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C^*-algebras of operators in non-archimedean Hilbert spaces

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Abstract. We show several examples of n.a. valued fields with involution. Then, by means of a field of this kind, we introduce "n.a. Hilbert spaces" in which the norm comes from a certain hermitian sesquilinear form. We study these spaces and the algebra of bounded operators which are defined on them and have an adjoint. Essential differences with respect to the usual case are observed.

Keywords: non-archimedean Hilbert space, non-archimedean C^*-algebra

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0. Introduction.

Several attempts have been made to define an analogous concept to the usual C^*-algebra among the non-archimedean (n.a.) normed algebras. The question of giving a definition of n.a. C^*-algebras in the language of the abstract theory of Banach spaces is explicitly presented in [9, p. 245]. Different kinds of algebras with some of the typical properties of the C^*-algebras have been studied [6], [7], [9], but "no analog of an involution is present" [7, p. 163]. The difficulty for the introduction of n.a. algebras with involution stems from the lack of examples of n.a. valued fields with non-trivial involution.

In this paper we present several examples of fields of this kind, and we use them to introduce n.a. Hilbert spaces. We study these spaces and the C^*-algebra (in the usual sense) of bounded operators on them which have an adjoint. The sequence space $c_0(K)$ is a n.a. Hilbert space and we study the associate C^*-algebra. We show several essential differences with the usual, real or complex case. For example, the Riesz-Fisher theorem is not valid in general, and $\{T \in L(E) \mid D(T^*) = E\} \neq L(E)$.

Notations and previous remarks.

Let K be a field. A non-archimedean (n.a.) valuation on K is a map $\alpha \in K \to |\alpha| \in \mathbb{R}$ such that for all $\alpha, \beta \in K$ it satisfies: $|\alpha| \geq 0$; $|\alpha| = 0$ if and only if $\alpha = 0$; $|\alpha\beta| = |\alpha|\ |\beta|$; and $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$. If K has a n.a. valuation, the set $\{|\alpha|, \alpha \in K, \alpha \neq 0\}$ is a multiplicative subgroup of $\mathbb{R}^+$. If it is a cyclic group, the valuation is called discrete; otherwise it is said to be dense.

Let X be a linear space over the field K. A non-archimedean norm on X is a norm which verifies the strong triangular inequality: $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. If X has a n.a. norm, it is called a n.a. normed space.

A n.a. normed algebra is a n.a. normed space A with a linear associative multiplication, satisfying $\|xy\| \leq \|x\|\ \|y\|$ for all $x, y \in A$.

A n.a. norm on a linear space X generates a metric defined by $d(x,y) = \|x-y\|$, which satisfies the ultrametric inequality $d(x,y) \le \max\{d(x,z), d(z,y)\}$ for all $x,y,z \in X$. In particular, a n.a. valuation on a field K induces in K the metric $d(\alpha,\beta) = |\alpha-\beta|$.

For a n.a. linear space the completeness is defined in the usual way. A n.a. complete normed space or algebra is called a n.a. Banach space or algebra. This completeness is not very useful in n.a. analysis. Its role is taken by the stronger concept of spherical completeness. A n.a. normed space X is said to be spherically complete if every collection of closed balls in X that is totally ordered by inclusion, has a nonempty intersection.

Let K be a field. An involution on K is a map $\alpha \in K \to \overline{\alpha} \in K$ that satisfies $\overline{\alpha+\beta} = \overline{\alpha} + \overline{\beta}$, $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$, $\overline{\overline{\alpha}} = \alpha$ for all $\alpha, \beta \in K$. If K has an involution, we call symmetric the elements of the set $\mathrm{S}(K) := \{\alpha \in K \mid \overline{\alpha} = \alpha\}$ and antisymmetric the elements of the set $\mathrm{AS}(K) := \{\alpha \in K \mid \overline{\alpha} = -\alpha\}$. Evidently, $K = \mathrm{S}(K) \oplus \mathrm{AS}(K)$, and $\alpha = \alpha_s + \alpha_a$ with $\alpha_s \in \mathrm{S}(K)$, $\alpha_a \in \mathrm{AS}(K)$ for all $\alpha \in K$ in a unique way. If there exists $0 \neq j \in \mathrm{AS}(K)$ (non-trivial case), we can write $\alpha = \alpha_s + j\alpha_j$ with $\alpha_s, \alpha_j \in \mathrm{S}(K)$ also in a unique way.

Let A be an algebra with involution $x \in A \to x^* \in A$ over the field K (i.e. $(x+y)^* = x^* + y^*$, $(\alpha x)^* = \overline{\alpha} x^*$, $(xy)^* = y^* x^*$, $x^{**} = x$ for all $x, y \in A$, $\alpha \in K$), and let $\mathrm{S}(A)$, $\mathrm{AS}(A)$ be respectively the symmetric and antisymmetric element sets of A. We also have $A = \mathrm{S}(A) \oplus \mathrm{AS}(A)$.

1. Non-archimedean valued fields with involution.

Let G be an additive subgroup of $\mathbb{R}$. The set

$$\mathbb{C}(x^G) := \left\{ \lambda = \sum_{r \in G} \alpha_r x^r,\ \alpha_r \in \mathbb{C},\ A_\lambda = \{r \in G \mid \alpha_r \neq 0\} \text{ is well ordered} \right\}$$

with the usual operations is a field, and $|\lambda| := e^{-\min A_\lambda}$ is a n.a. valuation on $\mathbb{C}(x^G)$ [7, p. 81–82].

Let us denote $K_1 = \mathbb{C}(x^{\mathbb{Q}})$, $K_2 = \mathbb{C}(x^{\mathbb{Z}})$, and $K_3 = \{\lambda \in K_1 \mid A_\lambda$ is finite or forms a sequence tending monotonously to infinite$\}$.

K_1 and K_2 are spherically complete fields, K_1 with dense valuation and K_2 with discrete valuation; and K_3 is complete but not spherically complete [7].

These examples are shown here to prove the existence of n.a. valued fields that allow a non-trivial involution. Fields similar to these ones have been considered in other contexts. For example, in [3] and [4], the author deals with linear spaces V over certain fields of formal power series, analogous to the above examples, in order to study the problem of classifying the measures on the orthomodular lattice $\mathscr{L}(V)$ of all linear subspaces of V.

In the quoted paper [3], it has been considered the subfield K_2' of K_2 given by $K_2' := \{\sum_{r\in\mathbb{Z}} \alpha_r x^r \in K_2 \mid \alpha_r \in \mathbb{R}\}$, with the valuation $w(\lambda) = -\log|\lambda|$, $w(0) = \infty$. We note that, since the author considers power series with real coefficients, our involution, described in the following Proposition 1.1, is reduced to the trivial one in this case.

The following result has a simple proof.

Proposition 1.1. *For $K = K_i$ $i = 1, 2, 3$, the mapping $\lambda = \sum_{r\in G} \alpha_r x^r \in K \to \overline{\lambda} = \sum_{r\in G} \overline{\alpha}_r x^r \in K$ is an isometric involution on K. Besides $|2_K| = 1$, and if we take $j = \sum_{r\in G} \alpha_r x^r$ with $\alpha_0 = i \in \mathbb{C}$, $\alpha_r = 0$ if $r \neq 0$, it follows $j \in \mathrm{AS}(K)$ and $|j| = 1$.*

If $|2| = 1$, the involution of K is isometric if and only if for all $\alpha \in K$ it is $|\alpha| = \max\{|\alpha_s|, |\alpha_a|\}$. If that occurs, then K is complete or spherically complete if and only if the same happens to $\mathrm{S}(K)$. Indeed, the map $\alpha \in \mathrm{S}(K) \to \alpha j \in \mathrm{AS}(K)$ is multiple of an isometry, thus $\mathrm{AS}(K)$ is complete or spherically complete if and only if $\mathrm{S}(K)$ is. $\mathrm{S}(K)$ being a field, by [5, III.4.7] it is complete or spherically complete if and only if the same happens to $\mathrm{S}(K)\times\mathrm{S}(K)$ with the valuation $|(a,b)| = \max\{|a|, |b|\}$, which is equivalent to be K complete or spherically complete because the mapping $\alpha \in K \to (\alpha_s, \alpha_a) \in \mathrm{S}(K) \times \mathrm{S}(K)$ is isometric.

From now on, K will be a field with characteristic different from 2, not trivially valued n.a., and with a non-trivial involution; i.e. there exists $j \in \mathrm{AS}(K) \setminus \{0\}$. Moreover E will be a linear space over the complete field K.

2. Non-archimedean inner products.

Let $\varphi(x, y)$ be a sesquilinear form in E. It is easy to prove that φ satisfies the polarization identity

$$\varphi(x, y) = \frac{1}{4}\left[q(x + y) - q(x - y) + jq(x - j^{-1}y) - jq(x + j^{-1}y)\right]$$

where q denotes the quadratic form associated with φ.

Definition 2.1. A non-archimedean inner product in E is a hermitian and anisotropic sesquilinear form $\varphi(x, y) \equiv (x, y)$ defined in E such that the associate mapping $\|\cdot\| : x \in E \to \|x\| := |(x, x)|^{1/2} \in \mathbb{R}$ is a non-archimedean norm. If besides $(E, \|\cdot\|)$ is complete, then E is said to be a n.a. Hilbert space.

The proof of the next theorem rests on a lemma of [2] that we apply to the field $\mathrm{S}(K)$.

Theorem 2.2. *If the involution of K is isometric and $|2| = 1$, then every hermitian and anisotropic sesquilinear form in E is a n.a. inner product that verifies:*

$$|(x, y)| \leq \|x\|\|y\| \quad \forall\, x, y \in E.$$

Proof: Let $x, y \in E$ be such that for all $\lambda \in K$, it is $\lambda x + y \neq 0$; then

$$P(\lambda) := (\lambda x + y, \lambda x + y) \neq 0.$$

In particular, if $(x, y) = a + b$ with $a \in \mathrm{S}(K)$, $b \in \mathrm{AS}(K)$, then the equation $P(\lambda) = 0$ has not roots in the (non-trivial) n.a. valued and complete field $\mathrm{S}(K)$. From here, by [2, Lemma 2, p. 54], and $|2| = 1$, we have $|a| \leq \|x\|\|y\|$.

Analogously, from $P(\lambda j) \neq 0$ for $\lambda \in \mathrm{S}(K)$ we obtain $|b| \leq \|x\|\|y\|$ and consequently

$$|(x, y)| = \max\{|a|, |b|\} \leq \|x\|\|y\|.$$

This conclusion is obvious if x, y are linearly dependent.

Now, for $x, y \in E$ it is

$$\|x+y\| \leq \left[\max\left(|(x,x)|, |(x,y)|, |(y,y)|\right)\right]^{1/2} = \max\left(\|x\|, \|y\|\right).$$

$\square$

Inner products defined through symmetric bilinear forms are studied in [2]. We shall use some of its results that are valid in our situation with the corresponding adaptation.

Let E be a n.a. inner product space and E' the dual space of E. For $M \subset E$, let us denote as usual $M^{\perp} := \{x \in E \mid (x, M) = 0\}$. Let be $h : y \in E \to f_y \in E'$ with $f_y(x) := (x, y)$ the Riesz-Fisher mapping. As a difference from the usual case, in general, h is not surjective.

Definition 2.3. $E_{\mathrm{RF}} := h(E)$ will be called the Riesz-Fisher dual of the space E.

If E is a n.a. inner product space over K such that $|2| = 1$, it is easy to prove that the Riesz-Fisher mapping and the canonical injection J of E on his bidual E'' are isometric, and besides $\|x\| = \|Jx\| = \|Jx\,|_{E_{\mathrm{RF}}}\|$.

In the next result we characterize the elements of E_{RF} in E'.

Proposition 2.4. *Let E be a n.a. inner product space over K, and let $f \in E' \setminus \{0\}$. Then $f \in E_{\mathrm{RF}}$ if and only if $N(f)^{\perp} \neq \{0\}$.*

PROOF: If $f = f_x \in E_{\mathrm{RF}}$, then $0 \neq x \in N(f)^{\perp}$. Conversely, if $0 \neq z \in N(f)^{\perp} \setminus \{0\}$, then $N(f) = N(f_z)$, hence $f = \lambda f_z = f_{\overline{\lambda} z}$ for some $\lambda \in K$. $\square$

Since E_{RF} is, in general, a proper subspace of E', we cannot introduce the adjoint operation as an involution on $L(E)$, which in the usual case gives $L(E)$ the C^*-algebra structure. In our case, we shall see in Section 4 that it is possible to define an adjoint operation in a certain subalgebra of $L(E)$ that will be thus structured as a non-commutative n.a. C^*-algebra.

3. The n.a. Hilbert space $c_0(K)$.

Let us denote by $c_0(K)$ the space of all sequences in K converging to zero and by e_n the unit vector basis of $c_0(K)$.

Proposition 3.1. *Suppose that K has an isometric involution, and $|2| = 1$. Then $((\alpha_n), (\beta_m)) := \sum_{n=1}^{\infty} \alpha_n \overline{\beta}_n \in K$ defines a hermitian and non-degenerate sesquilinear form in $c_0(K)$. Besides, if $(\ ,\ )$ is anisotropic, then $c_0(K)$ is a n.a. Hilbert space.*

PROOF: It is clear that $\sum_{n=1}^{\infty} \alpha_n \overline{\beta}_n$ converges ($\lambda_n \to 0$ implies $\sum \lambda_n$ converges [5]). Also it is easy to verify that $(\ ,\ )$ is a hermitian non-degenerate sesquilinear form. Let us suppose now that $(\ ,\ )$ is also anisotropic. For $x = (\alpha_n) \in c_0(K)$ we have

$$|(x,x)| = \Big|\sum_{1}^{\infty} \alpha_k \overline{\alpha}_k\Big| \leq \max_k |\alpha_k \overline{\alpha}_k| = \max_k |\alpha_k|^2 = \max_k |(e_k, x)|^2 \leq \|x\|^2 = |(x,x)|$$

then, $c_0(K)$ is a n.a. Hilbert space. $\square$

Let us note that for the fields K_i, $i = 1, 2, 3$, the mapping of the Proposition 3.1 is an inner product by [2, p. 62].

We give now a characterization for $c_0(K)_{\mathrm{RF}}$.

Theorem 3.2. *Suppose that K has a continuous involution, such that $c_0(K)$ is a n.a. Hilbert space, and let $f \in c_0(K)'$. Then, $f \in c_0(K)_{\mathrm{RF}}$ if and only if* $\lim_n f(e_n) = 0$. *Furthermore, $c_0(K)_{\mathrm{RF}}$ is a proper subspace of $c_0(K)' = \ell^\infty$.*

Proof: Let $f \in c_0(K)_{\mathrm{RF}}$ and $y = (y_n) \in c_0(K)$ such that $f(x) = (x, y)$ $\forall\, x \in c_0(K)$. Then $(f(e_n)) = ((e_n, y)) = (\overline{y}_n) \in c_0(K)$.

Let now $(f(e_n)) \in c_0(K)$ and take $y = (\overline{f(e_n)}) \in c_0(K)$. If $x = (x_n) \in c_0(K)$ then $(x, y) = \sum_{n=1}^\infty x_n \overline{\overline{f(e_n)}} = f(\sum_{n=1}^\infty x_n e_n) = f(x)$. Thus $f \in c_0(K)_{\mathrm{RF}}$.

Finally it is a simple routine to verify that the mapping

$$(x_n) \in c_0(K) \to f(x) := \sum_{n=1}^{\infty} x_n \in K$$

belongs to $c_0(K)'$ but $f(e_n) = 1$, hence $f \notin c_0(K)_{\mathrm{RF}}$. $\square$

4. n.a. C^*-algebras of operators.

Along this section, E will be a n.a. Hilbert space over K, and $L(E)$ the class of all continuous linear operators in E. For $T \in L(E)$, T' will denote the conjugate operator of T in E'. We shall also suppose that $|2| = 1$ (in K).

Definition 4.1. Given $T \in L(E)$ we define

$$D(T^*) := \{y \in E \mid \exists\, y^* \in E,\ (Tx, y) = (x, y^*) \text{ for all } x \in E\};$$
$$T^* : y \in D(T^*) \to y^* \in E \text{ and } A(E) := \{T \in L(E) \mid D(T^*) = E\}.$$

Theorem 4.2. (i) $A(E) = \{T \in L(E) \mid T'(E_{\mathrm{RF}}) \subset E_{\mathrm{RF}}\}$.

(ii) *$A(E)$ is a non-commutative unitary Banach algebra over K.*

(iii) *The map * is an involution on $A(E)$.*

(iv) *$A(E)$ is a n.a. C^*-algebra.*

Proof: (i): For $T \in L(E)$ we have

$$D(T^*) = E \Longleftrightarrow (T' \circ h)(E) \subset E_{\mathrm{RF}} \Longleftrightarrow T'(E_{\mathrm{RF}}) \subset E_{\mathrm{RF}}.$$

(ii): Only the completeness of $A(E)$ is not evident. To prove it, we shall show that $A(E)$ is closed in $L(E)$.

Let (T_n) be a Cauchy sequence on $A(E)$, and $T := \lim_n T_n \in L(E)$. Given $f_a = h(a) \in E_{\mathrm{RF}}$, for all $x \in E$ we have $T_n' f_a(x) = (T_n x, a) = (x, T_n^* a)$. Then

$$|(x, (T_n^* - T_m^*)a)| = |((T_n - T_m)x, a)| \le \|T_n - T_m\| \|x\| \|a\| \text{ for all } x \in E.$$

Taking $x = T_n^* a - T_m^* a \in E$, we obtain $\|T_n^* a - T_m^* a\| \leq \|T_n - T_m\|\|a\|$, therefore $(T_n^* a) \subset E$ is a Cauchy sequence (with respect to the norm) in E. If $b := \lim_n T_n^* a \in E$, we have $T' \circ f_a = f_b \in E_{\mathrm{RF}}$ and thus $T'(E_{\mathrm{RF}}) \subset E_{\mathrm{RF}}$.

(iii): For $T \in A(E)$ it is $T^* \in L(E)$, hence $T^{*\prime} \in L(E')$. Given now $f_a \in E_{\mathrm{RF}}$ ($a \in E$), for all $x \in E$ we have $(T^{*\prime} \circ f_a)x = f_{Ta}(x)$, hence $T^{*\prime}(E_{\mathrm{RF}}) \subset E_{\mathrm{RF}}$. The remaining properties are clear.

(iv): For $T \in A(E)$, $\|TT^*\| = \|T\|^2$ can be proved as in the usual case. □

Remark 4.3. The examples in the next section show that in general $A(E) \neq L(E)$, analogously as it happens in the case of the *-algebra of bounded operators with adjoint in a pre-Hilbert B-module over a complex B^*-algebra B (see [8]). This is an essential difference with the usual case.

5. The C^*-algebra of operators on the n.a. Hilbert space $c_0(K)$.

In this section, K will be a complete field with an isometric involution, where $|2| = 1$, and such that $c_0(K)$ is a n.a. Hilbert space with the inner product of Section 3. We shall denote $c_0(K)$ by c_0.

We need the following lemma which has a simple proof.

Lemma 5.1. *Let E be a n.a. inner product space over K, and $T \in L(E)$. Then*

$$\varphi_T : (x, y) \in E \times E \to \varphi_T(x, y) := (Tx, y) \in K$$

is a bounded sesquilinear form in E such that $\|\varphi_T\| = \|T\|$.

We characterize now the bounded sesquilinear forms in c_0 that arise from operators $T \in L(c_0)$, as stated in Lemma 5.1.

Lemma 5.2. *Let $\varphi : c_0 \times c_0 \to K$ be a bounded sesquilinear form. Then $\varphi = \varphi_T$ for some $T \in L(c_0)$ if and only if for all $x \in c_0$ it is $\lim_i \varphi(x, e_i) = 0$. In such a case, the operator T is unique.*

Proof: If $\varphi = \varphi_T$ with $T \in L(c_0)$, then for all $x \in c_0$ we have $Tx \in c_0$ and hence $\lim_i \varphi(x, e_i) = \lim_i (Tx)_i = 0$.

Let us suppose now that for all $x \in c_0$ it is $\lim_i \varphi(x, e_i) = 0$. The mapping

$$\gamma_x : y \in c_0 \to \gamma_x(y) := \overline{\varphi(x, y)} \in K$$

is a bounded linear form with $\|\gamma_x\| \leq \|\varphi\|\|x\|$. Since $\lim_i \gamma_x(e_i) = \lim \overline{\varphi(x, e_i)} = 0$, by (3.2) $\gamma_x \in (c_0)_{\mathrm{RF}}$ for all $x \in c_0$, thus there exists a unique $z_x \in c_0$ such that $\gamma_x(y) = (y, z_x)$ for all $y \in c_0$.

Now, the operator defined by $T : x \in c_0 \to Tx := z_x \in c_0$ clearly satisfies $\varphi_T = \varphi$. The unicity is obvious. □

Theorem 5.3. *Given $T \in L(c_0)$ we have $T \in A(c_0)$ if and only if for all $y \in c_0$, it is $\lim_i (Te_i, y) = 0$. Besides $A(c_0) \neq L(c_0)$.*

Proof: If $T \in A(c_0)$, for all $y \in c_0$ we have

$$\lim_i (Te_i, y) = \lim_i (e_i, T^* y) = \lim_i \overline{(T^* y)_i} = 0.$$

Conversely, if $\lim_i (Te_i, y) = 0$ for all $y \in c_0$, then by (5.1), $\psi(y,x) := (y, Tx) = \overline{\varphi_T(x,y)} \in K$ is a bounded sesquilinear form on c_0. For $x = e_i$ we have $\psi(y, e_i) = (y, Te_i) = \overline{(Te_i, y)} \to 0$. According to Lemma 5.2 there exists an operator $S \in L(c_0)$ such that for all $x, y \in c_0$; $\psi(y,x) = (Sx, y)$, or equivalently $(Tx, y) = (x, Sy)$, i.e. $T^* = S \in L(c_0)$.

In order to prove that $A(c_0) \neq L(c_0)$, we consider the operator

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \in c_0 \to Tx := \left(\sum_{i=1}^{\infty} a_i \right) e_1 \in c_0.$$

T is bounded with $\|T\| \leq 1$, so $T \in L(c_0)$. However, $T \notin A(c_0)$ because $\lim_i (Te_i, e_1) = (e_1, e_1) = 1 \neq 0$. □

Matricial representation of $T \in L(c_0(K))$.

Next we will give a matricial representation for operators of $L(c_0)$, and characterize the matrices that represent operators of $A(c_0)$. For that, every $T \in L(c_0)$ is represented by the infinite matrix $[\alpha_{ij}]$, where the i-th row of $[\alpha_{ij}]$ is the coordinate vector of Te_i.

Theorem 5.4. *Let $[\alpha_{ij}]$ be an infinite matrix of elements in K. Then:*

(i) $[\alpha_{ij}]$ *defines an operator $T \in L(c_0)$ if and only if it verifies*
  (i–1) $\lim_j \alpha_{ij} = 0$ *for every $i \in N$,*
  (i–2) $\mathrm{Sup}_{i,j \in N} |\alpha_{ij}| < \infty$.

(ii) $[\alpha_{ij}]$ *defines an operator $T \in A(c_0)$ if and only if it verifies* (i–1), (i–2) *and besides* $\lim_i \alpha_{ij} = 0$ *for every $j \in N$. In such a case, the adjoint operator T^* of T is represented by the adjoint matrix $[\overline{\alpha_{ji}}]$ of $[\alpha_{ij}]$.*

Proof: (i): If the matrix $[\alpha_{ij}]$ represents the operator $T \in L(c_0)$, then $Te_i = (\alpha_{ij} \mid j \in N) \in c_0$, hence (i–1) holds. Besides, for each $i \in N$ we have

$$\|Te_i\| = \mathrm{Sup}_j |\alpha_{ij}| \leq \|T\| \|e_i\| = \|T\|,$$

then $\mathrm{Sup}_{i,j} |\alpha_{ij}| \leq \|T\| < \infty$, and (i–2) holds. The converse is clear.

(ii): Let $T \in A(c_0) \subset L(c_0)$. By (5.3) we have $\lim_i (Te_i, e_j) = \lim_i \alpha_{ij} = 0$ for all $j \in N$. Let $T \in L(c_0)$ be now such that the associate matrix $[\alpha_{ij}]$ verifies $\lim_i \alpha_{ij} = 0$ for all $j \in N$. Then $\lim_i (Te_i, y) = 0$ for all $y \in c_0$, and due to (5.3) it results $T \in A(c_0)$.

Finally, let $[\alpha'_{ij}]$ be the matrix associated with T^*. We have

$$\alpha'_{ij} = (T^* e_i, e_j) = \overline{(Te_j, e_i)} = \overline{\alpha_{ji}}.$$

□

Now, we can show a subalgebra of $A(c_0)$ that is a n.a. C^*-subalgebra without unity. (Obviously any closed *-subalgebra of $A(c_0)$ is C^*-algebra).

Let us denote $A_1(c_0) := \{T \in L(c_0) \mid \lim_i Te_i = 0\}$.

Theorem 5.5. *$A_1(c_0)$ is a closed *-subalgebra of $A(c_0)$ without unity.*

Proof: By (5.3), $A_1(c_0)$ is subalgebra of $A(c_0)$. Let $(T_n) \subset A_1(c_0)$ be a convergent sequence with $T = \lim_n T_n \in A(c_0)$. Since

$$\|Te_i\| \leq \max\{\|Te_i - T_ne_i\|, \|T_ne_i\|\}$$

for every n, we have $\|Te_i\| \to 0$.

If $T \in A_1(c_0)$ and $[\alpha_{ij}]$ is the matrix of T, then $\lim_j T^*e_j = \lim_j \overline{\alpha_{ij}} = 0$ for each $i \in N$, and $T^* \in A_1(c_0)$. □

At the end, we present an example of an invertible operator in $L(c_0)$ that does not belong to $A(c_0)$. This shows that the regular group of A is a proper subgroup of that one of $L(c_0)$.

Example 5.6. Let $\alpha_{ij} = 1$ if $j \leq i$ and $\alpha_{ij} = 0$ if $j > i$. Due to (5.4), $[\alpha_{ij}]$ represents an injective operator $T \in L(c_0) \setminus A(c_0)$. Besides, if $y = (\beta_n) \in c_0$, then $x = (\beta_n - \beta_{n+1}) \in c_0$ satisfies $Tx = y$. Hence T is invertible. □

References

[1] Alvarez García J.A., *Involutions on non-archimedean fields and algebras*, Actas XIII Jornadas Hispano-Lusas de Matemáticas, Valladolid, 1988, to appear.

[2] Bayod Bayod J.M., *Productos internos en espacios normados no arquimedianos*, Doctoral dissertation, Universidad de Bilbao, 1976.

[3] Keller H.A., *Measures on orthomodular vector space lattices*, Studia Mathematica **88** (1988), 183–195.

[4] ———, *Measures on infinite-dimensional orthomodular spaces*, Foundations of Physics **20** (1990), 575–604.

[5] Monna A.F., *Analyse non-Archimedienne*, Springer-Verlag, 1970.

[6] Murphy G.J., *Commutative non-archimedean C^*-algebras*, Pacific J. Math. **78** (1978), 433-446.

[7] Narici L., Beckenstein E., Bachman G., *Functional Analysis and Valuation Theory*, Marcel Dekker, 1971.

[8] Paschke W.L., *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.

[9] Rooij A.C.M. Van, *Nonarchimedean Functional Analysis*, Marcel Dekker, 1978.

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