Oscillation properties for parabolic equations of neutral type

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Abstract. The oscillation of the solutions of linear parabolic differential equations with deviating arguments are studied and sufficient conditions that all solutions of boundary value problems are oscillatory in a cylindrical domain are given.

Keywords: partial differential equation, deviating argument, boundary problem, oscillation

Classification: 35B05, 35R10

In the last few years there has been much interest in studying the oscillatory behaviour of solutions of partial differential equations with deviating arguments. We refer the reader to Mishev & Bainov [1], [2], Yoshida [3], Georgiou & Kreith [4] and Cui [5]. However, only the papers [1] and [5] considered the oscillation for parabolic differential equations of neutral type.

The purpose of this paper is to establish some new oscillation criteria for the linear parabolic equation of neutral type of the form

$$\begin{aligned}\frac{\partial}{\partial t}\big[u-\sum_{i=1}^{l}\lambda_i(t)u(x,t-\tau_i)\big] &= a(t)\,\triangle\, u+\sum_{i=1}^{m}a_i(t)\,\triangle\, u(x,\rho_i(t)) \\ &\quad -\sum_{j=1}^{r}q_j(x,t)u(x,\sigma_j(t))\end{aligned} \tag{1}$$

$(x,t)\in\Omega\times(0,\infty)\equiv G$, where Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$, $u=u(x,t)$ and $\triangle$ is the Laplacian in Euclidean n-space R^n.

We assume throughout this paper that

(i) $a(t),a_i(t)\in C([0,\infty0;[0,\infty))$, $i=1,2,\dots,m$;
(ii) $q_j(x,t)\in C(\overline{G};[0,\infty))$, $q_j(t)=\min_{x\in\overline{\Omega}}q_j(x,t)$, $j=1,2,\dots,r$;
(iii) $\rho_i(t),\sigma_j(t)\in C([0,\infty);R)$ and $\lim_{t\to\infty}\rho_i(t)=\lim_{t\to\infty}\sigma_j(t)=\infty$, $i=1,2,\dots,m$; $j=1,2,\dots,r$, and $\tau_k=\text{const.}>0$, $k=1,2,\dots,l$;
(iv) $\lambda_k(t)\in C'([0,\infty);R)$, $k=1,2,\dots,l$.

Consider two kinds of boundary conditions:

$$u=\phi \quad \text{on}\ \ \partial\Omega\times[0,\infty), \tag{2}$$

$$\frac{\partial u}{\partial N}=\psi \quad \text{on}\ \ \partial\Omega\times[0,\infty), \tag{3}$$

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where N is the unit exterior normal vector to $\partial\Omega$ and ψ and ϕ are continuous functions on $\partial\Omega \times [0,\infty)$.

It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$\begin{cases} \triangle\, \omega + \lambda\omega = 0 & \text{in } \Omega \\ \omega = 0 & \text{on } \partial\Omega \end{cases}$$

is positive and the corresponding eigenfunction Φ is positive in Ω.

Our objective is to present conditions which imply that every solution $u(x,t)$ of (1) satisfying a certain boundary condition is oscillatory in $\Omega \times [0,\infty)$ in the sense that u has a zero in $\Omega \times [t,\infty)$ for any $t > 0$. Note that the condition for the oscillation of the problem (1), (2) for $\sigma_j(t) = t - \sigma_j$, $\rho_i(t) = t - \rho_i$, $\sigma_j \geq 0$, $\rho_i \geq 0$ and $\phi \equiv 0$ (σ_j and ρ_i are constants) has been obtained in the work of Mishev & Bainov [1], and Cui [5] has extended the results of [1].

Theorem 1. *Assume that the conditions* (i)–(iv) *hold, and let $u(x,t)$ be a positive solution of the problem* (1), (2) *in the domain G. Then the function*

$$V(t) = \int_\Omega u(x,t)\Phi(x)\,dx, \quad t \geq 0, \tag{4}$$

satisfies the differential inequality of neutral type

$$\begin{aligned} &\frac{d}{dt}\big[V(t) - \sum_{k=1}^{l} \lambda_k(t)V(t-\tau_k)\big] + \lambda_1 a(t)V(t) + \lambda_1 \sum_{i=1}^{m} a_i(t)V(\rho_i(t)) \\ &\quad + \sum_{j=1}^{r} q_j(t)V(\sigma_j(t)) \leq -\int_{\partial\Omega} \big[a(t)\phi(x,t) + \sum_{i=1}^{m} a_i(t)\phi(x,\rho_i(t))\big]\frac{\partial\Phi}{\partial N}\,dS \end{aligned} \tag{5}$$

for $t \geq t_0$, where t_0 is a sufficiently large positive number.

Proof: Suppose that $u(x,t)$ is a positive solution. Without loss of generality we may assume that $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. From the condition (iii), there exists a $t_1 \geq t_0$ so that $u(x,\rho_i(t)) > 0$, $u(x,\sigma_j(t)) > 0$ and $u(x,t-\tau_k) > 0$ in $\Omega \times [t_1,\infty)$ for $i = 1,2,\dots,m$; $j = 1,2,\dots,r$; $k = 1,2,\dots,l$.

Multiplying (1) by Φ, integrating (1) over Ω, we have that

$$\begin{aligned} &\frac{d}{dt}\Big[\int_\Omega u(x,t)\Phi\,dx - \sum_{k=1}^{l}\lambda_k(t)\int_\Omega u(x,t-\tau_k)\Phi\,dx\Big] \\ &= a(t)\int_\Omega \triangle\, u\Phi\,dx + \sum_{i=1}^{m} a_i(t)\int_\Omega \triangle\, u(x,\rho_i(t))\Phi\,dx \\ &\qquad - \sum_{j=1}^{r}\int_\Omega q_j(x,t)u(x,\sigma_j(t))\Phi\,dx \end{aligned} \tag{6}$$

for $t \geq t_1$.

It follows from Green's formula that

$$\int_\Omega \triangle u\Phi\, dx = \int_{\partial\Omega} (\frac{\partial u}{\partial N}\Phi - u\frac{\partial \Phi}{\partial N})\, dS + \int_\Omega u \triangle \Phi\, dx \\ = -\int_{\partial\Omega} \phi \frac{\partial \Phi}{\partial N}\, dS - \lambda_1 \int_\Omega u\Phi\, dx. \tag{7}$$

$$\int_\Omega \triangle u(x,\rho_i(t))\Phi\, dx = \int_{\partial\Omega} (\frac{\partial u}{\partial N}(x,\rho_i(t))\Phi(x) - u(x,\rho_i(t))\frac{\partial \Phi}{\partial N})\, dS \\ + \int_\Omega u(x,\rho_i(t)) \triangle \Phi\, dx \\ = -\int_{\partial\Omega} \phi(x,\rho_i(t)) \frac{\partial \Phi}{\partial N}\, dS - \lambda_1 \int_\Omega u(x,\rho_i(t))\Phi\, dx, \\ i = 1,2,\ldots,m; \tag{8}$$

$$\int_\Omega q_j(x,t)u(x,\sigma_j(t))\Phi(x)\, dx \geq q_j(t)\int_\Omega u(x,\sigma_j(t))\Phi\, dx, \quad j = 1,2,\ldots,r. \tag{9}$$

Thus, by (6)–(9) and (4), we obtain that

$$\frac{d}{dt}\big[V(t) - \sum_{k=1}^{l} \lambda_k(t)V(t-\tau_k)\big] + \lambda_1 a(t)V(t) + \lambda_1 \sum_{i=1}^{m} a_i(t)V(\rho_i(t)) \\ + \sum_{j=1}^{r} q_j(t)V(\sigma_j(t)) \\ \leq -a(t)\int_{\partial\Omega} \phi\frac{\partial \Phi}{\partial N}\, dS - \sum_{i=1}^{m} a_i(t)\int_{\partial\Omega} \phi(x,\rho_i(t))\frac{\partial \Phi}{\partial N}\, dS \\ = -\int_{\partial\Omega} \big[a(t)\phi(x,t) + \sum_{i=1}^{m} a_i(t)\phi(x,\rho_i(t))\big]\frac{\partial \Phi}{\partial N}\, dS$$

for $t \geq 1$. This completes the proof of Theorem 1. □

Theorem 2. *Let* (i)–(iv) *hold. If the differential inequalities of neutral type*

$$\frac{d}{dt}\big[V(t) - \sum_{k=1}^{l} \lambda_k(t)V(t-\tau_k)\big] + \lambda_1 a(t)V(t) + \lambda_1 \sum_{i=1}^{m} a_i(t)V(\rho_i(t)) \\ + \sum_{j=1}^{r} q_j(t)V(\sigma_j(t)) \leq -F(t) \tag{10}$$

$$\frac{d}{dt}\big[V(t) - \sum_{k=1}^{l} \lambda_k(t)V(t-\tau_k)\big] + \lambda_1 a(t)V(t) + \lambda_1 \sum_{i=1}^{m} a_i(t)V(\rho_i(t)) \\ + \sum_{j=1}^{r} q_j(t)V(\sigma_j(t)) \leq F(t) \tag{11}$$

have no ultimately positive solutions, where

$$F(t) = \int_{\partial\Omega} \big[a(t)\phi(x,t) + \sum_{i=1}^{m} a_i(t)\phi(x,\rho_i(t))\big] \frac{\partial \Phi}{\partial N}\, dS$$

then every solution of the problem (1), (2) *oscillates in* G.

PROOF: Suppose that the assertion is not true and $u(x,t)$ is a nonoscillatory solution. There exists a $t_1 > 0$ such that $u(x,t) \neq 0$ for any $(x,t) \in \Omega \times [t_1,\infty)$. If $u(x,t) > 0$ in $\Omega \times [t,\infty)$, then it follows from Theorem 1 that $V(t)$, which is defined by (4), is a positive solution of the inequality (10) for $t \geq t_1$, which contradicts the condition of the theorem.

If $u(x,t) < 0$ in $\Omega \times [t_1,\infty)$, let $W(x,t) = -u(x,t) > 0$ for $(x,t) \in \Omega \times [t_1,\infty)$, then $W(x,t)$ is an ultimately positive solution of the problem

$$\begin{cases} \frac{\partial}{\partial t}\big[u - \sum_{i=1}^{l} \lambda_i(t) u(x,t-\tau_i)\big] & = a(t) \vartriangle u + \sum_{i=1}^{m} a_i(t) \vartriangle u(x,\rho_i(t)) \\ & \quad - \sum_{j=1}^{r} q_j(x,t) u(x,\sigma_j(t)), \quad (x,t) \in G, \\ u(x,t) = -\phi(x,t), & (x,t) \in \partial\Omega \times [0,\infty), \end{cases}$$

and satisfies the inequality

$$\tag{12} \begin{aligned} &\frac{d}{dt}\Big[\int_\Omega W(x,t)\Phi\, dx - \sum_{i=1}^{l} \lambda_i(t) \int_\Omega W(x,t-\tau_i)\Phi\, dx\Big] + \lambda_1 a(t) \int_\Omega W(x,t)\Phi\, dx \\ &+\lambda_1 \sum_{i=1}^{m} a_i(t) \int_\Omega W(x,\rho_i(t))\Phi\, dx + \sum_{j=1}^{r} q_j(t) \int_\Omega W(x,\sigma_j(t))\Phi\, dx \leq F(t) \end{aligned}$$

for $t \geq t_2$, where $t_2 \geq t_1$ is a sufficiently large positive number.

Set

$$W(t) = \int_\Omega W(x,t)\Phi(x)\, dx, \quad t \geq t_2,$$

then $W(t)$ is a positive solution of the inequality (11) for $t \geq t_2$ from (12), which contradicts the condition of the theorem, too. This completes the proof of Theorem 2.

□

Theorem 3. *Let the conditions* (i)–(iv) *hold,* $\lambda_i(t) \leq 0$ $(i = 1,2,\dots,l)$ *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} F(s)\, ds = \infty,$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} F(s)\, ds = -\infty$$

for every sufficiently large number $t_0 \geq 0$. *Then every solution* $u(x,t)$ *of the problem* (1), (2) *is oscillatory in* G.

PROOF: By Theorem 2, we only need to prove that the inequalities (10) and (11) have no ultimately positive solutions. Assume for the contrary that $V(t)$ is a positive solution of the inequality (10) in $[t_0, \infty)$ for some $t_0 > 0$, then there exists a $t_1 \ge t_0$ such that $V(t) > 0$, $V(\sigma_j(t)) > 0$, $V(\rho_i(t)) > 0$ and $V(t - \tau_k) > 0$ for $t \ge t_1$, $j = 1, 2, \dots, r$; $i = 1, 2, \dots, m$; $k = 1, 2, \dots, l$.

From (10) we have

$$\frac{d}{dt}\big[V(t) - \sum_{i=1}^{l} \lambda_i(t) V(t - \tau_i)\big] \le -F(t), \quad t \ge t_1.$$

Hence

$$V(t) \le \big[V(t) - \sum_{i=1}^{l} \lambda_i(t) V(t - \tau_i)\big] \le -\int_{t_1}^{t} F(s)\, ds + \big[V(t_1) - \sum_{i=1}^{l} \lambda_i(t_1) V(t_1 - \tau_i)\big]$$

for $t \ge t_1$, and moreover

$$\liminf_{t\to\infty} V(t) \le -\limsup_{t\to\infty} \int_{t_1}^{t} F(s)\, ds + \big[V(t_1) - \sum_{i=1}^{l} \lambda_i(t_1) V(t_1 - \tau_i)\big] = -\infty,$$

which contradicts that $V(t)$ is a positive solution of (10) in $[t_0, \infty)$.

It is similar to prove that (11) has no ultimately positive solutions and we omit it. $\square$

Theorem 4. *Let* (i)–(iv) *hold,* $\lambda_i(t) \le 0$ $(i = 1, 2, \dots, l)$, *and*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \Big[\int_{\partial\Omega} (a(y)\psi(x, y) + \sum_{i=1}^{m} a_i(y)\psi(x, \rho_i(y))\, dS\Big]\, dy = \infty$$
$$\liminf_{t\to\infty} \int_{t_0}^{t} \Big[\int_{\partial\Omega} (a(y)\psi(x, y) + \sum_{i=1}^{m} a_i(y)\psi(x, \rho_i(y))\, dS\Big]\, dy = -\infty$$

for every sufficiently large number $t_0 \ge 0$. *Then every solution* $u(x, t)$ *of the problem* (1), (3) *is oscillatory in* G.

Proof: Suppose that the problem (1), (3) has a solution $u(x, t)$ which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 \ge 0$. We may suppose that $u(x, t) > 0$ in $\Omega \times [t_0, \infty)$. Then there exists a $t_1 \ge t_0$ so that $u(x, \rho_i(t)) > 0$, $u(x, \sigma_j(t)) > 0$ and $u(x, t - \tau_k) > 0$ in $\Omega \times [t_0, \infty)$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, r$; $k = 1, 2, \dots, l$.

Integrating (1) over Ω, we have

$$\begin{aligned}\frac{d}{dt}\Big[\int_\Omega u(x, t)\, dx - \sum_{k=1}^{l} \lambda_k(t) \int_\Omega u(x, t - \tau_k)\, dx\Big] &= a(t) \int_\Omega \triangle\, u\, dx \\ &+ \sum_{i=1}^{m} a_i(t) \int_\Omega \triangle\, u(x, \rho_i(t))\, dx - \sum_{j=1}^{r} \int_\Omega q_j(x, t) u(x, \sigma_j(t))\, dx\end{aligned} \tag{13}$$

for $t \geq t_1$.

Green's formula yields

$$\int_\Omega \triangle\, u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial N}\, dS = \int_{\partial\Omega} \psi\, dS, \tag{14}$$

$$\int_\Omega \triangle\, u(x,\rho_i(t))\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial N}(x,\rho_i(t))\, dS = \int_{\partial\Omega} \psi(x,\rho_i(t))\, dS, \quad i = 1,2,\ldots,m, \tag{15}$$

$$\int_\Omega q_j(x,t)u(x,\sigma_j(t))\, dx \geq q_j(t) \int_\Omega u(x,\sigma_j(t))\, dx, \quad j = 1,2,\ldots,r. \tag{16}$$

Then from (13)–(16) it follows that for $t \geq t_1$

$$\begin{aligned} &\frac{d}{dt}\big[H(t) - \sum_{k=1}^{l} \lambda_k(t) H(t-\tau_k)\big] + \sum_{j=1}^{r} q_j(t) H(\sigma_j(t)) \\ &\qquad \leq a(t) \int_{\partial\Omega} \psi\, dS + \sum_{i=1}^{m} a_i(t) \int_{\partial\Omega} \psi(x,\rho_i(t))\, dS, \end{aligned} \tag{17}$$

where

$$H(t) = \int_\Omega u(x,t)\, dx, \quad t \geq t_1.$$

Thus we have from (17) that

$$\begin{aligned} &\frac{d}{dt}\big[H(t) - \sum_{k=1}^{l} \lambda_k(t) H(t-\tau_k)\big] \\ &\qquad \leq a(t) \int_{\partial\Omega} \psi\, dS + \sum_{i=1}^{m} a_i(t) \int_{\partial\Omega} \psi(x,\rho_i(t))\, dS, \quad t \geq t_1. \end{aligned}$$

Hence

$$\begin{aligned} H(t) &\leq H(t) - \sum_{k=1}^{l} \lambda_k(t) H(t-\tau_k) \\ &\leq \int_{t_1}^{t} \big[\int_{\partial\Omega} (a(y)\psi(x,y) + \sum_{i=1}^{m} a_i(y)\psi(x,\rho_i(t)))\, dS\big]\, dy \\ &\quad + H(t_1) - \sum_{k=1}^{l} \lambda_k(t_1) H(t_1 - \tau_k), \quad t \geq t_1, \end{aligned}$$

and

$$\liminf_{t\to\infty} H(t) \leq \liminf_{t\to\infty} \int_{t_1}^{t} \big[\int_{\partial\Omega} (a(y)\psi(x,y) + \sum_{i=1}^{m} a_i(y)\psi(x,\rho_i(y)))\, dS\big]\, dy = -\infty,$$

which contradicts that $H(t) = \int_\Omega u(x,t)\, dx > 0$ for $t \geq t_1$. This completes the proof of the theorem. □

Example 1. We consider the problem

$$\frac{\partial}{\partial t}\big[u - e^{-t}u(x,t-\frac{\pi}{2})\big] = u_{xx} + e^{\pi}u_{xx}(x,t-\pi) - (e^t + e^{-\frac{\pi}{2}})e^{\frac{\pi}{2}-t}u(x,t-\frac{\pi}{2}) - (e^t - e^{-\frac{\pi}{2}})e^{\pi - t}u(x,t-\pi), \quad (x,t) \in (0,\pi)\times[0,\infty), \tag{18}$$

$$u(0,t) = u(\pi,t) = 3e^t\cos t, \quad t \geq 0. \tag{19}$$

Here $\phi(x,t) = 3e^t\cos t$, $\Omega = (0,\pi)$, $a(t) = 1$, $a_1(t) = e^{\pi}$, $\rho_1(t) = t-\pi$ and the eigenvalue problem

$$\begin{cases} \triangle\,\omega + \lambda\omega = 0 & \text{in } (0,\pi) \\ \omega = 0 & x = 0,\ x = \pi, \end{cases}$$

has an eigenvalue $\lambda_1 = 1$ and the corresponding eigenfunction $\Phi(x) = \sin x > 0$ in $(0,\pi)$. Hence

$$\begin{aligned} F(t) &= \int_{\partial\Omega}\big[a(t)\phi(x,t) + a_1(t)\phi(x,\rho_1(t))\big]\frac{\partial\Phi}{\partial N}\,dS \\ &= \big[1\cdot 3e^t\cos t + e^{\pi}\cdot 3e^{t-\pi}\cos(t-\pi)\big](1) \\ &+ \big[1\cdot 3e^t\cos t + e^{\pi}\cdot 3e^{t-\pi}\cos(t-\pi)\big](-1) \\ &= 0, \end{aligned}$$

and from (18) and (19) we have

$$\frac{d}{dt}\big[V(t) - e^{-t}V(t-\frac{\pi}{2})\big] + V(t) + e^{\pi}V(t-\pi) + (e^t + e^{-\frac{\pi}{2}})e^{\frac{\pi}{2}-t}V(t-\frac{\pi}{2}) + (e^t - e^{-\frac{\pi}{2}})e^{\pi-t}V(t-\pi) \leq 0 \tag{20}$$

for $t \geq 0$.

Since

$$\liminf_{t\to\infty}\int_{t-\frac{\sigma_i}{2}}^{t} h_i(t)\,dt > 0, \quad i = 2,3,4,$$

here $h_1(t) = 1$, $h_2(t) = e^{\pi}$, $h_3(t) = (e^t + e^{-\frac{\pi}{2}})e^{\frac{\pi}{2}-t}$, $h_4(t) = (e^t - e^{-\frac{\pi}{2}})e^{\pi-t}$, $\sigma_2 = \pi$, $\sigma_3 = \frac{\pi}{2}$, $\sigma_4 = \pi$, and

$$\liminf_{t\to\infty}\int_{t-\sigma_2}^{t} h_2(t)\,dt = \liminf_{t\to\infty}\int_{t-\pi}^{t} e^{\pi}\,dt > \frac{1}{e},$$

it follows that (20) has no ultimately positive solutions by [1, Theorem 3]. Then we find that the conditions of Theorem 2 are satisfied. It follows from Theorem 2 that every solution u of (18), (19) is oscillatory in $(0,\pi)\times(0,\infty)$. One such solution is $u(x,t) = e^t\cos t\sin x + 3e^t\cos t$.

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