Problems with nonlinear boundary value conditions

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Abstract. The existence and multiplicity results are shown for certain types of problems with nonlinear boundary value conditions.

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Introduction.

The purpose of this paper is to study several problems with nonlinear boundary value conditions. Mostly we study problems which are small perturbations of linear boundary value problems. The author was stimulated by the paper [1]; but in this paper we shall use several approaches to solve our problems: the implicit function theorem, the Mawhin coincidence degree theory, the Nielsen fixed point theory and when an unperturbed linear boundary value condition is a periodic one, we derive a Melnikov function for this problem [2].

Results.

We study

$$\begin{aligned} &x' = f_1(x) + \varepsilon . f_2(t,x) \\ &Ax(0) + Bx(T) = \varepsilon . \phi(x(0), x(T)), \end{aligned} \tag{$1-\varepsilon$}$$

where f_1, f_2, ϕ are continuous on R^m, $R \times R^m$, $R^m \times R^m$, respectively, $A, B \in \mathcal{L}(R^m)$, $T > 0$, $\varepsilon \in R$ is small.

Theorem 1. *Let us consider* $(1-\varepsilon)$ *for the case when* $f_1(c_0) = 0$ *for some* $c_0 \in R^m$, $Ac_0 + Bc_0 = 0$, f_1, f_2, ϕ *are* C^1*-smooth. If* $\det(A + B.e^{Df_1(c_0).T}) \neq 0$, *then* $(1-\varepsilon)$ *has a solution* x_ε *defined on* $[0,T]$ *for each* ε *small satisfying* $x_\varepsilon(.) \to c_0$ *as* $\varepsilon \to 0$.

Proof: We consider

$$\begin{aligned} &F_\varepsilon \colon C^1 \to C^0 \times R^m \\ &F_\varepsilon(x) = (x' - f_1(x) - \varepsilon f_2(t,x), Ax(0) + Bx(T) - \varepsilon\phi(x(0), x(T))). \end{aligned}$$

We see that

$$\begin{aligned} &F_0(c_0) = 0 \\ &D_x F_0(c_0)v = (v' - Df_1(c_0)v, Av(0) + Bv(T)). \end{aligned}$$

Thus $\ker D_xF_0(c_0) = \{v \mid v' = Df_1(c_0)v, Av(0) + Bv(T) = 0\}$ and by using our assumptions we have $\ker D_xF_0(c_0) = \{0\}$.

Let us solve

$$\begin{aligned} v' - Df_1(c_0)v &= r \\ Av(0) + Bv(T) &= w. \end{aligned}$$

Then $v(t) = \int\limits_0^t e^{Df_1(c_0)(t-s)} r(s)\, ds + e^{Df_1(c_0)t}c$. Hence

$$Ac + B.e^{Df_1(c_0)T}c = -B \int\limits_0^T e^{Df_1(c_0)(T-s)} r(s)\, ds$$

and the last equation we can solve in c. This completes the proof, since we can use the implicit function theorem. $\square$

Corollary 2. *Let $f_1 \equiv 0$. Then the conditions of Theorem 1 are $c_0 = 0$, $A + B$ is invertible.*

Now we consider $(1-\varepsilon)$ for the case $A = -B = Id$ and $x' = f_1(x)$ has an isolated T-periodic nonconstant solution $x_0(.)$. Hence $(1-\varepsilon)$ has the form

$$\begin{aligned} x' &= f_1(x) + \varepsilon.f_2(t, x) \\ x(0) - x(T) &= \varepsilon.\phi(x(0), x(T)). \end{aligned} \tag{$2-\varepsilon$}$$

Let ϕ, f_1, f_2 be C^2-smooth mappings. We note that $(2-0)$ has the family of solutions $\Gamma = \{x_0(. + c), c \in [0, T]\}$. We are interested in bifurcations of solutions of $(2-\varepsilon)$ from Γ for ε small. We apply the following theorem from [5, pp. 397]:

Theorem 3. *Let $F_\varepsilon\colon X \to Y$ be a C^2-smooth mapping, X, Y are Hilbert spaces and F_0 possesses a compact C^2-manifold $\mathcal{M}$ such that $F_0(\mathcal{M}) = 0$, $\ker D_xF_0(m) = T_m\mathcal{M}$, index $D_xF_0(m) = 0$, $D_xF_0(m)$ is a Fredholm operator for each $m \in \mathcal{M}$. Here $T_m\mathcal{M}$ is the tangent space of $\mathcal{M}$ at m. Let $P(m) \in \mathcal{L}(Y)$ be the orthogonal projection onto $(\mathrm{im}\, D_xF_0(m))^\perp$ for each $m \in \mathcal{M}$. Consider the map $M(m) = P(m).D_\varepsilon F_0(m)$, $M\colon \mathcal{M} \to Y$. If there is a $m_0 \in \mathcal{M}$ such that $M(m_0) = 0$, $DM(m_0)$ is injective. Then for any ε small the equation $F_\varepsilon(m) = 0$ has a solution near $\mathcal{M}$. We note that M can be considered as a map from $R^{\dim \mathcal{M}}$ into $R^{\dim \mathcal{M}}$ in the local coordinates.*

We shall derive M for the special case $(2-\varepsilon)$. We put $X = H^1([0, T], R^m)$, $Y = H^0([0, T], R^m) \times R^m$, $F_\varepsilon(x) = (x' - f_1(x) - \varepsilon.f_2(t, x), x(0) - x(T) - \varepsilon.\phi(x(0), x(T)))$ and $\mathcal{M} = \{x_0(. + c), c \in [0, T]\}$. Hence $\mathcal{M}$ is homeomorphic to a circle and

$$D_xF_0(m)v = (v' - Df_1(x_0(. + c)).v, v(0) - v(T)).$$

Since x_0 is an isolated T-periodic nonconstant solution of $(2-0)$ we have ker $D_xF_0(m) = T_m\mathcal{M}$ for each $m \in \mathcal{M}$. Now we derive im $D_xF_0(m)$ and thus let us solve

$$\begin{aligned} v' - Df_1(x_0(.+c))v &= r \\ v(0) - v(T) &= v_1 \in R^m. \end{aligned}$$

We put $w(t) = v(t) + \frac{t.v_1}{T}$, hence

$$\begin{aligned} &w' - Df_1(x_0(t+c))w = r + \frac{v_1}{T} - Df_1(x_0(t+c)).t.\frac{v_1}{T} \\ &w(0) = w(T). \end{aligned}$$

It is well-known that this equation has a solution if and only if

$$\int_0^T \tilde{x}_0(s+c).(r(s) + \frac{v_1}{T} - Df_1(x_0(s+c)).s.\frac{v_1}{T})\,ds = 0,$$

where $\tilde{x}_0$ is a nonzero T-periodic solution of $x' + (Df_1(x_0))^\top x = 0$.

Hence $(r, v_1) \in$ im $D_xF_0(x_0(.+c))$ if and only if

$$\begin{aligned} &\langle w(c), (r, v_1)\rangle_Y = \\ &\quad = \int_0^T \tilde{x}_0(s+c).r(s)\,ds + \frac{1}{T}\int_0^T \tilde{x}_0(s+c)(v_1 - Df_1(x_0(s+c))s.v_1)\,ds = 0, \end{aligned}$$

where $\langle .,.\rangle_Y$ is the scalar product on Y. Then

$$P(x_0(.+c))w_1 = \langle w(c), w_1\rangle_Y.\frac{1}{\| w(c) \|_Y}.w(c)$$

and

$$M(c) = \langle w(c), (-f_2(.,x_0(.+c)), -\phi(x_0(.+c), x_0(.+c)))\rangle_Y / \| w(c) \|_Y .w(c).$$

Now we shall use the fact: let $\frac{a(c)}{b(c)} = d(c)$, where a, b, d are real smooth functions, $b(c_0) \neq 0$. Then for $a(c_0) = 0$ it follows $d'(c_0) \neq 0$ if and only if $a'(c_0) \neq 0$. Thus instead of $M(c)$ we can consider the map

$$\begin{aligned} \bar{M}(c) =& \langle w(c), (f_2(.,x_0(.+c)), \phi(x_0(.+c), x_0(.+c)))\rangle_Y = \\ =& \int_0^T \tilde{x}_0(s+c).f_2(s, x_0(s+c))\,ds + \\ &+ \frac{1}{T}\int_0^T \tilde{x}_0(s+c).(\phi(x_0(s+c), x_0(s+c)) - \\ &- Df_1(x_0(s+c))s.\phi(x_0(s+c), x_0(s+c)))\,ds. \end{aligned}$$

Summing up we obtain

Theorem 4. *If there is $c_0 \in [0,T]$ such that $\bar{M}(c_0) = 0$, $\bar{M}'(c_0) \neq 0$, then for each ε small, $(2-\varepsilon)$ has a solution on $[0,T]$.*

Remark 5. We see that for $\phi \equiv 0$ $\bar{M}$ is the subharmonic Melnikov function [2] and thus $\bar{M}$ we can consider as a Melnikov function for $(2-\varepsilon)$.

Now we consider

$$\begin{aligned} &x' = f(t,x) \\ &Ax(0) + Bx(T) = \phi(x(0), x(T)), \end{aligned} \tag{3}$$

where f, ϕ are continuous. Let $G \subset R^m$ be an open bounded subset, $0 \in G$.

Theorem 6. *Assume that*

(i) $x' = \lambda f(t,x)$, $Ax(0) + Bx(T) = \lambda\phi(x(0), x(T))$ *has no solution for each* $\lambda \in (0,1)$ *satisfying*
$x(.) \subset \bar{G}$, $x(.) \cap \partial G \neq \emptyset$.

Moreover

(ii) $D = \{z \in R^m \mid Az + Bz = 0, z \in G\} \neq \{0\}$, $g(z) \neq 0$ *for each* $z \in \partial D$, *where*
$g(z) = JP(\phi(z,z) - B.\int_0^T f(s,z)\,ds)$
here $P\colon R^m \to (\operatorname{im}(A+B))^{\perp}$ *is a projection and*
$J\colon (\operatorname{im}(A+B))^{\perp} \to \{z, Az + Bz = 0\}$ *is an isomorphism.*

(iii) $\deg(g, D, 0) \neq 0$.

Then (3) *has a solution* x, $x(.) \subset G$.

Proof: We shall apply a theorem of Mawhin [3, p. 41]. We put

$$\begin{aligned} &X = C^0([0,T], R^m),\ Y = X \times R^m \\ &Lx = (x', Ax(0) + Bx(T)) \\ &N(x) = (f(.,x), \phi(x(0), x(T))) \\ &\Omega = \{x \in X, x(.) \in G\}. \end{aligned}$$

By our assumptions $Lx = \lambda N(x)$, $\lambda \in (0,1)$ has no solution on $\partial\Omega$. We compute $\ker L = \{x' = 0,\ Ax(0) + Bx(T) = 0\} = \{x \mid x = \text{constant} = c_1,\ Ac_1 + Bc_1 = 0\}$. Now $\operatorname{im} L = \{(v,w) \mid x' = v,\ Ax(0) + Bx(T) = w\}$. But

$$x(t) = \int_0^t v(s)ds + c_1,\ Ac_1 + B\int_0^T v(s)\,ds + Bc_1 = w,$$

$$Ac_1 + Bc_1 = w - B.\int_0^T v(s)\,ds.$$

This equation has a solution if and only if

$$P(w - B.\int_0^T v(s)\,ds) = 0.$$

Hence

$$\text{im } L = \{(v, w),\ P(w - B.\int_0^T v(s)\,ds) = 0\}.$$

Thus dim coker im $L = \dim \ker L \neq 0$. We take

$$\bar{P}(v, w) = (0, P(w - B.\int_0^T v(s)\,ds)).$$

Then im $(I - \bar{P}) = \text{im } L$.

Finally consider the map

$$J.\bar{P}.N/\ker L \cap \Omega \to 0 \times R^m \circlearrowleft R^m$$

defined in the following way

$$g(z) = J.P(\phi(z, z) - B.\int_0^T f(s, z)\,ds),\ z \in D.$$

Since $g(z) \neq 0$ for $z \in \partial D$ and $\deg(g, D, 0) \neq 0$ we see that also the last assumption of the theorem of Mawhin is satisfied. The proof is finished. □

Theorem 7. *Let us consider*

$$\begin{aligned} &x' = \varepsilon.f(t, x) \\ &Ax(0) + Bx(T) = \varepsilon.\phi(x(0), x(T)) \end{aligned} \tag{4}$$

and assume the existence of G *as in Theorem* 6 *possessing the properties* (ii), (iii). *Then* (4) *has a solution for each* ε *small.*

Proof: The proof is similar as for Theorem 6. □

Theorem 7 expresses only the existence result. Now we shall apply a theorem of [4] to show a multiplicity result.

Theorem 8 ([4])**.** *Let* $X \subset Y$ *be Banach spaces,* X *is compactly embedded into* Y. *Consider* $Lx = \varepsilon N(x)$, *where* $L\colon X \to Y$ *is continuous, linear, Fredholm with index* $L = 0$, $\ker L \neq \{0\}$ *and* $N\colon Y \to Y$ *maps bounded sets into bounded sets, continuous. Moreover we assume that the map* $\Pi(z) = z + JPN(z)$ *is* μ*-retractible onto* S *with a retraction* π, *where* J *is an isomorphism from* im P *onto* $\ker L$, $P\colon Y \to Y$ *is a projection,* im $(I - P) = \text{im } L$, S *is a compact, nonempty, locally contractible subset of* $\ker L$, $\mu > 0$. *Then the equation* $Lx = \varepsilon.N(x)$ *has for each* ε *small at least* $N(\pi.\Pi)$ *solutions. Here* $N(\pi.\Pi)$ *is the Nielsen number of the map* $\pi.\Pi/S\colon S \to S$.

Theorem 9. *Consider* (4) *and assume that there is S a compact, nonempty, locally contractible subset of* $\{c \in R^m,\ Ac + Bc = 0\} = W$ *and the map*

$$\psi(z) = z + JP(\phi(z,z) - B\int_0^T f(s,z)\,ds),\ \psi\colon W \to W$$

is μ-retractible onto S with respect to π. Then (4) *has at least $N(\pi.\psi)$ solutions for each ε small.* (*The operators J, P are from Theorem* 6.)

PROOF: We put

$$X = C^1([0,T], R^m),\ Y = C^0([0,T], R^m) \times R^m,$$

L, N as in the proof of Theorem 6. It is clear that $\Pi = \psi$ and thus the assertion follows by Theorem 8. □

Example 1. Consider

$$(5-\varepsilon) \qquad \begin{aligned} &x_1' = \varepsilon.f_1(t,x_1,x_2),\ 0 \le t \le T \\ &x_2' = \varepsilon.f_2(t,x_1,x_2) \\ &a_1x_1(0) + a_2x_2(0) = \varepsilon.\phi_1(x_1(0),x_2(0)) \\ &b_1x_1(T) + b_2x_2(T) = \varepsilon.\phi_2(x_1(T),x_2(T)). \end{aligned}$$

In this case

$$A = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}.$$

Hence

$$A + B = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}.$$

Applying Corollary 2 we obtain

Proposition 10. *If $a_1.b_2 - a_2.b_1 \neq 0$ and f_1, f_2, ϕ_1, ϕ_2 are C^1-smooth then $(5-\varepsilon)$ has a solution for each ε small tending to 0 as $\varepsilon \to 0$.*

Consider the case $a_1.b_2 - a_2.b_1 = 0$, $a_1^2 + a_1^2 \neq 0 \neq b_1^2 + b_2^2$. Then (see Theorem 9)

$$\begin{aligned} W =& \{(c_1,c_2) \mid a_1c_1 + a_2c_2 = 0,\ b_1c_1 + b_2c_2 = 0\} \\ &= \{c.(a_2,-a_1),\ c \in R\} \\ &(\mathrm{im}\,(A+B))^\perp = \{c.(b_1,-a_1),\ c \in R\}. \end{aligned}$$

Hence

$$\begin{aligned} &P(v_1,v_2) = \frac{(v_1b_1 - v_2a_1)}{b_1^2 + a_1^2}(b_1,-a_1) \\ &J(c.(b_1,-a_1)) = c.(b_1^2 + a_1^2).(a_2,-a_1). \end{aligned}$$

Thus

$$\begin{aligned} g(c) =& b_1\phi_1(c.a_2, -c.a_1) + b_1.a_1 \int_0^T f_1(s, c.a_2, -c.a_1)\, ds \\ & -a_1.\phi_2(c.a_2, -c.a_1) + b_2.a_1 \int_0^T f_2(s, c.a_2, -c.a_1)\, ds, \end{aligned}$$

since $\dim W = 1$.

Proposition 11. *Let f_1, f_2, ϕ_1, ϕ_2 be continuous and*

$$\limsup_{|c|\to\infty} g(c)/c > 0 \quad \text{or} \quad \liminf_{|c|\to\infty} g(c)/c < 0.$$

Then $(5-\varepsilon)$ has a solution for each ε small.

Proof: The assertion follows by Theorem 7. □

Example 2. Consider

$$(6-\varepsilon) \qquad \begin{aligned} & x_1' = \varepsilon.f_1(t, x_1, x_2),\ 0 \le t \le T \\ & x_2' = \varepsilon.f_2(t, x_1, x_2) \\ & a_1x_1(0) + a_2x_1(T) = \varepsilon.\phi_1(x_1(0), x_2(0)) \\ & b_1x_2(0) + b_2x_2(T) = \varepsilon.\phi_2(x_1(T), x_2(T)). \end{aligned}$$

In this case

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix}.$$

Hence

$$A + B = \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & b_1 + b_2 \end{pmatrix}.$$

According to Corollary 2 we obtain

Proposition 12. *If $(a_1+a_2)(b_1+b_2) \neq 0$ and f_1, f_2, ϕ_1, ϕ_2 are C^1-smooth then $(6-\varepsilon)$ has a solution for each ε small tending to 0 as $\varepsilon \to 0$.*

Let $a_1 = -a_2 \neq 0$, $b_1 + b_2 \neq 0$. Then (see Theorem 9)

$$\begin{aligned} & W = \{(c, 0),\ c \in R\} \\ & P(v_1, v_2) = (v_1, 0) \\ & (\operatorname{im}(A+B))^{\perp} = \{(c, 0),\ c \in R\} \\ & J((c, 0)) = c. \end{aligned}$$

Thus

$$g(c) = \phi_1(c, 0) - a_2. \int_0^T f_1(s, c, 0)\, ds. \tag{7}$$

Proposition 13. *Let f_1, f_2, ϕ_1, ϕ_2 be continuous and*

$$\limsup_{|c|\to\infty} g(c)/c > 0 \quad \text{or} \quad \liminf_{|c|\to\infty} g(c)/c < 0.$$

Then $(6-\varepsilon)$ has a solution for each ε small. Here g is defined by (7).

Lastly, consider $a_1 = -a_2 \neq 0$, $b_1 = -b_2 \neq 0$. Then (see Theorem 9) $W = R^2$, $P = J = Id$ and

$$\psi(c_1, c_2) =$$

$$(c_1 + \phi_1(c_1, c_2) - a_2 \int_0^T f_1(s, c_1, c_2)\, ds, c_2 + \phi_2(c_1, c_2) - b_2 \int_0^T f_2(s, c_1, c_2)\, ds) \tag{8}$$

$$\psi \colon R^2 \to R^2.$$

Applying Theorem 9 we obtain

Proposition 14. *Let f_1, f_2, ϕ_1, ϕ_2 be continuous and S be a compact, locally contractible subset of R^2. If the map ψ defined by (8) is μ-retractible onto S with respect to a retraction π then $(6-\varepsilon)$ has at least $N(\pi.\psi)$ solutions for any ε small.*

To be more concrete we take $S = A_{r,p} = \{z \in R^2, r \leq |z| \leq p\}$ for fixed $0 < r < p$. We have constructed in [4] a family of mappings q for each $m \in \mathcal{N} \setminus \{1\}$ satisfying $N(\rho_{r,p}.q) = m-1$, where $\rho_{r,p}$ is the usual retraction on $A_{r,p}$ (see [4]) and q is μ-retractible onto $A_{r,p}$ with respect to $\rho_{r,p}$ for some $\mu > 0$.

If $T = 1 = a_2 = b_2$ and

$$\begin{aligned} f_i(s, c_1, c_2) &= 2.q_i(c_1, c_2).s \\ \phi_i(c_1, c_2) &= 2.q_i(c_1, c_2) - c_i, \quad i = 1, 2, \end{aligned} \tag{9}$$

where $q = (q_1, q_2)$. Then easy computations show that the map ψ from (8) has the form $\psi = q$ and $\pi = \rho_{r,p}$. Summing up we have

Proposition 15. *Consider the special case (9) of the problem discussed in Proposition 14. Then in this case $(6-\varepsilon)$ has at least $m-1$ solutions for each ε small.*

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