Small functions and iterative methods

Michal Fečkan

Abstract. Iterative methods based on small functions are used both to show local surjectivity of certain operators and a fixed point property of mappings on scales of complete metric spaces.

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1. Introduction.

This paper will be concerned with the study of an equation $F(x) = y$. We shall use iteration methods based on small functions introduced by V. Pták [1] .

In the first part of Section 2, we present two theorems which guarantee local surjectivity for certain operators. In the second part, we generalize Pták's nondiscrete mathematical induction on scales of metric spaces. Then we use this result to a generalized implicit function theorem [2]; [3] obtaining a relatively simple proof.

2. Results.

Firstly we give two theorems ensuring surjectivity of certain operators.

Theorem 2.1. *Let $F \in C^1(B_1, Y)$, where $B_1 = \{x \in X, |x| \leq 1\}$, X, Y are Banach spaces, $m > 0$, $y_1 \in Y$, and $A\colon B_1 \to L(X,Y)$ be such that*

i) *The inverse $A^{-1}(x)$ of $A(x)$ exists for any $x \in B_1$, $|A^{-1}(x)| \leq c$, and the mapping A is Lipschitz.*
ii) $|y_1 - y_0| \leq m$, $y_0 = F(0)$.
iii) $|DF(x) - A(x)| \leq g(|x|)$ *for all* $x \in B_1$.
iv) *g is nondecreasing on $[0,1]$,* $c.m + c.r(c.m) + c.r(c.r(c.m)) + \cdots < 1$, *where* $Dr = g,\ r(0) = 0$.

Then there exists an $x \in B_1$ such that $F(x) = y_1$.

Proof: We shall use the continuation method. Consider the following equation

$$x' = A^{-1}(x)(y_1 - y_0),$$
$$x(0) = 0.$$

We have $|x(t)| \leq \int_0^t c.m \leq t.c.m \leq c.m < 1$. Hence $x(t) \in B_1$ for each $t \in [0,1]$. Since

$$\frac{d}{dt}F(x(t)) = DF(x).x'(t) = DF(x).A^{-1}(x)(y_1 - y_0),$$

$$F(x(1)) = y_0 + \int_0^1 \frac{d}{dt}F(x(t))\,dt = y_0 + \int_0^1 DF(x).A^{-1}(x)(y_1 - y_0)\,dt.$$

Hence

$$\begin{aligned}|F(x(1)) - y_1| =& |y_0 - y_1 + \int_0^1 DF(x).A^{-1}(x)(y_1 - y_0)\,dt| \\ &\leq \int_0^1 |y_0 - y_1 + DF(x).A^{-1}(x)(y_1 - y_0)|\,dt \\ &\leq \int_0^1 g(|x(t)|).c.|y_0 - y_1|\,dt \\ &\leq \int_0^1 g(t.c.m).c.m\,dt = r(c.m).\end{aligned}$$

Finally

$$|F(x(1)) - y_1| \leq r(c.m) \text{ and } |x(1)| \leq c.m < 1.$$

For $x(1)$, $F(x(1)) = z_1$, we can apply the above method to obtain $x(2)$, $F(x(2)) = z_2$ with the properties

$$\begin{aligned}&|z_2 - y_1| \leq r(c.r(c.m)),\\ &|x(2) - x(1)| \leq c.r(c.m).\end{aligned}$$

By the assumptions of the theorem we have

$$c.m + c.r(c.m) + c.r(c.r(c.m)) + \cdots < 1$$

and hence $x(n) \to x \in B_1$ and $F(x) = y_1$, where $x(n)$ is constructed by the induction in the above way. This completes the proof. □

We recall that a function w on $T = \{t,\ 0 < t < t_0\}$ is a *small function* [1, p. 224] if $w\colon T \to T$ and the sum

$$W(t) = t + w(t) + w(w(t)) + w(w(w(t))) + \cdots$$

is finite for each $t \in T$.

Now we shall study the local surjectivity of operators on more general spaces.

Theorem 2.2. *Let X be a Fréchet space with the F-norm $|.|$ and we consider a mapping $g\colon U \to X$, $U = \{x \in X, |x| < \delta\}$ with the following properties*

a) *g is continuous.*

b) $|g(x) - g(y) - (x - y)| \le w(|x - y|)$,

where $w\colon R_+ = [0, \infty) \to R_+$, $w(0) = 0$, and w is a nondecreasing small function such that $W(\delta_1) < \delta$ for some $\delta_1, 0 < \delta_1 < \delta$.

Then there is a neighbourhood V of 0 such that the restriction g/V is an open mapping.

Proof: We define a sequence $\{z_k\}$ as follows:

$$
\begin{aligned}
&z_1 = x \text{ for } |x| < \delta - W(\delta_1),\\
&z_{i+1} = z_i + (y - g(z_i)) \text{ for } |y - g(x)| = h_1 < \delta_1.
\end{aligned}
$$

We see that for $i > 1$

$$
\begin{aligned}
|z_{i+1} - z_i| = |y - g(z_i)| &= |y - g(z_{i-1}) + g(z_{i-1}) - g(z_i)|\\
&= |z_i - z_{i-1} + g(z_{i-1}) - g(z_i)| \le w(|y - g(z_{i-1})|).
\end{aligned}
$$

Hence for $i \ge 1$

$$
\begin{aligned}
&|z_{i+1}| \le |z_i| + |y - g(z_i)|,\\
&|z_{i+1} - x| \le |z_i - x| + |y - g(z_i)|.
\end{aligned}
$$

Putting $h_i = |y - g(z_i)|$ we obtain

$$
\begin{aligned}
&h_i \le w(h_{i-1}),\\
&|z_{i+1}| \le |x| + h_1 + h_2 + \cdots + h_i \le |x| + W(h_1),\\
&|z_{i+1} - x| \le h_1 + h_2 + \cdots + h_i \le W(h_1).
\end{aligned}
$$

We have shown that $B(g(x), \delta_1) \subset g(B(x, W(\delta_1)))$, where $B(z, r) = \{x \in X, |x - z| < r\}$. This ends the proof. $\square$

Now we shall generalize the meaning of small functions. Let T be a positive real number and $w(t, s) = (w_1(t, s), w_2(t, s))$ be a mapping of a set $(0, T) \times S$ into itself, where S is a metric space and $w_1(t, s) \in R$, $w_2(t, s) \in S$. For any nonnegative integer n define

$$
\begin{aligned}
s_{n+1} &= w_2(t_n, s_n),\ t_0 = t,\\
t_{n+1} &= w_1(t_n, s_n),\ s_0 = s.
\end{aligned}
$$

We put $W(t, s) = t_0 + t_1 + \cdots$, and the function w is said to be small on $(0, T') \times S$, $(0 < T' \le T)$ if $W(t, s) < \infty$ and $\{s_n\}$ is a convergent sequence in S for all $(t, s) \in (0, T') \times S$. If $S = \{x\}$ then we obtain the definition from [1] .

Now we consider a scale of complete metric spaces $(X_s, d_s)_{s\in S}$ with the following properties:
for each $s \in S$ there exists $s' \in S$ and a neighbourhood $U(s) \subset S$, $s \in U(s)$ such that for each $s_1 \in U(s)$ it holds

$$(+) \qquad X_{s'} \supset X_{s_1},\ d_{s'}(x,y) \le d_{s_1}(x,y) \text{ for } x, y \in X_{s_1}.$$

Let I be an interval of the form $I = \{t,\ 0 < t < t_0\}$ for a positive t_0. For each $t \in I$ and $s \in S$ let $Z(t,s)$ be a subset of X_s. Suppose that these sets are not all empty on each orbit $\{(t_i, s_i)\}$ of w and

$$\begin{aligned} Z(t,s) &\subset U_{w_2(t,s)}(Z(w_1(t,s), w_2(t,s)), t) \\ &= \{x \in X_{w_2(t,s)},\ d_{w_2}(Z(w_1,w_2),x) < t\}. \end{aligned}$$

We have the following theorem:

Theorem 2.3. *Suppose that the subsets $Z(t,s)$ have the above properties and, moreover, $w = (w_1, w_2)$ is small (see the above definition of smallness). Further, for s_0, which is a limit of $\{s_n\}$, we find $s_0' \in S$ by the property $(+)$. Then*

$$Z(0) = \bigcap_{t>0, n\ge 1} \overline{\cup_{r<t} \cup_{p>n} Z(r, s_p)}$$

exists in $X_{s_0'}$ and

$$Z(t_i, s_i) \subset U_{s_0'}(Z(0), W(t,s))$$

for $i \gg 1$.

Proof: If $z \in Z(t_i, s_i) \subset U_{s_{i+1}}(Z(t_{i+1}, s_{i+1}), t_i)$ then there exists

$$z_1 \in Z(t_{i+1}, s_{i+1})$$

such that

$$d_{s_{i+1}}(z, z_1) < t_i.$$

In the same way we can find $z_2 \in Z(t_{i+2}, s_{i+2})$ such that $d_{s_{i+2}}(z_1, z_2) < t_{i+1}$. If we consider that $s_i \in U(s_0)$ for $i \ge i_0$ then

$$\begin{aligned} d_{s_0'}(z, z_2) &\le d_{s_0'}(z, z_1) + d_{s_0'}(z_1, z_2) \\ &\le d_{s_{i+1}}(z, z_1) + d_{s_{i+2}}(z_1, z_2) \\ &< t_i + t_{i+1}. \end{aligned}$$

By the induction we have $\{z_n\}_1^\infty \subset X_{s_0'}$ such that

$$d_{s_0'}(z, z_n) < W(t,s) \text{ and } z_n \in Z(t_{i+n}, s_{i+n}).$$

The proof is finished. $\square$

Corollary 2.4. *Let $(X_s, |.|_s)$, $(0 < s < \infty)$ be a scale of Banach spaces such that $X_s \supset X_{s'}$, $|u|_s \leq |u|_{s'}$ for $s' \geq s$. Let $Z(t_i, s_i)$ be subsets of X_{s_i}, $0 < s < s_{i+1} \leq s_i$, $i = 1, 2, \cdots$ and suppose*

$$\emptyset \neq Z(t_i, s_i) \subset U_{t_{i+1}}(Z(t_{i+1}, s_{i+1}), t_i),$$
$$t_1 + t_2 + \cdots < \infty$$

for a sequence $\{t_i\}$. Then $\emptyset \neq Z(0) \subset X_s$.

Theorem 2.5. *Let $(X_s, d_s)_{s\in S}$ be a scale with the above properties and let $G\colon \cup X_s \to \cup X_s = X$ be a mapping such that*

(1) *X has a complete metric d such that $d(x, y) \leq d_s(x, y)$ for each s, $x, y \in X_s$.*
(2) *$G\colon X_s \to X$ is continuous*
(3) *There exist $g\colon S \to S$, $w\colon T \times S \to T$ such that for each $s \in S$ it holds $G\colon X_s \to X_{g(s)}$, $X_s \subset X_{g(s)}$, $d_{g(s)}(Gx, Gy) \leq w(d_s(x, y), s)$ and $w(., s)$ is increasing.*

If $(g(.), w(., g(.)))$ is small, i.e., $\{s_n\}$ is convergent in S and $t_1 + t_2 + \cdots < \infty$ for any $s_{n+1} = g(s_n)$, $t_{n+1} = w(t_n, s_{n+1})$. Then $G\colon X \to X$ has a fixed point.

Proof: Put $Z(t, s) = \{x \in X_s, d_{g(s)}(x, Gx) < t\} \subset X_s \subset X_{g(s)}$. By the assumptions of the theorem we obtain: if $x \in Z(t, s)$ then

$$d_{g(s)}(x, Gx) < t \text{ and}$$
$$d_{g^2(s)}(Gx, G^2x) \leq w(d_{g(s)}(Gx, x), g(s)) < w(t, g(s)).$$

Thus

$$Z(t, s) \subset U_{g(s)}(Z(w(t, g(s)), g(s)), t).$$

Applying the above theorem we obtain the proof. □

Finally, we give a proof of a generalized implicit function theorem from [3] based on the above results.

Theorem 2.6. *For each number s, $0 \leq s \leq 1$, we are given a Banach space $(Y_s, |.|_s)$ with the following properties:*
1. *$Y_{s'} \supset Y_s$, and $|.|_{s'} \leq |.|_s$ for $s' \leq s$.*
2. *Let R be a positive number and set $R_s = \{u \in Y_s, |u|_s < R\}$. Let f be a mapping defined on R_0 with values in Y_0 such that f maps each R_s into Y_0. Suppose that the following conditions are satisfied:*
a) *$f\colon R_s \to Y_0$ is continuous for each s.*
b) *For each $u \in \cup_{0<s} R_s$ there exists a mapping $f'(u)\colon \cup_{s>0} Y_s \to \cup_{s>0} Y_s$ such that for each $s' < s$, $u \in R_s$ implies $f'(u)Y_s \subset Y_{s'}$, and*

$$|f(u + v) - f(u) - f'(u)v|_{s'} \leq K_1(s - s').|v|_s^2$$

whenever u and $u+v$ belong to R_s.
c) *If $u \in R_s$, there exists $v \in \cap_{s'<s} Y_{s'}$ such that for each $s' < s$*

$$|f'(u)v + f(u)|_s \le K_2(s-s').|f(u)|_{s'}^2,\ |v|_{s'} \le K_3(s-s').|f(u)|_s,$$

where K_i are positive nonincreasing functions defined on the interval (0,1].
If there are $S(1) \in R_+$ and $s_n \in (0,1)$, $(0 < s < s_{n+1} < s_n)$ such that for

$$\begin{aligned} K(n) &= K_1(w_n).K_3^2(w_n) + K_2(w_n), \\ \tilde{K}_3(n) &= K_3(w_n), \\ w_n &= (s_n - s_{n+1})/2, \end{aligned}$$

the following expression comes true:

$$\begin{aligned} &\tilde{K}_3(1)S(1) + \sum_{1<i} \tilde{K}_3(i).S^{2^{i-1}}(1).K^{2^{i-2}}(1)\cdots K(i-1) < R - |u_0|_{s_1}, \\ &|f(u_0)|_{s_1} < S(1),\ u_0 \in R_{s_1}. \end{aligned}$$

Then there exists $u \in R_s$ such that $f(u) = 0$.

Proof: We follow the proof from [2]; [3]. We put for $i \in N \cup \{\infty\}$

$$\begin{aligned} k(i) &= k(i+1) + \tilde{K}_3(i).S(i),\ k(\infty) = 0, \\ S(i+1) &= K(i).S^2(i). \end{aligned}$$

The above condition guarantees that such $k(i)$ exists for each i. We set

$$Z(i, s_i) = \{u \in R_{s_i},\ |f(u)|_{s_i} < S(i) \text{ and } |u - u_0|_{s_i} < R - |u_0|_{s_1} - k(i) - d\}$$

for each $i \in N$ and $d > 0$ small fixed. Then using the same arguments as in [2]; [3] we obtain: for each $u \in Z(i, s_i)$ there exists $\tilde{u} \in R_{s_{i+1}}$ with the properties

$$|f(\tilde{u})|_{s_{i+1}} < K(i).S^2(i),\ |\tilde{u} - u|_{s_{i+1}} < \tilde{K}_3(i)S(i).$$

Indeed, by c) there is $v \in \cap_{s'<s_i} Y_{s'}$ such that

$$\begin{aligned} &|f'(u)v + f(u)|_{s_i} \le K_2(w_i).|f(u)|_{(s_i+s_{i+1})/2}^2, \\ &|v|_{(s_i+s_{i+1})/2} \le K_3(w_i).|f(u)|_{s_i}. \end{aligned}$$

Since

$$\begin{aligned} &|v|_{(s_i+s_{i+1})/2} \le \tilde{K}_3(i).|f(u)|_{s_i} < \tilde{K}_3(i).S(i), \\ &|u|_{(s_i+s_{i+1})/2} \le |u - u_0|_{s_i} + |u_0|_{s_1} < R - k(i), \end{aligned}$$

thus

$$|u+v|_{(s_i+s_{i+1})/2} < \tilde{K}_3(i).S(i) + R - k(i) = R - k(i+1) < R,$$

i.e.,

$$u + v \in R_{(s_i+s_{i+1})/2}.$$

According to b) we have

$$|f(u+v) - f(u) - f'(u)v|_{s_{i+1}} \leq K_1(w_i).|v|^2_{(s_i+s_{i+1})/2}.$$

Thus

$$\begin{aligned}
|f(u+v)|_{s_{i+1}} &\leq K_2(w_i).|f(u)|^2_{(s_i+s_{i+1})/2} + K_1(w_i).K_3^2(w_i).|f(u)|^2_{s_i} \\
&\leq \big(K_2(w_i) + K_1(w_i).K_3^2(w_i)\big).|f(u)|^2_{s_i} \\
&< K(i).S^2(i), \\
|v|_{s_{i+1}} &< K_3(w_i).S(i) = \tilde{K}_3(i).S(i).
\end{aligned}$$

We take $\tilde{u} = u + v$.

Hence

$$|\tilde{u} - u_0|_{s_{i+1}} < R - |u_0|_{s_1} - k(i) + \tilde{K}_3(i).S(i) - d \leq R - |u_0|_{s_1} - k(i+1) - d.$$

Finally, we have

$$Z(i, s_i) \subset U_{s_{i+1}}(Z(i+1, s_{i+1}), \tilde{K}_3(i).S(i)).$$

Since $\sum_i \tilde{K}_3(i).S(i) < \infty$, we can apply Corollary 2.4. From the definition of $Z(0)$ we have: $u \in Z(0) \Rightarrow |u - u_0|_s \leq R - |u_0|_{s_1} - d$. Thus $|u|_s < R$. We conclude by a) that: $u \in Z(0) \subset R_s \Rightarrow f(u) = 0$. The proof is completed. $\square$

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Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, Bratislava, Czechoslovakia

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