$) is an element of the group algebra FH of H over F, recall that the *augmentation* of α is defined by $aug\,\alpha = \sum c_i$. Note that the group of *normalized units* $V(FH) = \{\alpha \in FH : aug\,\alpha = 1\}$ is a p-primary subgroup of the unit group $U(FH)$ of FH. In fact, as is easily seen, $U(FH) = F^* \times V(FH)$ is a direct product of the multiplicative group of F with the group of normalized units.

In this brief note we consider a condition on H which guarantees that H itself is a direct factor of $U(FH)$, but first we summarize some known results in Theorem 1 below. Note that H is a direct factor of $U(FH)$ if and only if H is a direct factor of $V(FH)$.

Theorem 1. *In each of the following cases, H is a direct factor of $U(FH)$ and, if F is perfect, the complementary factor of H in $V(FH)$ is totally projective.*

(a) ([M2]) *H is totally projective.*
(b) ([HU]) *H is a coproduct of groups with the cardinality of each factor not exceeding $\aleph_1$.*

We shall prove a version of Theorem 1 for the class of p-primary A-groups, a class which includes the totally projective groups and Warfield's S-groups [W] as subclasses. The relevant definitions are included below, but at this point we hasten to add that the importance of the class of A-groups is due to the fact that it is the largest class of p-primary abelian groups that have been satisfactorily classified by numerical invariant and a corresponding existence theorem. This was done by P. Hill [H2]. We now state our main result.

Theorem 2. *Suppose F is a field of characteristic $p \neq 0$ and H is a p-primary abelian group. If H is an A-group, then H is a direct factor of $U(FH)$.*

Our method in proving Theorem 2 will involve applications of W. May's result quoted above as Theorem 1 (a), P. Hill's classification of A-groups, and the computation of the Ulm-Kaplansky invariants of $V(FH)$ by T.Zh. Mollov [M]. Unfortunately, our methods will yield no information on the complement of H in $V(FH)$ in case F is perfect.

In the sequel, F always denotes a field of characteristic $p \neq 0$ and all groups are multiplicative abelian groups. Once the reader makes the appropriate translation to the multiplicative setting, all abelian group notation and terminology not explicitly defined herein are in agreement with Fuchs [F]. For example, $H[p] = \{h \in H : h^p = 1\}$.

2. A-groups and invariants.

If G is a p-group and σ is an ordinal we define G^σ inductively as follows. Set $G^0 = G$. If σ is isolated, define $G^\sigma = (G^{\sigma-1})^p$ and, if σ is a limit, $G^\sigma = \bigcap_{\alpha<\sigma} G^\alpha$. If μ is the smallest ordinal with $G^\mu = G^{\mu+1}$, recall that μ is the *length* of G. If G has length μ, then G^μ is the maximal divisible subgroup of G and we write $G^\mu = G^\infty$ with the convention that $\sigma < \infty$ for all ordinals σ. Observe that if σ is an ordinal and F is perfect, then $V(FG)^\sigma = V(FG^\sigma)$. Therefore, in this case, the lengths of $V(FG)$ and G are the same.

Suppose that G is a p-group of limit length μ. We call an isotype subgroup H of G *almost balanced* in G if $(G/H)^\sigma = G^\sigma H/H$ for all $\sigma < \mu$. A pair of p-groups (H, G) is a *μ-elementary pair* and H is a *μ-elementary A-group* if G is totally projective (of length μ), H is almost balanced in G, and G/H is totally projective. We do not require that totally projective groups be reduced. By [H2, Theorem 1], every μ-elementary A-group is totally projective if μ is cofinal with ω_0. Finally, an *A-group* is a coproduct of μ-elementary A-groups for various limit ordinals μ not cofinal with ω_0.

Suppose $H = \coprod_{i\in I} H_i$ is an A-group with (H_i, G_i) a $\mu(i)$-elementary pair for distinct limit ordinals $\mu(i)$ not cofinal with ω_0. We recall from [H2] that the A-invariants $f_H(\alpha, \beta)$ of H are defined as follows. For each $i \in I$, set $E_i = (G_i/H_i)^{\mu(i)}$. Then, if all dimensions are computed over the field with p elements,

$$f_H(\alpha,\beta) = \begin{cases} \dim(H^\alpha[p]/H^{\alpha+1}[p]), \text{ if } \alpha < \infty \text{ and } \beta = 0. \\ \dim(H^\infty[p]), \text{ if } \alpha = \infty \text{ and } \beta = 0. \\ \dim(E_i^\alpha[p]/E_i^{\alpha+1}[p]), \text{ if } \alpha < \infty \text{ and } \beta = \mu(i). \\ \dim(E_i^\infty[p]), \text{ if } \alpha = \infty \text{ and } \beta = \mu(i). \end{cases}$$

In all other cases, $f_H(\alpha, \beta) = 0$.

By [H2, Lemma B], $f_H(\alpha, \beta)$ is an isomorphism invariant of H, independent of the choices of the G_i's, and two A-groups are isomorphic if and only if they have the same A-invariants [H2, Theorem 3].

In order to apply the theory of A-groups to our direct factor problem, we need some additional notation. If H is a subgroup of the p-group G, the natural map $G \to G/H$ induces a group-epimorphism $V(FG) \to V(F(G/H))$ with kernel we denote by $K(FH)$. Our first result follows from Lemma 3 in the author's paper [U] and a result of [M2] which states that $V(FG)$ is totally projective if and only if G itself is totally projective.

Proposition 1. *Suppose F is perfect and μ is a limit ordinal. Then (H, G) is a μ-elementary pair if and only if $(K(FH), V(FG))$ is a μ-elementary pair.*

Proof: By [U, Lemma 3], H is almost balanced in G if and only if $K(FH)$ is almost balanced in $V(FG)$. That G (respectively, G/H) is totally projective if and only if $V(FG)$ (respectively, $V(FG)/K(FH) \cong V(F(G/H))$) is totally projective follows from Theorem 1 (a) and [M2, Proposition 9]. □

3. The direct factor theorem.

In [M1, Lemma 2], W. May proved that H is a direct factor $V(FH)$ if H is isomorphic to a direct factor of $V(FH')$ for some group H'. We shall find the following modification of this result to be useful.

Proposition 2. *Suppose A and H are subgroups of a p-group G and A is isomorphic to a direct factor of $K(FH)$. Then, A is a direct factor of $V(FA)$. In particular, if H is isomorphic to a direct factor of $K(FH)$, then H is a direct factor of $V(FH)$.*

Proof: Suppose $K(FH) = B \times C$ for some subgroups B and C. If $\varphi : A \to B$ is an isomorphism, Then $B \subseteq U(FG)$ implies that φ induces an F-algebra homomorphism $f : FA \to FG$ with $f \mid A = \varphi$. Note that $f(V(FA)) \subseteq K(FH)$. Indeed, suppose $\alpha = c_1a_1 + \cdots + c_na_n$ where each $a_i \in A$, $c_i \in F$ and $\sum c_i = 1$. Then, $f(\alpha) = \sum c_i\varphi(a_i)$ with each $\varphi(a_i) \in B \subseteq K(FH)$. From this it follows easily that $f(\alpha) \in K(FH)$ as desired. Finally, if $\pi : K(FH) \to B$ is the projection along C, the composition $\varphi^{-1}\pi f : V(FA) \to A$ restricts to the identity map on A. Therefore, A is a direct factor of $V(FA)$. □

If G is a p-group and σ is an ordinal, let $f_\sigma(G)$ denote the σ-th Ulm-Kaplansky invariant of G. If F is both infinite and perfect and if G has length μ, it was shown by Mollov [M] that

$$f_\sigma(V(FG)) = \begin{cases} |F||G^\sigma|, & \text{if } \ \sigma < \mu. \\ |F||G^\infty|, & \text{if } \ \sigma = \infty \ \text{ and } G^\infty \neq 1. \\ 0, & \text{if } \ \mu \leq \sigma < \infty. \\ 0, & \text{if } \ \sigma = \infty \ \text{ and } G^\infty = 1. \end{cases}$$

Proposition 3. *Suppose μ is a limit ordinal not cofinal with ω_0. If H is a μ-elementary A-group, then H is a direct factor of $V(FH)$.*

Proof: Select a totally projective group G of length μ such that (H, G) is a μ-elementary pair and let κ be an infinite cardinal with $\kappa > |G|$. Let $\overline{F}$ be a perfect extension field of F with $|\overline{F}| \geq \kappa$. One way to construct such an $\overline{F}$ is to take a set of commuting indeterminates $X = \{X_\alpha\}_{\alpha \in I}$ over F with $|I| \geq \kappa$ and let $\overline{F}$ be an algebraic closure of the function field $F(X)$.

We now use the result of [M] mentioned above to compute the A-invariants of $K(\overline{F}H)$ and $K(\overline{F}H)\times H$. Let λ be the length of H and observe that $f_\sigma(H)\le |H|<\kappa$ if $\sigma<\lambda$ and $f_\sigma(H)=0$ if $\lambda\le\sigma<\infty$. Since $V(\overline{F}H)$ is isotype in $K(\overline{F}H)$, $\kappa\le|\overline{F}|=f_\sigma(V(\overline{F}H))\le f_\sigma(K(\overline{F}H))$ for all $\sigma<\lambda$. Moreover, $f_\infty(K(\overline{F}H))=f_\infty(K(\overline{F}H))+f_\infty(H)$ since either $f_\infty(H)=0$ or else $f_\infty(H)<\kappa\le f_\infty(V(\overline{F}H))\le f_\infty(K(\overline{F}H))$. We conclude that the Ulm-Kaplansky invariants of $K(\overline{F}H)$ and $K(\overline{F}H)\times H$ are equal.

If $\overline{\lambda}$ is the length of $(G/H)^\mu$, observe that $f_\sigma((G/H)^\mu)\le|G|<\kappa$ if $\sigma<\overline{\lambda}$ and $f_\sigma((G/H)^\mu)=0$ if $\overline{\lambda}\le\sigma<\infty$. Moreover, for every $\sigma<\overline{\lambda}$, $\kappa\le|\overline{F}|=f_\sigma(V(\overline{F}(G/H)^\mu))=f_\sigma((V(\overline{F}G)/K(\overline{F}H))^\mu)$. Note also that the Ulm-Kaplansky invariants at ∞ of $(G/H)^\mu$ and $V(\overline{F}(G/H)^\mu)$ have sum equal to $f_\infty(V(\overline{F}(G/H)^\mu))$. It follows that the Ulm-Kaplansky invariants of $(V(\overline{F}G)/K(\overline{F}H))^\mu$ and $(V(\overline{F}G)/K(\overline{F}H))^\mu\times(G/H)^\mu$ are the same for all σ. We conclude that the A-invariants of $K(\overline{F}H)$ and $K(\overline{F}H)\times H$ are equal.

Since both $K(\overline{F}H)$ and $K(\overline{F}H)\times H$ are μ-elementary A-groups by Proposition 1, $K(\overline{F}H)\cong K(\overline{F}H)\times H$. It now follows from Proposition 2 that H is a direct factor of $V(\overline{F}H)$. Since $V(FH)$ is a subgroup of $V(\overline{F}H)$ which contains H, we have that H is also a direct factor of $V(FH)$. □

We are now in position to prove our main result.

Proof of Theorem 2: As observed in the introduction, it suffices to show that H is a direct factor of $V(FH)$. Since H is an A-group, $H=\coprod_{i\in I}H_i$ where each H_i is a $\mu(i)$-elementary A-group for distinct limit ordinals $\mu(i)$ not cofinal with ω_0. Define a homomorphism $f: V(F(\coprod H_i))\to\coprod V(FH_i)$ as follows. Given $\alpha=c_1h_1+\cdots+c_nh_n$ with $c_1,\dots,c_n\in F$, $\sum c_i=1$, and $h_1,\dots,h_n\in\coprod H_i$, let h_{ji} denote the component of h_j in H_i, $1\le j\le n$. Define $f(\alpha)$ to be the element in the coproduct $\coprod V(FH_i)$ whose component in $V(GH_i)$ is $c_1h_{1i}+\cdots+c_nh_{ni}$. By Proposition 3, for each i there exists a projection $\pi_i: V(FH_i)\to H_i$. These projections induce a map $\pi:\coprod V(FH_i)\to\coprod H_i=H$. Clearly $\pi f: V(FH)\to H$ restricts to the identity map on H. Therefore, H is a direct factor of $V(FH)$ and the proof is complete. □

We recall from [H1] that if (H,G) is an ω_1-elementary pair with $(G/H)^{\omega_1+1}=1$, then H is a coproduct of ω_1-elementary A-groups each of cardinality not exceeding $\aleph_1$. Therefore, in this case, Theorem 1 (b) applies and we conclude that if F is perfect, the complementary factor of H in $V(FH)$ is totally projective. We conjecture that this holds for arbitrary μ-elementary A-groups (and hence for A-groups generally).

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