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The category of uniform spaces as a completion of the category of metric spaces

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Abstract. A criterion for the existence of an initial completion of a concrete category **K** universal w.r.t. finite products and subobjects is presented. For **K** = metric spaces and uniformly continuous maps this completion is the category of uniform spaces.

Keywords: universal completion, metric space, uniform space

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Introduction.

We investigate initial completions of concrete categories, i.e. initially complete categories $\mathbf{K}^*$ such that $\mathbf{K}$ is a full, concrete subcategory of $\mathbf{K}^*$. Recall that $\mathbf{K}^*$ is said to be a universal initial completion of $\mathbf{K}$, see [H], provided that (a) $\mathbf{K}$ is closed under initial sources in $\mathbf{K}^*$ and (b) for each initially complete $\mathbf{L}$ and each concrete functor $F : \mathbf{K} \to \mathbf{L}$ preserving initial sources there exists an extension to an initial-sources preserving functor $F^* : \mathbf{K}^* \to \mathbf{L}$, unique up-to natural isomorphism. More in general, let Δ be a collection of sources in the base-category, then a <u>Δ-universal initial completion</u> of $\mathbf{K}$, see [E], is an initial completion $\mathbf{K}^*$ such that

(a) $\mathbf{K}$ is closed in $\mathbf{K}^*$ under initial sources carried by Δ-sources (shortly: initial Δ-sources)

and

(b) for each initially complete category $\mathbf{L}$ and each concrete functor $F : \mathbf{K} \to \mathbf{L}$ preserving initial Δ-sources there exists an extension to an initial-sources preserving functor $F^* : \mathbf{K}^* \to \mathbf{L}$, unique up-to a natural isomorphism.

A description of the Δ-universal initial completion (as a category of "Δ-complete sources") has been presented in [E]. Unfortunately, the conclusion made in that paper that $\mathbf{K}^*$ is always legitimate (i.e. lives in the universe U of classes), formulated in Theorem 4, is false: For example, if $\mathbf{K}$ is a discrete, large category, then the Δ-universal completion is illegitimate, being codable by the collection of all subclasses of $\mathbf{K}$. In the case Δ = all sources or $\Delta = \emptyset$, the legitimacy of the Δ-universal completion is characterized in [AHS].

In the present paper we concentrate on the case

$$\Delta_{fm} = \text{all finite products and monomorphisms.}$$

(Or, equivalently, all finite monosources.) We prove that each fibre-small, hereditary concrete category has a fibre-small (thus, legitimate) Δ_{fm}-universal initial completion. For

$\mathbf{K} = \mathbf{Met}$, metric spaces and uniformly continuous maps

this completion is

$\mathbf{K}^* = \mathbf{Unif}$, uniform spaces and uniformly continuous maps.

In this special case the same result also holds for Δ = all finite sources, or Δ = all countable sources, but it does not hold for Δ = all sources. In fact, a concrete description of the universal initial completion of **Met** is not known.

The paper has been inspired by Zdeněk Frolík who asked us about a categorical motivation of uniform spaces from the point of view of metric spaces. This paper is, most unfortunately, the end of a long, fruitful, and happy collaboration of the authors: Jan Reiterman died precisely when it was completed.

I. Δ-universal completion.

Recall from [A] that a construct (i.e. a concrete category over **Set**) is called hereditary provided that every subset of the underlying set of any object K gives rise to an initial subobject of K. This can be generalized to concrete categories over a base category $\mathbf{X}$ (i.e. pairs consisting of a category $\mathbf{K}$ and a faithful, amnestic functor $|\ | : \mathbf{K} \to \mathbf{X}$) provided that a fixed factorization structure $(\mathbf{E}, \mathbf{M})$ for $\mathbf{X}$-morphisms is given:

Definition. A concrete category $\mathbf{K}$ over an $(\mathbf{E}, \mathbf{M})$-base category is called hereditary provided that given an object K of $\mathbf{K}$, every $\mathbf{M}$-morphism $m : X \to |K|$ has an initial lift.

Recall that a concrete category is fibre-small provided that for each object X of the underlying category the fibre $\{K \in \mathbf{K} \mid |K| = X\}$ is a set.

Theorem. *Let $\mathbf{X}$ be an $\mathbf{E}$-co-wellpowered $(\mathbf{E}, \mathbf{M})$-category, and let Δ be a collection of sources containing all $\mathbf{M}$-maps (considered as singleton sources). Then each hereditary, fibre-small concrete category over $\mathbf{X}$ has a fibre-small Δ-universal initial completion.*

Proof: The Δ-universal initial completion has been described in the proof of Theorem 4 of [E] (for the case of Γ = all sources and P = forgetful functor of $\mathbf{K}$) as the category of all Δ-complete sources in $\mathbf{K}$. It is our task to show that this category is fibre-small (thus, legitimate).

For each Δ-complete source $\sigma = (X \xrightarrow{f_i} |S_i|)_{i \in I}$ and each i we factorize $f_i = m_i \cdot e_i$ ($m_i \in \mathbf{M},\ e_i \in \mathbf{E}$) and denote by $m_i : S_i' \to S_i$ the initial morphism induced by m_i (which exists since $\mathbf{K}$ is hereditary). From the fact that Δ contains $\{m_i\}$ it follows that $X \xrightarrow{e_i} |S_i'|$ is a member of the (Δ-complete) source σ. Thus,

$$\sigma' = (X \xrightarrow{e_i} |S_i'|)_i \in I$$

is a subsource of σ. This subsource fully determines σ (in other words, given two Δ-complete sources $\sigma_1 \neq \sigma_2$, it follows that $\sigma_1' \neq \sigma_2'$). In fact, from the Δ-completeness it follows that for each $i \in I$ and each morphism $h : S'_i \to S$ in $\mathbf{K}$ we have $X \xrightarrow{he_i} |S|$ in σ; consequently, σ is precisely the source of all composites of the members of σ' with morphisms of $\mathbf{K}$. Thus, to prove the fibre-smallness, it is sufficient to observe that for each object X of $\mathbf{X}$, all the possible sources σ' form a set. In fact, since $\mathbf{X}$ is $\mathbf{E}$-co-wellpowered, we have a set of representatives for all e_i's, and for each such a representative $e_i : X \to Y$ we have (since $\mathbf{K}$ is fibre-small) only a set of possible objects S'_i with $|S'_i| = Y$. Thus, there exists a set A of representative structured maps $e_i : X \to |S'_i|$, and the collection of all Δ-complete sources with the domain X can, obviously, be coded by the set of all subsets of A. □

Remark. For constructs $\mathbf{K}$, i.e. concrete categories over $\mathbf{Set}$, the Δ-universal initial completion $\mathbf{K}^*$ can be described as follows.

Objects on the underlying set X are all collections σ of $\mathbf{K}$-objects K with the underlying sets $|K| = X/\sim$ (where $\sim$ is an equivalence relation on X) such that:

(1) If $K \in \sigma$ then $K' \in \sigma$ whenever $|K'| = X/\sim$ and the canonical map $c : X \to |K'|$ of $|K'|$ (assigning to each $x \in X$ the equivalence class of x) factorizes as the canonical map of $|K|$ followed by a $\mathbf{K}$-morphism $K \to K'$;

(2) For each initial Δ-source $(S \xrightarrow{f_i} S_i)_{i \in I}$ in $\mathbf{K}$

$$\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[d] \\
|K_i| \arrow[d, "h_i"'] & {|K| = X/\ker h} \arrow[d, hook] \\
S_i & S \arrow[l, "f_i"']
\end{tikzcd}$$

and each map $h : X \to |S|$ such that every $f_i h$ factorizes as the canonical map of $|K_i|$ for some $K_i \in \sigma$ followed by a $\mathbf{K}$-morphism $h_i : K_i \to S_i$ it follows that σ contains the initial lift of the inclusion $X/\ker h \hookrightarrow |S|$.

Morphisms from σ to σ' are maps $f : |\sigma| \to |\sigma'|$ such that for each $K' \in \sigma'$ (with the canonical map $c : |\sigma'| \to |K'|$) σ contains the initial lift of the inclusion $|\sigma|/\sim \hookrightarrow |K|$, where $\sim$ is the kernel equivalence of c.f.

II. A completion of the category of metric spaces.

Proposition. *The category* $\mathbf{Unif}$ *is a* Δ_{fm}*-universal initial completion of the category* $\mathbf{Met}$.

Proof: We apply the general result above to the special case of $\Delta = \Delta_{fm}$ and $\mathbf{K} = \mathbf{Met}$. The objects of Δ_{fm}-universal initial completion $\mathbf{K}^*$ with the underlying set X are collections σ of metric spaces $K = (X/\sim, d_K)$, where $\sim$ is an equivalence on X, such that (1) and (2) are satisfied. Each K induces a pseudometric d_K^* on the set X by $d_K^*(x,y) = d_K([x],[y])$ (where $x \mapsto [x]$ denotes the canonical map). The conditions (1) and (2) guarantee that the resulting set of pseudometrics on X forms a uniformity (defined as a collection of pseudometrics). Conversely, for each uniformity on X and each pseudometric d of that uniformity we have a corresponding metric space on the set $X/\sim$ where $x \sim y$ means that $d(x,y) = 0$. The axioms of a uniformity guarantee that the resulting set of metric spaces is an object of $\mathbf{K}^*$. The morphisms of $\mathbf{K}^*$ correspond precisely to the uniformly continuous maps. Thus, **Unif** is a Δ_{fm}-universal initial completion of **Met**. $\square$

Remark. Since **Met** is, obviously, closed under countable initial sources in **Met**, we can also say that for Δ = all countable sources, **Unif** is a Δ-universal initial completion of **Met**. However, the universal completion (Δ = all sources) is different:

Example of an initial source in **Met** which is not initial in **Unif**.

Let δ be the discrete metric (with value 1) on the set N of natural numbers. Let $\mathbf{F}$ be a free ultrafilter on N containing the set E of even numbers. For each $F \in \mathbf{F}$ let δ_F be the following metric on N:

$$\delta_F(x,y) = \begin{cases} \frac{1}{n} & \text{if } \{x,y\} = \{2n-1, 2n\} \text{ with } 2n \in F \\ 0 & \text{if } x = y \\ 1 & \text{else.} \end{cases}$$

As proved in [PRRS], the set of all δ_F, $F \in \mathbf{F}$, is a base of a uniformity σ on N which is an atom in the fibre of N (i.e. the only strictly finer uniformity is that induced by δ). Consequently, the following source

$$((N,\sigma) \xrightarrow{id} (N,\delta_F))_{F\in\mathbf{F}}$$

is initial in **Unif**. We will show that, nevertheless,

$$((N,\delta) \xrightarrow{id} (N,\delta_F))_{F\in\mathbf{F}}$$

is initial in **Met**. In fact, let (X,δ_0) be a metric space, and let $f : X \to N$ be a map such that $f : (X,\delta_0) \to (N,\delta_F)$ is uniformly continuous for each $F \in \mathbf{F}$. We will prove that the image of f, considered as a uniform subspace of (N,σ), is discrete — thus, $f : (X,\delta_0) \to (N,\delta)$ is uniformly continuous.

Suppose that image of f is not discrete. Then it is isomorphic to (N,σ). [In fact, every uniform subspace A of (N,σ) is isomorphic to (N,σ) or is discrete, according to whether the set $\{2n \mid n \in N,\ 2n \in A,\ 2n+1 \in A\}$ is a member of $\mathbf{F}$ or not.] Thus, we can assume that f is surjective. Since σ is an atom, it follows that f is a final morphism. However, (N,σ) is not a quotient of a metric space, since it is not generated by a single pseudometric — this is a contradiction.

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