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On oriented vector bundles over CW-complexes of dimension 6 and 7

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Abstract. Necessary and sufficient conditions for the existence of n-dimensional oriented vector bundles ($n = 3, 4, 5$) over CW-complexes of dimension ≤ 7 with prescribed Stiefel-Whitney classes $w_2 = 0$, w_4 and Pontrjagin class p_1 are found. As a consequence some results on the span of 6 and 7-dimensional oriented vector bundles are given in terms of characteristic classes.

Keywords: CW-complex, oriented vector bundle, characteristic classes, Postnikov tower

Classification: 57R22, 57R25, 55R25

1. Introduction.

It is well known fact that every two-dimensional oriented vector bundle is uniquely determined by its Euler class. Using the difference cocycles, complete characterization of oriented vector bundles over CW-complexes of dimension 4 has been obtained in [2]. A natural question is when oriented vector bundles over a fixed CW-complex can be uniquely determined by their characteristic classes. Since n-dimensional oriented vector bundles over a CW-complex X are in one-to-one correspondence with homotopy classes in $[X, BSO(n)]$, we can investigate the mapping which assigns characteristic classes to every homotopy class from $[X, BSO(n)]$. Now the problem of characterization consists of two tasks: (i) To describe the image of the mapping mentioned above. (ii) To find conditions under which the mapping is injective. In some cases the latter problem is solved in [3] and [8]. For dim $X = n = 3, 4, 6, 7, 8$, Woodward in [14] has given the solution of both problems. Two classification theorems for dimensions 6 and 7 obtained by Woodward's approach are contained in Section 2.

We will consider oriented vector bundles with second Stiefel-Whitney class $w_2 = 0$ which are in one-to-one correspondence with homotopy classes in $[X, BSpin(n)]$. What we are going to do is the description of characteristic classes which are associated with such bundles for $n = 3, 4, 5$ over a CW-complex X of dimension ≤ 7. In fact, we find necessary and sufficient conditions for cohomology classes of X to be characteristic classes of such bundles, and so we solve problem (i) for the above dimensions. The results of this kind and their proofs form Section 3. The case of oriented vector bundles with $w_2 \neq 0$ is much more complicated and we are going to treat it elsewhere.

The maximal number of linearly independent sections in a vector bundle ξ is defined to be the span of ξ. Thomas in [13] has given essentially complete computation of the span of tangent bundles of oriented smooth manifolds up to the dimension 7. The results described above enable us to prove similar statements for general spin vector bundles of dimensions 6 and 7 over CW-complexes of the same dimension — see Section 4. To obtain results for dimension 3 and 4 is easy, the approach is almost the same as for tangent bundles in [13]. The case of dimension 5 is treated separately in [1].

2. Preliminaries.

We will consider only oriented vector bundles. They will be denoted by Greek letters. ε will stand for trivial one-dimensional vector bundle. For every CW-complex X, the mapping $\beta_2 : H^*(X,\mathbb{Z}_2) \to H^*(X,\mathbb{Z})$ is Bockstein homomorphism associated with exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. The mappings $i_* : H^*(X,\mathbb{Z}_2) \to H^*(X,Z_4)$ and $\rho_k : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z}_k)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and reduction mod k, respectively. We will use $w_j(\xi)$ for the j-th Stiefel-Whitney class of the vector bundle ξ, $p_1(\xi)$ for the first Pontrjagin class and $e(\xi)$ for the Euler class. The letters w_j, p_1, e will stand for characteristic classes of the universal $Spin(n)$-bundle over $BSpin(n)$. Eilenberg-MacLane space with n-th homotopy group G will be denoted $K(G,n)$ and ι_n will stand for the fundamental class in $H^n(K(G,n),G)$.

An important role in our considerations play the Steenrod squares Sq^i and the Pontrjagin square $\mathfrak{P}$, an operation from $H^{2k}(X,\mathbb{Z}_2)$ into $H^{4k}(X,\mathbb{Z}_4)$, and the following relations:

$$\rho_2\mathfrak{P}x = x^2 \tag{1}$$

$$\rho_4 p_1(\xi) = \mathfrak{P}w_2(\xi) + i_*w_4(\xi) \tag{2}$$

$$w_6(\xi) = Sq^2 w_4(\xi) + w_2(\xi)w_4(\xi) \tag{3}$$

the first being an easy consequence of definition, the second being proved in [4] and [7] and the last one being a special case of Wu formula.

We say that $x \in H^*(X,\mathbb{Z})$ is an element of order k ($k = 2,3,4,\dots$) if and only if $x \neq 0$ and k is the least positive integer such that $kx = 0$ (if it exists). Some results will involve the following hypotheses:

Condition (A). *$H^4(X,\mathbb{Z})$ has no element of order 4.*

Condition (B). *$H^6(X,\mathbb{Z})$ has no element of order 2.*

Condition (C). *Every element in $H^6(X,\mathbb{Z})$ of order 2 is in the form $\beta_2 Sq^2 y$ for some $y \in H^3(X,\mathbb{Z}_2)$.*

Obviously, (B) implies (C). Using results and remarks in [14] we obtain

Proposition 1. *Let X be a connected CW-complex of dimension ≤ 7 and suppose*

$$\gamma : [X, BSO(6)] \to H^2(X,\mathbb{Z}_2) \oplus H^4(X,\mathbb{Z}_2) \oplus H^4(X,\mathbb{Z}) \oplus H^6(X,\mathbb{Z})$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), p_1(\xi), e(\xi))$. Then

(i) *$\mathrm{im}\,\gamma = \{(a,b,c,d) \mid \rho_4 c = \mathfrak{P}a + i_*b\ ,\ \rho_2 d = Sq^2 b + ab\}$*

(ii) *γ is injective if and only if the conditions (A) and (C) are satisfied.*

Proof: It can be carried out by similar methods as in [14]. See also [1] for some details. □

Proposition 2. *Let X be a connected CW-complex of dimension ≤ 7 and suppose*

$$\gamma : [X, BSO(7)] \to H^2(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z})$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), p_1(\xi))$. Then

(i) *im $\gamma = \{(a, b, c) \mid \rho_4 c = \mathfrak{P}a + i_* b\}$*

(ii) *γ is injective if and only if the condition* (A) *is satisfied.*

Proof: The same as for the proof of Proposition 1 applies to it. □

3. Existence of vector bundles with prescribed characteristic classes.

In this section we derive the main results. For a CW-complex X, the question is whether a given set of cohomological classes with different coefficients forms characteristic (Stiefel-Whitney, Pontrjagin, Euler) classes for some n-dimensional oriented vector bundle. Any set of cohomological classes determines (up to homotopy) a mapping $f : X \to C$ where C is a product of Eilenberg-MacLane spaces. Simultaneously, the characteristic classes define a mapping $\alpha : BSO(n) \to C$ (or $BSpin(n) \to C$ if we require $w_2 = 0$) which can be taken to be a fibration. Now the problem of existence of an n-dimensional oriented vector bundle with prescribed characteristic classes is nothing else than the problem of lifting the mapping f in the fibration α.

$$\begin{tikzcd} & BSO(n) \arrow[d, "\alpha"] \\ X \arrow[ur] \arrow[r, "f"] & C \end{tikzcd}$$

Classical approach to such a kind of problems is the method of the Postnikov tower (see [9]). In several special cases we make the Postnikov tower and compute k-invariants. The computation becomes easier if we consider only oriented vector bundle with $w_2 = 0$.

First, we are going to deal with the existence of a 3-dimensional oriented vector bundle over a connected CW-complex X of dimension ≤ 7 with prescribed $w_2 = 0$ and p_1. In the beginning we show how $H^*(BSpin(3))$ in low dimensions looks like.

$BSpin(3)$ is 3-connected, $\pi_4(BSpin(3)) = \mathbb{Z}$. That is why $H^4(BSpin(3), \mathbb{Z}) = \mathbb{Z}$. $p_1 \in H^4(BSpin(3), \mathbb{Z})$ is a non zero element. If we consider p_1 as a mapping $BSpin(3) \to K(\mathbb{Z}, 4)$, we get that $p_{1*} : \pi_4(BSpin(3)) \to \mathbb{Z}$ is a multiplication by 4. It follows from two facts: (i) $\rho_4 p_1 = 0$ (see (2)) implies the existence of just one $u \in H^4(BSpin(3), \mathbb{Z})$ such that $p_1 = 4u$. (ii) There is a 3-dimensional oriented vector bundle over S^4 with $p_1 = 4$ which we obtain from 4-dimensional oriented vector bundle with $p_1 = 4$ and Euler class $e = 0$. The existence of such a bundle follows from [14]. $u_* \in \mathrm{Hom}(\pi_4(BSpin(3)), \mathbb{Z}) \cong H^4(BSpin(3), \mathbb{Z}) \cong \mathbb{Z}$ is a generator and it yields that u is a generator in $H^4(BSpin(3), \mathbb{Z})$. Hence $\rho_2 u \neq 0$. According to [6] we know that $H^*(BSpin(3), \mathbb{Z}_2) = \mathbb{Z}_2[\rho_2 u]$.

Now we construct the Postnikov tower for the mapping $\alpha : BSpin(3) \to K(\mathbb{Z}, 4)$, $\alpha^*(\iota_4) = u$. We can consider this mapping as a fibration with a fibre V. From the long homotopy exact sequence we get that V is 4-connected, $\pi_5(V) \cong \mathbb{Z}_2$, $\pi_6(V) \cong \mathbb{Z}_2$. The first invariant lies in $H^6(K(\mathbb{Z}, 4), \mathbb{Z}_2) \cong \mathbb{Z}_2$ and from the Serre exact sequence we get that it is equal to $Sq^2\rho_2\iota_4$ because $\alpha^* : H^6(K(\mathbb{Z}, 4), \mathbb{Z}_2) \to H^6(BSpin(3), \mathbb{Z}_2)$ is zero.

$$\begin{tikzcd}
\bar{F} \arrow[r] & V \arrow[r, "\bar{\beta}_1"] \arrow[d] & K(\mathbb{Z}_2, 5) \arrow[d, "i_1"] & \\
F \arrow[r] & B\,Spin(3) \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & E_1 \arrow[d, "\pi_1"] & \\
 & K(\mathbb{Z}, 4) \arrow[r, equal] & K(\mathbb{Z}, 4) \arrow[r, "Sq^2\rho_2\iota_4"] & K(\mathbb{Z}_2, 6)
\end{tikzcd}$$

Let E_1 be the first stage of the Postnikov tower. Let the new mappings be denoted according to the diagram and consider $\beta_1 : BSpin(3) \to E_1$ as a fibration with a fibre F. This fibre is homotopy equivalent to the homotopy fibre $\bar{F}$ of the mapping $\bar{\beta}_1$. That is why computing homotopy groups of $\bar{F}$ we get that F is 5-connected and $\pi_6(F) \cong \mathbb{Z}_2$. Hence β_1 is a 6-equivalence. The next invariant φ is a transgression of the generator of $H^6(F, \mathbb{Z}_2) \cong \mathbb{Z}_2$. From the Serre exact sequence for the fibration $F \to BSpin(3) \to E_1$ we get

$$\xrightarrow{0} H^6(F, \mathbb{Z}_2) \xrightarrow{\tau} H^7(E_1, \mathbb{Z}_2) \longrightarrow H^7(BSpin(3), \mathbb{Z}_2) \cong 0.$$

That is why φ is the only generator of $H^7(E_1, \mathbb{Z}_2)$. Using the Serre exact sequence for the fibration $K(\mathbb{Z}_2, 5) \to E_1 \to K(\mathbb{Z}, 4)$, we obtain $i_1^*\varphi = Sq^2\iota_5$.

Let E_2 be the second stage of the Postnikov tower. The mapping $\beta_2 : BSpin(3) \to E_2$ is a 7-equivalence.

$$\begin{tikzcd}
 & B\,Spin(3) \arrow[r, "\beta_2"] \arrow[dr, "\beta_1"] \arrow[ddr, "\alpha"'] & E_2 \arrow[d, "\pi_2"] & \\
 & & E_1 \arrow[r, "\varphi"] \arrow[d, "\pi_1"] & K(\mathbb{Z}_2, 7) \\
X \arrow[uur, "\xi"] \arrow[rr, "f"] & & K(\mathbb{Z}, 4) \arrow[r, "Sq^2\rho_2\iota_4"] & K(\mathbb{Z}_2, 6)
\end{tikzcd}$$

Now we are in a position to give an easy proof of the following

Theorem 1. *Let X be a connected CW-complex of dimension ≤ 7 and let $P \in H^4(X, \mathbb{Z})$. Then an oriented 3-dimensional vector bundle ξ over X with*

$$w_2(\xi) = 0 \qquad , \qquad p_1(\xi) = P$$

exists if and only if there is $U \in H^4(X,\mathbb{Z})$ such that

(i) $P = 4U$
(ii) $Sq^2\rho_2 U = 0$
(iii) $0 \in \Phi(U)$

where Φ is the secondary cohomology operation from $H^4(X,\mathbb{Z})$ into $H^7(X,\mathbb{Z}_2)$ associated with the relation $Sq^2 \circ Sq^2\rho_2 = 0$.

Proof: Let there be $U \in H^4(X,\mathbb{Z})$ satisfying (i)–(iii). If we denote $f : X \to K(\mathbb{Z},4)$ the mapping generated by U, we can lift it to a mapping $\xi : X \to BSpin(3)$ in the Postnikov tower. Moreover, $\xi^*(u) = U$ and $\xi^*(p_1) = P$. That is why ξ determines a 3-dimensional vector bundle with the prescribed characteristic classes.

Conversely, let $\xi : X \to BSpin(3)$ represent an oriented vector bundle with $w_2(\xi) = 0$ and $p_1(\xi) = P$. The choice $U = \xi^*(u) = (\alpha\circ\xi)^*\iota_4$ ensures (i). Further, we know that the mapping $\alpha\circ\xi$ can be lifted to ξ in the Postnikov tower. This implies (ii) and (iii). □

Remark. The "if" part of the above theorem was already known to Thomas [13], but his proof did not give any possibility to obtain also the "only if" part. □

Now we will investigate the existence of 4-dimensional oriented vector bundles over a connected CW-complex of dimension ≤ 7 with prescribed $w_2 = 0$, e (or w_4) and p_1. We will proceed in the same way as in the case of 3-dimensional vector bundles.

$BSpin(4)$ is 3-connected, $\pi_4(BSpin(4)) \cong \mathbb{Z}\oplus\mathbb{Z}$. That is why $H^4(BSpin(4),\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$. p_1, $e \in H^4(BSpin(4),\mathbb{Z})$ are non zero elements. We are going to find generators of $H^4(BSpin(4),\mathbb{Z})$. Since $\rho_2 p_1 = 0$ (see (1), (2)) and the mapping $H^4(BSpin(4),\mathbb{Z}) \xrightarrow{2\times} H^4(BSpin(4),\mathbb{Z})$ is injective, there is just one element $u \in H^4(BSpin(4),\mathbb{Z})$ such that

$$p_1 = 2u\,.$$

Further, according to (2), $\rho_4(2u+2e) = \rho_4(p_1+2e) = 0$ and it implies that there is just one $v \in H^4(BSpin(4),\mathbb{Z})$ satisfying

$$e = 2v - u\,. \tag{4}$$

We can show that u and v form generators of $H^4(BSpin(4),\mathbb{Z})$. Considering the isomorphisms

$$[BSpin(4), K(\mathbb{Z},4)] \cong H^4(BSpin(4),\mathbb{Z}) \cong \mathrm{Hom}\,(\pi_4(BSpin(4)),\mathbb{Z})$$

given by the prescription $g \mapsto g_*$, it is sufficient to prove that u_*, v_* generate the group $\mathrm{Hom}\,(\pi_4(BSpin(4)),\mathbb{Z})$. Generators of $\pi_4(BSpin(4))$ are mappings $\tau, \sigma : S^4 \to BSpin(4)$ which induce tangent and Hopf fibrations over S^4, respectively. The relations

$$p_{1*}(\tau) = p_1(\tau) = 0\,, \qquad e_*(\tau) = e(\tau) = 2$$
$$p_{1*}(\sigma) = p_1(\sigma) = -2\,, \qquad e_*(\sigma) = e(\sigma) = 1$$

imply

$$u_*(\tau) = 0\ ,\ u_*(\sigma) = -1\ ,\quad v_*(\tau) = 1\ ,\ v_*(\sigma) = 0\,.$$

Next we know that $H^4(BSpin(4), \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where w_4 is one of the generators. According to [6], $H^n(BSpin(4), \mathbb{Z}_2) \cong 0$ for $n = 5, 6, 7$. The relation $\rho_2 e = w_4$ and (4) imply $\rho_2 u = w_4$. If $\rho_2 v$ were 0 or w_4, it would be $v = 2y$ or $u + v = 2y$ which would be a contradiction with the fact that u and v are generators. That is why $\rho_2 v$ is the second generator of $H^4(BSpin(4), \mathbb{Z}_2)$.

We build a Postnikov tower for the mapping $\alpha : BSpin(4) \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)$ given by $\alpha^*(\iota_4 \otimes 1) = u$, $\alpha^*(1 \otimes \iota_4) = v$. We can consider this mapping as a fibration with a fibre V. From the homotopy exact sequence we get that V is 4-connected and $\pi_5(V) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\pi_6(V) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The first invariants can be easily obtained from the Serre exact sequence for the fibration $V \to BSpin(4) \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)$. They are $Sq^2 \rho_2 \iota_4 \otimes 1$ and $1 \otimes Sq^2 \rho_2 \iota_4$.

$$\begin{tikzcd}
\bar{F} \arrow[r] & V \arrow[r, "\bar{\beta}_1"] \arrow[d] & K(\mathbb{Z}_2,5) \times K(\mathbb{Z}_2,5) \arrow[d, "i_1"] & \\
F \arrow[r] & B\,Spin(4) \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & E_1 \arrow[d, "\pi_1"] & \\
& K(\mathbb{Z},4) \times K(\mathbb{Z},4) \arrow[r, equal] & K(\mathbb{Z},4) \times K(\mathbb{Z},4) \arrow[r, "Sq^2\rho_2\iota_4\otimes 1", "1\otimes Sq^2\rho_2\iota_4"'] & K(\mathbb{Z}_2,6) \times K(\mathbb{Z}_2,6)
\end{tikzcd}$$

Let E_1 be the first stage of the Postnikov tower and let the new mappings be denoted according to the diagram. Consider $\beta_1 : BSpin(4) \to E_1$ as a fibration with a fibre F, which is homotopy equivalent with the homotopy fibre $\bar{F}$ of the mapping $\bar{\beta}_1$. That is why computing the homotopy groups of $\bar{F}$ we get that F is 5-connected (i.e. β_1 is a 6-equivalence) and $\pi_6(F) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The next invariants φ, ψ are given by transgression in the Serre exact sequence for the fibration β_1. Since $H^7(BSpin(4), \mathbb{Z}_2) = 0$, we get that φ and ψ form a basis of $H^7(E_1, \mathbb{Z}_2)$. The Serre exact sequence for $K(\mathbb{Z}_2, 5) \times K(\mathbb{Z}_2, 5) \xrightarrow{i_1} E_1 \xrightarrow{\pi_1} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)$ yields that φ and ψ can be chosen in such a way that $i_1^*\varphi = Sq^2\iota_5 \otimes 1$, $i_1^*\psi = 1 \otimes Sq^2\iota_5$.

Let E_2 be the second stage. The mapping $\beta_2 : BSpin(4) \to E_2$ is a 7-equivalence. It enables us to prove

Theorem 2. *Let X be a connected CW-complex of dimension ≤ 7 and let $P, E \in H^4(X, \mathbb{Z})$. Then an oriented 4-dimensional vector bundle ξ over X with*

$$w_2(\xi) = 0\ ,\qquad p_1(\xi) = P\ ,\qquad e(\xi) = E$$

exists if and only if there are $U, V \in H^4(X, \mathbb{Z})$ such that

(i) $P = 2U\ ,\ E = 2V - U$

(ii) $Sq^2\rho_2 U = 0$, $Sq^2\rho_2 V = 0$
(iii) $0 \in \Phi(U)$, $0 \in \Phi(V)$

where Φ is the secondary cohomology operation from $H^4(X,\mathbb{Z})$ into $H^7(X,\mathbb{Z}_2)$ associated with the relation $Sq^2 \circ Sq^2\rho_2 = 0$.

PROOF: It proceeds in a similar way as the proof of Theorem 1. The choice of U and V is given by $\xi^*(u)$ and $\xi^*(v)$. We will show only that φ and ψ determine the secondary operation Φ. This operation is defined by the only element $\lambda \in H^7(E,\mathbb{Z}_2)$ where $K(\mathbb{Z}_2,5) \to E \to K(\mathbb{Z},4)$ is the fibration given by $Sq^2\rho_2\iota_4$. However, the definition of E_1 implies that $E_1 = E \times E$ and $\varphi = \lambda \otimes 1$, $\psi = 1 \otimes \lambda$. □

Consequence. *Let X be a connected CW-complex of dimension ≤ 7 and let $P \in H^4(X,\mathbb{Z})$, $W \in H^4(X,\mathbb{Z}_2)$. Then an oriented 4-dimensional vector bundle ξ over X with*

$$w_2(\xi) = 0\ , \qquad w_4(\xi) = W\ , \qquad p_1(\xi) = P$$

exists if and only if there is $U \in H^4(X,\mathbb{Z})$ such that

(i) $P = 2U$, $\rho_2 U = W$
(ii) $Sq^2 W = 0$
(iii) $0 \in \Phi(U)$

where Φ is the secondary cohomology operation from $H^4(X,\mathbb{Z})$ into $H^7(X,\mathbb{Z}_2)$ associated with the relation $Sq^2 \circ Sq^2\rho_2 = 0$.

PROOF: Let such a vector bundle ξ exist. Denote its Euler class by E. According to Theorem 2 there are U, V satisfying (i), (ii), (iii) of Theorem 2. Moreover, $\rho_2 U = \rho_2 E = w_4(\xi) = W$. So (i), (ii) and (iii) of Consequence are satisfied.

Conversely, let (i), (ii) and (iii) of Consequence be fulfilled. Then U and $V = 0$ satisfy (i), (ii) and (iii) of Theorem 2. Hence there is a vector fibration ξ with $w_2(\xi) = 0$, $e(\xi) = -U$ and $p_1(\xi) = P$. Then also $w_4(\xi) = \rho_2 e(\xi) = \rho_2 U = W$. □

The existence of 5-dimensional oriented vector bundles with prescribed $w_2 = 0$, w_4 and p_1 can be dealt with in a very similar way as in the case of Theorem 1 and 2, so we give only a conclusion.

Theorem 3. *Let X be a connected CW-complex of dimension ≤ 7 and let $P \in H^4(X,\mathbb{Z})$, $W \in H^4(X,\mathbb{Z}_2)$. Then an oriented 5-dimensional vector bundle ξ over X with*

$$w_2(\xi) = 0\ , \qquad w_4(\xi) = W\ , \qquad p_1(\xi) = P$$

exists if and only if there is $U \in H^4(X,\mathbb{Z})$ such that

(i) $P = 2U$, $\rho_2 U = W$
(ii) $Sq^2 W = 0$
(iii) $0 \in \Phi(U)$

where Φ is the secondary cohomology operation from $H^4(X,\mathbb{Z})$ into $H^7(X,\mathbb{Z}_2)$ associated with the relation $Sq^2 \circ Sq^2\rho_2 = 0$.

4. Span of some vector bundles.

In this section we derive several consequences of the results obtained above concerning the span of 6 and 7-dimensional oriented vector bundles over a CW-complex of the same dimension. Moreover, we compare our results with the results already known.

Corollary 1. *Let X be a connected CW-complex of dimension 6 and let ξ be an oriented 6-dimensional vector bundle over X with $w_2(\xi) = 0$.*

(1) *span $\xi \geq 1$ if and only if $e(\xi) = 0$.*

If Conditions (A) and (C) are satisfied then

(2) *span $\xi \geq 2$ if and only if $e(\xi) = 0$ and there is a $U \in H^4(X, \mathbb{Z})$ such that $2U = p_1(\xi)$, $\rho_2 U = w_4(\xi)$.*

(3) *span $\xi \geq 3$ if and only if $e(\xi) = 0$, $w_4(\xi) = 0$ and there is a $U \in H^4(X, \mathbb{Z})$ such that $4U = p_1(\xi)$, $Sq^2 \rho_2 U = 0$.*

(4) *span $\xi \geq 4$ if and only if $e(\xi) = 0$, $w_4(\xi) = 0$ and there is a $U \in H^2(X, \mathbb{Z})$ such that $\rho_2 U = 0$, $U^2 = p_1(\xi)$.*

(5) *span $\xi = 6$ if and only if $e(\xi) = 0$, $w_4(\xi) = 0$, $p_1(\xi) = 0$.*

Proof: In this and the next proofs we will often use the following relations for the direct sum of oriented vector bundles:

$$w_k(\sigma \oplus n\varepsilon) = w_k(\sigma)\ ,\ p_1(\sigma \oplus n\varepsilon) = p_1(\sigma)\ ,\ e(\sigma \oplus n\varepsilon) = 0$$

(1) is well known and is included only for completeness.

(2) ($\Rightarrow$) Let $\xi = \sigma \oplus 2\varepsilon$, where σ is a 4-dimensional oriented vector bundle over X. Then $w_2(\sigma) = 0$, $w_4(\sigma) = w_4(\xi) = W$, $p_1(\sigma) = p_1(\xi) = P$. Due to Consequence of Theorem 2, there is $U \in H^4(X, \mathbb{Z})$ such that $2U = p_1(\xi)$, $\rho_2 U = w_4(\xi)$. Moreover, $e(\xi) = 0$.

($\Leftarrow$) According to Consequence of Theorem 2, for $U \in H^4(X, \mathbb{Z})$, $Sq^2 \rho_2 U = Sq^2 w_4(\xi) = \rho_2 e(\xi) = 0$ there is an oriented 4-dimensional vector bundle σ over X with

$$w_2(\sigma) = 0\ ,\ w_4(\sigma) = \rho_2 U = w_4(\xi)\ ,\ p_1(\sigma) = 2U = p_1(\xi)\,.$$

Due to Proposition 1, $\sigma \oplus 2\varepsilon = \xi$ because both vector bundles have the same characteristic classes.

(3) ($\Rightarrow$) Let $\xi = \sigma \oplus 3\varepsilon$, where σ is a 3-dimensional oriented vector bundle over X. Then $e(\xi) = 0$, $w_4(\xi) = w_4(\sigma) = 0$ and $w_2(\sigma) = w_2(\xi) = 0$, $p_1(\sigma) = p_1(\xi) = P$. Due to Theorem 1, there is $U \in H^4(X, \mathbb{Z})$ such that $4U = p_1(\xi)$, $Sq^2 \rho_2 U = 0$.

($\Leftarrow$) For $U \in H^4(X, \mathbb{Z})$, $4U = p_1(\xi)$, $Sq^2 \rho_2 U = 0$, Theorem 1 implies the existence of an oriented 3-dimensional vector bundle σ over X with $w_2(\sigma) = 0$, $p_1(\sigma) = 4U = p_1(\xi)$. Then Proposition 1 gives $\sigma \oplus 3\varepsilon = \xi$ since the characteristic classes of both vector bundles coincide.

(4) ($\Rightarrow$) Let $\xi = \sigma \oplus 4\varepsilon$, where σ is a 2-dimensional oriented vector bundle over X. Then $e(\xi) = 0$, $w_4(\xi) = w_4(\sigma) = 0$ and for $U = e(\sigma)$ we get $\rho_2 U = w_2(\sigma) = w_2(\xi) = 0$ and $U^2 = p_1(\sigma) = p_1(\xi)$.

($\Leftarrow$) For $U \in H^2(X,\mathbb{Z})$ there is an oriented 2-dimensional vector bundle σ over X such that $e(\sigma) = U$. Then $w_2(\sigma) = \rho_2 U = 0$, $p_1(\sigma) = U^2 = p_1(\xi)$ and Proposition 1 ensures that $\sigma \oplus 4\varepsilon = \xi$ since the characteristic classes of both vector bundles are the same.

(5) follows immediately from Proposition 1. □

Corollary 2. *Let X be a connected CW-complex of dimension 7 and let ξ be an oriented 7-dimensional vector bundle over X with $w_2(\xi) = 0$.*

(1) *span $\xi \geq 1$ if and only if $e(\xi) = 0$.*

If Condition (A) is satisfied then

(2) *span $\xi \geq 2$ if and only if $w_6(\xi) = 0$ and there is a $U \in H^4(X,\mathbb{Z})$ such that $2U = p_1(\xi)$, $\rho_2 U = w_4(\xi)$ and $0 \in \Phi(U)$.*

(3) *span $\xi \geq 3$ if and only if $w_6(\xi) = 0$ and there is a $U \in H^4(X,\mathbb{Z})$ such that $2U = p_1(\xi)$, $\rho_2 U = w_4(\xi)$ and $0 \in \Phi(U)$.*

(4) *span $\xi \geq 4$ if and only if $w_4(\xi) = 0$ and there is a $U \in H^4(X,\mathbb{Z})$ such that $4U = p_1(\xi)$, $Sq^2\rho_2 U = 0$, $0 \in \Phi(U)$.*

(5) *span $\xi \geq 5$ if and only if $w_4(\xi) = 0$ and there is a $U \in H^2(X,\mathbb{Z})$ such that $\rho_2 U = 0$, $U^2 = p_1(\xi)$.*

(6) *span $\xi = 7$ if and only if $w_4(\xi) = 0$, $p_1(\xi) = 0$.*

Remark. Notice that under Condition (A) span $\xi \geq 2$ implies span $\xi \geq 3$.

Proof: (2) ($\Rightarrow$) Let $\xi = \sigma \oplus 2\varepsilon$, where σ is a 5-dimensional oriented vector bundle over X. Then $w_6(\xi) = w_6(\sigma) = 0$ and $w_2(\sigma) = w_2(\xi) = 0$. Due to Theorem 3, there is a $U \in H^4(X,\mathbb{Z})$ such that $2U = p_1(\sigma) = p_1(\xi)$, $\rho_2 U = w_4(\sigma) = w_4(\xi)$, $0 \in \Phi(U)$.

($\Leftarrow$) For $U \in H^4(X,\mathbb{Z})$, $Sq^2\rho_2 U = Sq^2 w_4(\xi) = w_6(\xi) = 0$, $0 \in \Phi(U)$, Theorem 3 gives the existence of an oriented 5-dimensional vector bundle σ over X with

$$w_2(\sigma) = 0\ ,\ w_4(\sigma) = \rho_2 U = w_4(\xi)\ ,\ p_1(\sigma) = 2U = p_1(\xi)\,.$$

Then due to Proposition 2, $\sigma \oplus 2\varepsilon = \xi$ since the characteristic classes of both vector bundles coincide.

(3) ($\Rightarrow$) Let $\xi = \sigma \oplus 3\varepsilon$, where σ is an oriented 4-dimensional vector bundle over X. Then $w_6(\xi) = 0$, $w_2(\sigma) = 0$ and due to Consequence of Theorem 2, there is a $U \in H^4(X,\mathbb{Z})$ such that $2U = p_1(\sigma) = p_1(\xi)$, $\rho_2 U = w_4(\sigma) = w_4(\xi)$ and $0 \in \Phi(U)$.

($\Leftarrow$) For $U \in H^4(X,\mathbb{Z})$, $Sq^2\rho_2 U = Sq^2 w_4(\xi) = w_6(\xi) = 0$, $0 \in \Phi(U)$, Consequence of Theorem 2 ensures the existence of an oriented 4-dimensional vector bundle σ over X with

$$w_2(\sigma) = 0\ ,\ w_4(\sigma) = \rho_2 U = w_4(\xi)\ ,\ p_1(\sigma) = 2U = p_1(\xi)\,.$$

Then due to Proposition 2, $\sigma \oplus 2\varepsilon = \xi$ since both vector bundles have identical characteristic classes.

(4) can be obtained using Theorem 1 and Proposition 2 in the same way as the previous proofs.

(5) follows from the characterization of 2-dimensional oriented vector bundles and Proposition 2 in a similar way.

(6) is an easy consequence of Proposition 2. □

Remark. General results on the cases when the span of $4k+2$ and $4k+3$-dimensional vector bundles is $\geq 2, 3$ or 4 are given in [5], [10], [11] and [12] in terms of Stiefel-Whitney classes and secondary cohomology operations. However, some of our results cannot be derived from these general theorems even if we consider X to be a manifold and the span equal to 3 and 4. (For example, (3) in Corollary 1, (3) and (4) in Corollary 2.)

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