On uniformly nonsquare points and nonsquare points of Orlicz spaces*

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Abstract. For Orlicz spaces endowed with the Orlicz norm and the Luxemburg norm, the criteria for uniformly nonsquare points and nonsquare points are given.

Keywords: Orlicz space, uniformly nonsquare point, nonsquare point

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R. James in [1] and J. Schäffer in [2] introduced conceptions of uniformly nonsquare, locally uniformly nonsquare and nonsquare Banach spaces, respectively. In this paper, we introduce the notions of uniformly nonsquare point and nonsquare point, and give criteria for them in Orlicz spaces.

Let $S(X)$ be the unit sphere of Banach space X. $x \in S(X)$ is called a uniformly nonsquare point in the sense of Schäffer (we write S-UNSP, for simplicity) provided that there is $\delta_x > 0$ such that for every $y \in S(X)$,

$$\mathrm{Max}\{\|x+y\|, \|x-y\|\} \geq 1 + \delta_x;$$

$x \in S(X)$ is called a (S)-nonsquare point (S-NSP) if for every $y \in S(X)$

$$\mathrm{Max}\{\|x+y\|, \|x-y\|\} > 1;$$

$x \in S(X)$ is called a uniformly nonsquare point in the sense of James (J-UNSP) provided that there is $\delta_x > 0$ such that for every $y \in S(X)$,

$$\mathrm{Min}\{\|x+y\|, \|x-y\|\} \leq 2 - \delta_x;$$

$x \in S(X)$ is called a (J)-nonsquare point (J-NSP) if for every $y \in S(X)$,

$$\mathrm{Min}\{\|x+y\|, \|x-y\|\} < 2.$$

Let $M(u)$ and $N(v)$ be a pair of complemented N-functions, we use L_M to express the Orlicz space generated by $M(u)$,

$$L_M = \{x(t) : \exists\, \lambda > 0,\ R_M(\lambda x) < \infty\},$$

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and its subspace E_M,

$$E_M = \{x(t) : \forall\, \lambda > 0,\ R_M(\lambda x) < \infty\},$$

where $R_M(x) = \int_G M(x(t))\, d\mu$ is called the modulo of x over a finite nonatomic measure space (G, Σ, μ).

We denote by $L_M = [L_M(G), \|.\|_M]$ and $L_M = [L_M(G), \|.\|_{(M)}]$ (see [3], [6]) the Orlicz spaces endowed with the Orlicz norm and the Luxemburg norm, respectively. $M \in \Delta_2$ means that $M(u)$ satisfies the Δ_2-condition for large u, and $M \in \nabla_2$ means that $N \in \Delta_2$.

S. Chen and Y. Wang testified in [4] that L_M always is (S)-locally uniformly nonsquare, so every point on $S(L_M)$ is an S-UNSP, and so S-NSP. S. Chen verified in [5] that a point on $S(L_{(M)})$ is an S-UNSP iff $M \in \Delta_2$. We give the criteria for the five other cases and list them as follows:

$\|x\| = 1$	S-UNSP	S-NSP	J-UNSP	J-NSP
L_M	always [4]	always [4]	$M \in \nabla_2$	always
$L_{(M)}$	$M \in \Delta_2$ [5]	$R_M(x) = 1$	$\exists\, \lambda > 1,\ R_M(\lambda x) < \infty$	$\exists\, \lambda > 1,\ R_M(\lambda x) < \infty$

Replacing L_M and $L_{(M)}$ by l_M and $l_{(M)}$ in the table, we have the same results in Orlicz sequence spaces as in Orlicz function spaces, and so we omit them here.

Theorem 1. *For $x \in S(L_{(M)})$, TFAE:*

(1) *x is a (S)-nonsquare point,*
(2) $R_M(x) = 1$.

Proof: (1) $\Rightarrow$ (2). Suppose $R_M(x) < 1$. Then we know that $M \notin \Delta_2$, i.e., there exist $u_n \nearrow +\infty$, such that $M((1 + \frac{1}{n})u_n) > 2^n M((1 + \frac{1}{2n})u_n)$.

Take $c > 0$ such that $\mu G_c > 0$, where $G_c = \{t \in G : |x(t)| \le c\}$. Passing to a subsequence, if necessary, we can assume that $c \le u_n/2n$ for every n. Take disjoint subsets $\{G_n\}_n \subset G_c$ such that

$$M((1 + \frac{1}{2n})u_n)\mu G_n = \frac{1}{2^n}, \qquad (n = 1, 2, \dots).$$

Take an integer n' such that $\sum_{n=n'}^{\infty} \frac{1}{2^n} < 1 - R_M(x)$. Set

$$y(t) = \begin{cases} u_n, t \in G_n,\ n = n', n'+1, n'+2, \dots \\ 0,\ \text{otherwise.} \end{cases}$$

Then $R_M(y) = \sum_{n=n'}^{\infty} M(u_n)\mu G_n \le \sum_{n=n'}^{\infty} \frac{1}{2^n} \le 1$.

For an arbitrary $\lambda > 1$, denote $m = \left[\frac{1}{\lambda - 1}\right] + n'$. Then we have

$$R_M(\lambda y) = \sum_{n=n'}^{\infty} M(\lambda u_n)\mu G_n \ge \sum_{n=m}^{\infty} M((1 + \frac{1}{n})u_n)\mu G_n = \infty,$$

i.e., $\|y\|_{(M)} = 1$.

Notice that for $\varepsilon = 1$ or $\varepsilon = -1$, we have

$$\begin{aligned} R_M(x+\varepsilon y) &= R_M(x\chi_{G\setminus\bigcup_{n=n'}^{\infty} G_n}) + R_M((x+\varepsilon y)\chi_{\bigcup_{n=n'}^{\infty} G_n}) \\ &\le R_M(x) + R_M((|x|+|y|)\chi_{\bigcup_{n=n'}^{\infty} G_n}) \\ &\le R_M(x) + \sum_{n=n'}^{\infty} M((1+\frac{1}{2n})u_n)\mu G_n \le 1. \end{aligned}$$

On the other hand, for an arbitrary $\lambda > 1$, denoting $m = \left[n' + \frac{3\lambda}{2(\lambda-1)}\right]$, we have

$$\begin{aligned} R_M(\lambda(x+\varepsilon y)) &\ge R_M(\lambda(x+\varepsilon y)_{\bigcup_{n=n'}^{\infty} G_n}) \\ &\ge R_M(\lambda(|y|-|x|)\chi_{\bigcup_{n=n'}^{\infty} G_n}) \\ &\ge \sum_{n=n'}^{\infty} M(\lambda(1-\frac{1}{2n})u_n)\mu G_n \\ &\ge \sum_{n=m'}^{\infty} M((1+\frac{1}{n})u_n)\mu G_n = \infty, \end{aligned}$$

whence $\|x+y\|_{(M)} = 1$, $\|x-y\|_{(M)} = 1$, which contradicts the fact that x is an (S)-nonsquare point.

(2) $\Rightarrow$ (1). Suppose that x is not an (S)-nonsquare point, i.e., there is $y \in S(L_{(M)})$ such that $\|x+y\|_{(M)} = 1$ and $\|x-y\|_{(M)} = 1$. Then

$$R_M(x+y) + R_M(x-y) \le 2 = 2R_M(x),$$

i.e.,

$$R_M(x) - \frac{1}{2}(R_M(x+y) + R_M(x-y)) \ge 0.$$

Since $x = \frac{x+y+x-y}{2}$, from the convexity of $M(u)$, we have

$$R_M(x) - \frac{1}{2}(R_M(x+y) + R_M(x-y)) \le 0.$$

Thus

$$R_M(\frac{x+y+x-y}{2}) = \frac{1}{2}(R_M(x+y) + R_M(x-y)),$$

so $M(u)$ is affine on the segments $\langle x(t)+y(t),\ x(t)-y(t)\rangle$ ($t \in G$, μ-a.e.). Since $M(u)$ is an N-function, we deduce that $|x(t)| \ge |y(t)|$ ($t \in G, \mu$-a.e.). So $2|y(t)| \le |x(t) + y(t)|$, or $2|y(t)| \le |x(t) - y(t)|$. Therefore, $R_M(2y) \le R_M(x+y) + R_M(x-y) \le 2$, and from $\|y\|_{(M)} = 1$ we get $R_M(y) = 1$.

Replace x by y in the preceding, we get that $M(u)$ is affine on the segments $\langle y(t)+x(t),\ y(t)-x(t)\rangle$ ($t \in G, \mu$-a.e.). Hence, for μ-a.e. $t \in G$, $M(u)$ is affine on $\langle x(t)-y(t),\ x(t)+y(t)\rangle$ and $\langle x(t)+y(t),\ y(t)-x(t)\rangle$, which contradicts $\|x-y\|_{(M)} = 1$. □

Corollary 1. *Any point $x \in S(E_{(M)})$ is an (S)-nonsquare one.*

Corollary 2. *$L_{(M)}$ is (S)-nonsquare iff $M \in \Delta_2$.*

Theorem 2. *For $x \in S(L_{(M)})$,* TFAE:

(1) *x is a (J)-uniformly nonsquare point,*
(2) *x is a (J)-nonsquare point,*
(3) *$R_M(\lambda x) < \infty$ for some $\lambda > 1$.*

Proof: (3) $\Rightarrow$ (1). Take $c > 1$ large enough such that $R_M(x\chi_{G_1}) \geq \frac{7}{8} R_M(x)$, where $G_1 = \{t \in G : \frac{1}{c} \leq |x(t)| \leq c\}$. Choose d, $d > 2c$, in such way that $\frac{M(c)}{M(d)} \leq \frac{1}{8} R_M(x)$. Set $\sigma = \mathrm{Sup}_{1/c \leq u \leq d}(2M(\frac{u}{2})/M(u))$, $0 < \sigma < 1$. Denoting $\delta = \frac{3}{8}(1-\sigma)R_M(x)$ and taking $\varepsilon > 0$ small enough, we get

$$R_M((1+\varepsilon)x) \leq R_M(x) + \frac{3}{8}(1-\sigma)R_M(x) = R_M(x) + \delta.$$

In the following, we shall show that for any $y \in S(L_{(M)})$, it holds

$$\mathrm{Min}\{\|\frac{x+y}{2}\|_{(M)},\ \|\frac{x-y}{2}\|_{(M)}\} \leq 1 - \frac{\varepsilon}{2(1+\varepsilon)}\,. \tag{$*$}$$

Denote $G_2 = \{t \in G : |y(t)| \leq d\}$. Then

$$M(d)\mu(G \setminus G_2) \leq R_M(y\chi_{G\setminus G_2}) \leq R_M(y) \leq 1, \text{ i.e., } \mu(G\setminus G_2) \leq \frac{1}{M(d)}\,.$$

Thus

$$R_M(x\chi_{G_1\setminus G_2}) \leq M(c)\mu(G_1 \setminus G_2) \leq M(c)\mu(G\setminus G_2) \leq \frac{M(c)}{M(d)} \leq \frac{1}{8}R_M(x).$$

Defining $D = G_1 \cap G_2$, we get

$$\frac{7}{8}R_M(x) \leq R_M(x\chi_{G_1}) = R_M(x\chi_{G_1\setminus G_2}) + R_M(x\chi_D) \leq \frac{1}{8}R_M(x) + R_M(x\chi_D),$$

i.e.,

$$R_M(x\chi_D) \geq \frac{3}{4}R_M(x). \tag{1}$$

Hence

$$\begin{aligned}
&2 + \delta - R_M\Big(\frac{(1+\varepsilon)x+y}{2}\Big) - R_M\Big(\frac{(1+\varepsilon)x-y}{2}\Big)\\
&\geq R_M(x) + \delta + R_M(y) - R_M\Big(\frac{(1+\varepsilon)x+y}{2}\Big) - R_M\Big(\frac{(1+\varepsilon)x-y}{2}\Big)\\
&\geq R_M((1+\varepsilon)x) + R_M(y) - \Big[R_M\Big(\frac{(1+\varepsilon)x+y}{2}\Big) + R_M\Big(\frac{(1+\varepsilon)x-y}{2}\Big)\Big]\\
&\geq R_M((1+\varepsilon)x\chi_D) + R_M(y\chi_D)\\
&\quad - \Big[R_M\Big(\frac{(1+\varepsilon)x+y}{2}\chi_D\Big) + R_M\Big(\frac{(1+\varepsilon)x-y}{2}\chi_D\Big)\Big]\,.
\end{aligned} \tag{2}$$

Denote $D_1 = \{t \in D : x(t)\ y(t) \geq 0\}$ and $D_2 = D \setminus D_1$. Then

$$\begin{aligned}
&R_M\big(\frac{(1+\varepsilon)x+y}{2}\chi_D\big) + R_M\big(\frac{(1+\varepsilon)x-y}{2}\chi_D\big) \\
&= R_M\big(\frac{(1+\varepsilon)x+y}{2}\chi_{D_1}\big) + R_M\big(\frac{(1+\varepsilon)x+y}{2}\chi_{D_2}\big) \\
&\quad + R_M\big(\frac{(1+\varepsilon)x-y}{2}\chi_{D_1}\big) + R_M\big(\frac{(1+\varepsilon)x-y}{2}\chi_{D_2}\big) \\
&\leq \frac{R_M((1+\varepsilon)x\chi_{D_1}) + R_M(y\chi_{D_1})}{2} + R_M\big(\frac{\max(|(1+\varepsilon)x|,|y|)}{2}\chi_{D_2}\big) \\
&\quad + R_M\big(\frac{\max(|(1+\varepsilon)x|,|y|)}{2}\chi_{D_1}\big) + \frac{R_M((1+\varepsilon)x\chi_{D_2}) + R_M(y\chi_{D_2})}{2} \\
&= \frac{R_M((1+\varepsilon)x\chi_D) + R_M(y\chi_D)}{2} + R_M\big(\frac{\max(|(1+\varepsilon)x|,|y|)}{2}\chi_D\big).
\end{aligned}$$

While $t \in D$, $\frac{1}{c} \leq \frac{1+\varepsilon}{c} \leq \max(|(1+\varepsilon)x|,|y|) \leq d$, we have

$$\begin{aligned}
&R_M\big(\frac{(1+\varepsilon)x+y}{2}\chi_D\big) + R_M\big(\frac{(1+\varepsilon)x-y}{2}\chi_D\big) \\
&\leq \frac{1}{2}(R_M((1+\varepsilon)x\chi_D) + R_M(y\chi_D)) + \frac{\sigma}{2}R_M(\max(|(1+\varepsilon)x|,|y|)\chi_D) \\
&\leq \frac{(1+\sigma)}{2}(R_M((1+\varepsilon)x\chi_D) + R_M(y\chi_D)).
\end{aligned}$$

Combining (1) and (2), we get

$$\begin{aligned}
&2 + \delta - R_M\big(\frac{(1+\varepsilon)x+y}{2}\big) - R_M\big(\frac{(1+\varepsilon)x-y}{2}\big) \\
&\geq \frac{1-\sigma}{2}(R_M((1+\varepsilon)x\chi_D) + R_M(y\chi_D)) \\
&\geq \frac{1-\sigma}{2}R_M((1+\varepsilon)x\chi_D) \geq \frac{3}{8}(1-\sigma)R_M(x) = \delta,
\end{aligned}$$

i.e.,

$$2 - R_M\big(\frac{(1+\varepsilon)x+y}{2}\big) - R_M\big(\frac{(1+\varepsilon)x-y}{2}\big) \geq 0.$$

Thus

$$\mathrm{Min}\{R_M\big(\frac{(1+\varepsilon)x+y}{2}\big),\ R_M\big(\frac{(1+\varepsilon)x-y}{2}\big)\} \leq 1.$$

If $R_M\big(\frac{(1+\varepsilon)x+y}{2}\big) \leq 1$, we have $\|\frac{(1+\varepsilon)x+y}{2}\|_{(M)} \leq 1$, i.e., $\|\frac{x+\frac{y}{1+\varepsilon}}{2}\|_{(M)} \leq \frac{1}{1+\varepsilon}$. Notice that

$$\Big|\|\frac{x+y}{2}\|_{(M)} - \|\frac{x+\frac{y}{1+\varepsilon}}{2}\|_{(M)}\Big| \leq \|\frac{x+y}{2} - \frac{x+\frac{y}{1+\varepsilon}}{2}\|_{(M)} = \frac{1}{2}(1-\frac{1}{1+\varepsilon}) = \frac{\varepsilon}{2(1+\varepsilon)}.$$

Therefore we get

$$\Big\|\frac{x+y}{2}\Big\|_{(M)} \le \frac{1}{1+\varepsilon} + \frac{\varepsilon}{2(1+\varepsilon)} = \frac{2+\varepsilon}{2(1+\varepsilon)} = 1 - \frac{\varepsilon}{2(1+\varepsilon)}\,.$$

If $R_M\big(\frac{(1+\varepsilon)x-y}{2}\big) \le 1$, we have similarly

$$\Big\|\frac{x-y}{2}\Big\|_{(M)} \le 1 - \frac{\varepsilon}{2(1+\varepsilon)}\,.$$

(1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (3). Suppose that $R_M(\lambda x) = \infty$ for any $\lambda > 1$. Take $\xi_1 > \xi_2 > \dots$ with $\xi_n \to 1$.
Since $R_M(\xi_1 x) = \infty$, $\exists\, c_1 > 0$, $R_M(\xi_1 x \chi_{G_1}) \ge 1$ where $G_1 = \{t \in G : |x(t)| \le c_1\}$, since $R_M(\xi_1 x \chi_{G \setminus G_1}) = \infty$, $\exists\, c_1' > 0$, $R_M(\xi_1 x \chi_{G_1'}) \ge 1$ where $G_1' = \{t \in G \setminus G_1 : |x(t)| \le c_1'\}$, since $R_M(\xi_2 x \chi_{G \setminus G_1 \setminus G_1'}) = \infty$, $\exists\, c_2 > 0$, $R_M(\xi_2 x \chi_{G_2}) \ge 1$ where $G_2 = \{t \in G \setminus G_1 \setminus G_1' : |x(t)| \le c_2\}$, since $R_M(\xi_2 x \chi_{G \setminus G_1 \setminus G_1' \setminus G_2}) = \infty$, $\exists\, c_2' > 0$, $R_M(\xi_2 x \chi_{G_2'}) \ge 1$ where $G_2' = \{t \in G \setminus G_1 \setminus G_1' \setminus G_2 : |x(t)| \le c_2'\} \dots$
Continuing this process in such a way, we get the disjoint subsets $G_1, G_1', G_2, G_2', \dots$ satisfying

$$R_M(\xi_n x \chi_{G_n}) \ge 1, \quad R_M(\xi_n x \chi_{G_n'}) \ge 1 \quad (n = 1, 2, \dots).$$

Set

$$y = x\chi_{G_1 \cup G_2 \cup \dots}\,, \quad z = x\chi_{G_1' \cup G_2' \cup \dots}\,.$$

Then $x = y + z$, $yz = 0$, $R_M(y) \le R_M(x) \le 1$, $R_M(z) \le R_M(x) \le 1$. But for any integer m,

$$R_M(\xi_m y) = \sum_{n=1}^{\infty} R_M(\xi_m x \chi_{G_n}) \ge \sum_{n=m}^{\infty} R_M(\xi_n x \chi_{G_n}) = \infty,$$

so $\|y\|_{(M)} = 1$. Similarly, $\|z\|_{(M)} = 1$. Set $x' = y - z$. From $|x(t)| = |x'(t)|$, we get $\|x'\|_{(M)} = \|x\|_{(M)} = 1$. On the other hand

$$\Big\|\frac{x+x'}{2}\Big\|_{(M)} = \|y\|_{(M)} = \Big\|\frac{x-x'}{2}\Big\|_{(M)} = \|z\|_{(M)} = 1,$$

which contradicts the fact that x is a (J)-nonsquare point. $\square$

Corollary 1. *Every point $x \in S(E_{(M)})$ is a (J)-uniformly nonsquare one, and so also a (J)-nonsquare.*

Corollary 2. *$L_{(M)}$ is (J)-locally uniformly nonsquare ((J)-nonsquare) iff $M \in \Delta_2$.*

Proof: When $M \in \Delta_2$, $L_{(M)} = E_{(M)}$, it is Corollary 1. When $M \notin \Delta_2$, take y as in the proof of Theorem 1, (1) $\Rightarrow$ (2), which is also not a (J)-uniformly nonsquare point. From $\|y\|_{(M)} = 1$, we get that $L_{(M)}$ is not (J)-locally uniformly nonsquare. □

Theorem 3. *For $x \in S(L_M)$, TFAE:*

(1) *x is a (J)-uniformly nonsquare point,*

(2) *$M \in \nabla_2$.*

Proof: (2) $\Rightarrow$ (1). See [4].

(1) $\Rightarrow$ (2). Take $d > 0$, $\mu G_d > 0$, where $G_d = \{t \in G : |x(t)| \le d\}$. For any integer n, choose $y_n \in E_M$, $\|y_n\|_{(M)} = 1$ and $\int_G x(t)y_n(t)\,d\mu > 1 - \frac{1}{n}$. If supposing $M \notin \nabla_2$ (equivalently $N \notin \Delta_2$), there exists $v_n > 0$ large enough such that

(i) $N(v_n)\mu G_d > \frac{1}{n}$,

(ii) when $e \subset G$, $\mu e \le \frac{1}{nN(v_n)}$, then $\int_{G\setminus e} x(t)y_n(t)\,d\mu > 1 - \frac{1}{n}$,

(iii) $N((1+\frac{1}{n})v_n) > nN(v_n)$.

By (i), there is $G_n \subset G_d$ such that $N(v_n)\mu G_n = \frac{1}{n}$. By (ii), we get $\int_{G\setminus G_n} xy_n\,d\mu > 1 - \frac{1}{n}$. Notice that $R_N(v_n\chi_{G_n}) = N(v_n)\mu G_n = \frac{1}{n}$,

$$R_N\big((1+\frac{1}{n})v_n\chi_{G_n}\big) = N\big((1+\frac{1}{n})v_n\big)\mu G_n > 1,$$

whence we have $1 \ge \|v_n\chi_{G_n}\|_{(N)} \ge \frac{1}{1+\frac{1}{n}}$.

Since $v_n\chi_{G_n}$ is a simple function of $L_{(N)}$, there exists $u_n\chi_{G_n} \in L_M$, satisfying $\|u_n\chi_{G_n}\|_M = 1$ and such that

$$\int_G u_n\chi_{G_n} \cdot v_n\chi_{G_n}\,d\mu = u_nv_n\mu G_n = \|v_n\chi_{G_n}\|_{(N)} \ge \frac{1}{1+\frac{1}{n}}.$$

Set $y'_N(t) = \frac{1}{1+\frac{1}{n}}(v_n\chi_{G_n}(t) + y_n(t)\chi_{G\setminus G_n}(t))$. Then

$$R_N(y'_n) \le \frac{1}{1+\frac{1}{n}}(N(v_n)\mu G_n + R_m(y_n)) = 1.$$

So, we have

$$\begin{aligned}
\|u_n\chi_{G_n} + x\|_M &\ge \int_G (u_n\chi_{G_n}(t) + x(t))y'_n(t)\,d\mu \\
&\ge \frac{1}{1+\frac{1}{n}}\,\Big(\int_{G_n}(u_n + x(t))v_n\,d\mu + \int_{G\setminus G_n} x(t)y_n(t)\,d\mu\Big) \\
&\ge \frac{1}{1+\frac{1}{n}}\,\Big(u_nv_n\mu G_n - dv_n\mu G_n + \int_{G\setminus G_n} x(t)y_n(t)\,d\mu\Big) \\
&\ge \frac{1}{1+\frac{1}{n}}\,\Big(\frac{1}{1+\frac{1}{n}} - \frac{d}{n} + 1 - \frac{1}{n}\Big),
\end{aligned}$$

whence $\lim_{n\to\infty} \|u_n\chi_{G_n} + x\|_M = 2$.

Replace $y'_n(t)$ by $y''_n(t) = \frac{1}{1+\frac{1}{n}}(v_n\chi_{G_n}(t) - y_n(t)\chi_{G\setminus G_n}(t))$. We get $\lim_{n\to\infty} \|u_n\chi_{G_n} - x\|_M = 2$, which is a contradiction with the fact that x is a (J)-uniformly nonsquare point.

□

Corollary 1. *L_M is (J)-locally uniformly nonsquare iff $M \in \nabla_2$.*

Theorem 4. *Every point $x \in S(L_M)$ is a (J)-nonsquare point.*

PROOF: For $x, y, \in S(L_M)$. There are $k, h > 0$ such that

$$\|x\|_M = \frac{1}{k}(1 + R_M(kx)), \quad \|y\|_M = \frac{1}{h}(1 + R_M(hy)).$$

Assume that $\|x \pm y\|_M = 2$. Then

$$\begin{aligned} 2 = \frac{1}{k}(1 + R_M(kx)) + \frac{1}{h}(1 + R_M(hy)) \geq \\ \geq \frac{k+h}{k \cdot h} \left(1 + R_M\left(\frac{h}{k+h}kx \pm \frac{k}{k+h}\right)\right) \geq \|x \pm y\|_M = 2, \end{aligned}$$

i.e.,

$$M\left(\frac{h}{k+h}kx(t) \pm \frac{k}{k+h}hy(t)\right) = \frac{h}{k+h}M(kx(t)) + \frac{k}{k+h}M(hy(t)) \quad (t \in G,\ \mu\text{-a.e.}),$$

so $M(u)$ is affine on $\langle hy(t), kx(t)\rangle$ and $\langle kx(t), -hy(t)\rangle$ $(t \in G, \mu\text{-a.e.})$, which contradicts the fact that $M(u)$ is an N-function.

□

Corollary. *L_M is always (J)-nonsquare.*

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