On the preservation of separation axioms in products*

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Abstract. We give sufficient and necessary conditions to be fulfilled by a filter Ψ and an ideal Λ in order that the Ψ-quotient space of the Λ-ideal product space preserves T_k-properties ($k = 0, 1, 2, 3, 3\frac{1}{2}$) ("in the sense of the Łos theorem"). Tychonoff products, box products and ultraproducts appear as special cases of the general construction.

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0. Introduction.

The aim of the paper is to answer the following question: given a family of topological spaces, "sufficiently" many of them having the T_k-property ($k = 0, 1, 2, 3, 3\frac{1}{2}$), under what condition (filter-) quotient space of ideal product space will preserve that property. And while we solve that problem numerous new questions come from our considerations, too obvious that we would find it necessary to state them explicitly.

Through the paper $\{X_i \mid i \in I\}$ will be a nonempty family of nonempty sets. $\mathbf{X}_i$ will be a topological space with "ground" set X_i and topology $\mathcal{O}_i$, arbitrary unless otherwise stated. The notation, so far the well-known notions are considered, is standard.

Preliminaries are given more because of the coherence of the text and introducing some notation.

Our methatheory is ZFC (which does not mean that we need necessarily all its axioms).

1. Preliminaries.

For $\Psi \subseteq P(I)$, $\sim_\Psi$ is a well-known relation on $\prod_{i\in I} X_i$ determined by Ψ:

$$f \sim_\Psi g \ \text{ iff } \ \{i \in I \mid f(i) = g(i)\} \in \Psi.$$

For the sake of brevity we will denote the set $\{i \in I \mid f(i) = g(i)\}$ by $I_{f,g}$. Of course, $I_{f,g} = I_{g,f}$. It holds the following

Lemma 1.1. *For $\Psi \subseteq P(I)$, Ψ is a filter on I if and only if for any family of nonempty sets $\{X_i \mid i \in I\}$, $\sim_\Psi$ is an equivalence relation on $\prod_{i\in I} X_i$.*

For a filter Ψ on I and $f \in \prod_{i\in I} X_i$, $[f]$ will be the equivalence class of the relation $\sim_\Psi$ containing f (i.e. $[f] = \{g \in \prod_{i\in I} X_i \mid f \sim_\Psi g\}$). For $A \subseteq \prod_{i\in I} X_i$ we put $A^* = \bigcup_{f\in A}[f]$.

q will be the canonical mapping from $\prod_{i\in I} X_i$ onto the quotient set $\prod_{i\in I} X_i/\sim_\Psi$.

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Lemma 1.2. *Let A, A_α, $\alpha < \beta$ be subsets of $\prod_{i\in I} X_i$ and let Ψ be a filter on I. Then it holds:*

(a) $(A^*)^* = A^*$;
(b) $(\bigcup_{\alpha<\beta} A_\alpha)^* = \bigcup_{\alpha<\beta} A_\alpha^*$;
(c) $A^* = q^{-1}(q(A))$, $q(A^*) = q(A)$;
(d) *if* $B \subseteq \prod_{i\in I} X_i/\sim_\Psi$, *then* $q^{-1}(B) = (q^{-1}(B))^*$;
(e) *if* $A \subseteq q^{-1}(B)$, *then* $A^* \subseteq q^{-1}(B)$.

Lemma 1.3. *Let Ψ be a filter on I and let $A = \prod_{i\in I} A_i$, $B = \prod_{i\in I} B_i$, $\emptyset \neq A_i, B_i \subseteq X_i$. It holds:*

(a) $A^* \subseteq B^*$ *iff* $\{i \in I \mid A_i \subseteq B_i\} \in \Psi$;
(b) $A^* = B^*$ *iff* $\{i \in I \mid A_i = B_i\} \in \Psi$;
(c) $f \in A^*$ *iff* $\{i \in I \mid f(i) \in A_i\} \in \Psi$;
(d) $A^* = \prod_{i\in I} X_i$ *iff* $\{i \in I \mid A_i = X_i\} \in \Psi$;
(e) $A^* \cap B^* = \emptyset$ *iff* $\{i \in I \mid A_i \cap B_i \neq \emptyset\} \notin \Psi$.

Proof: (a) Assume $A^* \subseteq B^*$, that is $q(A) \subseteq q(B)$ (1.2 (c)) or, in other words, $\forall f \in A\ \exists g \in B\ f \sim_\Psi g$ and let $J = \{i \in I \mid A_i \subseteq B_i\}$ and $f \in A$ be such that $f(i) \in A_i - B_i$ for any $i \in J^c$. By assumption, for some $g \in B$, $I_{f,g} \in \Psi$ and since $I_{f,g} \subseteq J$, $J \in \Psi$ too.

The converse is equally trivial.

(e) Let $A^* \cap B^* = \emptyset$. That is equivalent to $q(A) \cap q(B) = \emptyset$, therefore no element from A is in relation $\sim_\Psi$ with any element from B and, surely, $\{i \in I \mid A_i \cap B_i \neq \emptyset\} \in \Psi$ would be in contradiction with it.

Analogously, $A^* \cap B^* \neq \emptyset$ or, equivalently $q(A) \cap q(B) \neq \emptyset$, implies $\exists f \in A$ $\exists g \in B$ such that $I_{f,g} \in \Psi$ and, clearly, $I_{f,g} \subseteq \{i \in I \mid A_i \cap B_i \neq \emptyset\}$. □

Lemma 1.4. *Let Ψ be a filter on I, $A \in \Psi$, $L \subseteq I$, $B_i \subseteq X_i$ and let π_j be the projection $\prod_{i\in I} X_i \to X_j$. Then:*

$$\Big(\bigcap_{i\in L} \pi_i^{-1}(B_i)\Big)^* = \Big(\bigcap_{i\in L\cap A} \pi_i^{-1}(B_i)\Big)^*.$$

Proof: $\bigcap_{i\in L} \pi_i^{-1}(B_i) = \prod_{i\in I} C_i$, $\bigcap_{i\in L\cap A} \pi_i^{-1}(B_i) = \prod_{i\in I} D_i$,

$$\text{where } \ C_i = \begin{cases} B_i & i \in L \\ X_i & \text{otherwise} \end{cases}, \quad D_i = \begin{cases} B_i & i \in L\cap A \\ X_i & \text{otherwise} \end{cases}.$$

Now, $\{i \in I \mid C_i = D_i\} \supseteq A$ and by 1.3 (b) the assertion follows. □

2. More preliminaries.

Lemma 2.1. *Let $\emptyset \neq \Lambda \subseteq P(I)$. Then it holds: $\forall L_1, L_2 \in \Lambda\ \exists L_3 \in \Lambda\ L_1 \cup L_2 \subseteq L_3$ if and only if for any family of topological spaces $\{(X_i, \mathcal{O}_i) \mid i \in I\}$, $B^\Lambda = \{\bigcap_{i\in L} \pi_i^{-1}(O_i) \mid O_i \in \mathcal{O}_i, L \in \Lambda\}$ is a base for a topology on $\prod_{i\in I} X_i$.*

If $(\emptyset \neq)\Lambda \subseteq P(I)$ satisfies the above condition and $\Lambda\downarrow$ is ideal generated by Λ ($\Lambda\downarrow = \{L \subseteq I \mid L$ is a subset of some set from $\Lambda\}$), then $B^\Lambda = B^{\Lambda\downarrow}$.

Definition 2.2. *Topology on $\prod X_i$ with the base B^Λ, Λ an ideal on I, will be called Λ-ideal product topology and be denoted by $\mathcal{O}^\Lambda$. The topological space $(\prod_{i\in I} X_i, \mathcal{O}^\Lambda)$ will be denoted by $\prod^\Lambda \mathbf{X}_i$ and its quotient space, determined by the mapping $q : \prod_{i\in I} X_i \to \prod_{i\in I} X_i/\sim_\Psi$, by $\prod_\Psi^\Lambda \mathbf{X}_i$, while $\mathcal{O}_\Psi^\Lambda$ will be the corresponding topology.*

From now on, Ψ will be a filter and Λ an ideal on I. Instead of $\prod_{i\in I} X_i$ we will simply write $\prod X_i$ (in general we will not point out any index set if it is clear from the context what set is in question).

Lemma 2.3. *If $O \in \mathcal{O}^\Lambda$ then $O^* \in \mathcal{O}^\Lambda$.*

Proof: Let $B = \bigcap_{i\in L} \pi_i^{-1}(O_i) \in B^\Lambda$ and $f \in B^*$. Then for some $g \in B$, $I_{f,g} \in \Psi$ and $f \in U = \bigcap_{i\in L\cap I_{f,g}} \pi_i^{-1}(O_i) \subseteq U^* = B^*$ (1.4). □

Corollary 2.4. *The mapping q of the topological space $\prod^\Lambda \mathbf{X}_i$ onto the topological space $\prod_\Psi^\Lambda \mathbf{X}_i$ is open.*

Proof: Obviously, q is open if and only if for any $O \in \mathcal{O}^\Lambda$, $q^{-1}(q(O)) = O^*$ is open. □

Lemma 2.5. *Let $A \subseteq I$. Then $\Psi_A = \{B\cap A \mid B \in \Psi\}$ and $\Lambda_A = \{B\cap A \mid B \in \Lambda\}$ are, respectively, filter and ideal on A.*

The corresponding topological spaces (for the index set A) will be denoted by $\prod^{\Lambda_A} \mathbf{X}_i$ and by $\prod_{\Psi_A}^{\Lambda_A} \mathbf{X}_i$.

Lemma 2.6. *Let $A \in \Psi$. Then the mapping ϕ of topological space $\prod_\Psi^\Lambda \mathbf{X}_i$ onto topological space $\prod_{\Psi_A}^{\Lambda_A} \mathbf{X}_i$, defined by $\phi[f] = [f\,|_A]_A$, is a homeomorphism, where*

$$[f\,|_A]_A = \{g \in \prod_{i\in A} X_i | f|_A \sim_{\Psi_A} g\}.$$

Proof: ϕ is obviously a bijection.

Further, let r be mapping of $\prod_{i\in I}^\Lambda \mathbf{X}_i$ onto $\prod_{i\in A}^{\Lambda_A} \mathbf{X}_i$ defined by $r(f) = f\,|_A$, and let q_A be the canonical mapping of $\prod_{i\in A}^{\Lambda_A} \mathbf{X}_i$ onto $\prod_{\Psi_A}^{\Lambda_A} X_i$. Trivially, r is continuous and open, whence ϕ is continuous and open, too. For ϕ is continuous (open) if and only if $\phi\circ q = q_A \circ r$ is continuous (open). □

3. The $(\Lambda\Psi)$-condition.

Definition 3.1. *For a filter Ψ and an ideal Λ on I the condition*

$$\forall A \in \Psi\ \forall B \notin \Psi\ \exists L \in \Lambda (L \subseteq A\cap B^c \wedge L^c \notin \Psi)$$

will be called the $(\Lambda\Psi)$-condition or, more simply, just $(\Lambda\Psi)$.

Theorem 3.2. *Let Λ be an ideal and Ψ a filter on I. Then the $(\Lambda\Psi)$-condition holds if and only if $\forall\, k \in \{0,1,2,3,3\frac{1}{2}\}$ and for any family $\{(X_i,\mathcal{O}_i) \mid i \in I\}$, $\{i \in I \mid (X_i,\mathcal{O}_i)$ is a T_k-space$\} \in \Psi$ implies $\prod_\Psi^\Lambda \mathbf{X}_i$ is a $T_k$space.*

Proof: $(\Leftarrow)$ Suppose the $(\Lambda\Psi)$-condition is not satisfied; thus there exist some sets $A \in \Psi$, $B \notin \Psi$, such that for any $L \in \Lambda$, $L \subseteq A \cap B^c \Rightarrow L^c \in \Psi$. Let, for $i \in A$, $\mathbf{X}_i$ be a two-element discrete space, otherwise $\mathbf{X}_i$ is a two-element indiscrete space (in both cases the "ground" set is $\{0,1\}$). Then $\{i \in I \mid \mathbf{X}_i$ is a T_k-space$\} \in \Psi$, while $\prod_\Psi^\Lambda \mathbf{X}_i$ is not a T_k-space (for any $k \in \{0,1,2,3,3\frac{1}{2}\}$). For if $f(i) = 1$ for each $i \in I$ and $g(i) = 1$ for $i \in B$ only, it holds: $g \notin [f]$, $g \in \overline{[f]}$ i.e. $[f] = q^{-1}(\{[f]\})$ is not closed. Therefore $\{[f]\}$ is not a closed set in $\prod_\Psi^\Lambda \mathbf{X}_i$ and, consequently, $\prod_\Psi^\Lambda \mathbf{X}_i$ is not a T_1-space. One (similar) step more and it follows that $\prod_\Psi^\Lambda \mathbf{X}_i$ is not a T_0-space either. So it remains to be proved $g \in \overline{[f]}$. Assume $g \in \bigcap_{i\in L} \pi_i^{-1}(O_i)$, $L \in \Lambda$. Clearly, due to the choice of topological spaces, $\bigcap_{i\in L} \pi_i^{-1}(O_i) = \bigcap_{i\in L\cap A} \pi_i^{-1}(O_i)$. Let us put $L_1 = L \cap A$, $L_2 = L_1 \cap B^c$ and let

$$h(i) = \begin{cases} 1 & i \in L_2^c \\ 0 & \text{otherwise.} \end{cases}$$

Since $L_2^c \in \Psi$, $h \in [f]$ and it is not difficult to see that $h \in \bigcap_{i\in L_1} \pi_i^{-1}(O_i)$ ($L_1 = (L_1 \cap B) \cup (L_1 \cap B^c) = (L_1 \cap B) \cup L_2$ and if $i \in L_1 \cap B$, $h(i) = g(i) = 1$, if $i \in L_2$ $h(i) = g(i) = 0$).

$(\Rightarrow)$ In the coming considerations it is assumed that $(\Lambda\Psi)$ holds.

(0) Let $A = \{i \in I \mid (X_i,\mathcal{O}_i)$ is a T_0-space$\} \in \Psi$ and let $[f], [g]$ be two different elements from $\prod_\Psi^\Lambda \mathbf{X}_i$. Then for some $L \in \Lambda$, $L \subseteq A \cap I_{f,g}^c$ and $L^c \notin \Psi$. For any $i \in L$, $f(i) \neq g(i)$, and in the topological space $(X_i,\mathcal{O}_i)$ there exists an open set containing either $f(i)$ but not $g(i)$ (in that case we put $i \in L_f$) or vice versa ($i \in L_g$). $L = L_f \cup L_g$ and at least one of the sets L_f^c, L_g^c does not belong to Ψ. Suppose $L_f^c \notin \Psi$ and let $O = \bigcap_{i\in L_f} \pi_i^{-1}(O_i)$ where $O_i \in \mathcal{O}_i$ is such that $f(i) \in O_i$, $g(i) \notin O_i$. Now, $f \in \bigcap_{i\in L_f} \pi_i^{-1}(O_i)$ (hence, surely, $[f] \in q(\bigcap_{i\in L_f} \pi_i^{-1}(O_i))$) but $[g] \notin q(\bigcap_{i\in L_f} \pi_i^{-1}(O_i))$ ($[g] = [h]$ for some $h \in \bigcap_{i\in L_f} \pi_i^{-1}(O_i)$ would imply $I_{g,h} \in \Psi$ and $I_{g,h} \subseteq L_f^c$, contradiction). □

(1) Suppose $A = \{i \in I \mid (X_i,\mathcal{O}_i)$ is T_1 $\} \in \Psi$, $f, g \in \prod X_i$, $[f] \neq [g]$ and let $L \in \Lambda$ be such that $L \subseteq A \cap I_{f,g}^c$ and $L^c \notin \Psi$. Then for any $i \in L$ there is some open set $O_i \in \mathcal{O}_i$ containing $g(i)$ but not $f(i)$. It follows directly that $[f] \notin q(\bigcap_{i\in L} \pi_i^{-1}(O_i))$. We conclude that $\{[f]\}$ is a closed set in $\prod_\Psi^\Lambda X_i$. □

(2) Let $A = \{i \in I \mid (X_i,\mathcal{O}_i)$ is T_2 $\} \in \Psi$ and $[f] \neq [g]$. For any $i \in L$, where L is an element from Λ satisfying $L \subseteq A \cap I_{f,g}^c$ and $L^c \notin \Psi$, there exist disjoint open sets $U_i, V_i \in \mathcal{O}_i$ containing, respectively, $f(i)$ and $g(i)$ and, as in the previous case, it is easy to see that $q(\bigcap_{i\in L} \pi_i^{-1}(U_i)) \cap q(\bigcap_{i\in L} \pi_i^{-1}(V_i)) = \emptyset$. □

(3) Assume $A = \{i \in I \mid (X_i, \mathcal{O}_i) \text{ is } T_3\ \} \in \Psi$. We already know that the space $\prod_\Psi^\Lambda \mathbf{X}_i$ is a T_1-space. Thus only its regularity is to be proved.

Let $[f] \in q(O)$, $O \in \mathcal{O}^\Lambda$. Hence $f \in [f] = q^{-1}(\{[f]\}) \subset q^{-1}(q(O)) = U$. By 1.2 (c), $U = U^*$. We show and it finishes the proof that there exists some open set W and some (closed) set F such that $[f] \subseteq W = W^* \subseteq F \subseteq F^* \subseteq U^* \subseteq U$ where we have that F^* is closed. For then:

$$[f] \in \{[f]\} = q([f]) \subseteq q(W) = q(W^*) \subseteq q(F^*) \subseteq q(U) = q(O)$$

and $q(F^*)$ is closed ($q^{-1}(q(F^*)) = F^*$ — 1.3 (c)).

Let $f \in B = \bigcap_{i\in L} \pi_i^{-1}(O_i) \subseteq U$, $L \in \Lambda$. If $B_1 = \bigcap_{i\in L\cap A} \pi_i^{-1}(O_i)$, then $B^* = B_1^*$ (1.4). $L \cap A = \emptyset$ implies $B_1^* = U = \prod X_i$ and this case is trivial. If $L \cap A \neq \emptyset$ then for any $i \in L \cap A$, $(X_i, \mathcal{O}_i)$ is a T_3-space and $f(i) \in O_i$, therefore for some open set $V_i \in \mathcal{O}_i$ it holds:

$f(i) \in V_i \subseteq \overline{V_i} \subseteq O_i$. Let $V = \bigcap_{i\in L\cap A} \pi_i^{-1}(V_i)$, $F = \bigcap_{i\in L\cap A} \pi_i^{-1}(\overline{V_i})$ and $W = V^*$. Now

$$[f] \subseteq W = W^* = V^* \subseteq F^* \subseteq B^* = B_1^* \subseteq U = U^*.$$

Let us note that F is closed (if $g \notin F$ then for some $i \in L \cap A$, $g(i) \in \overline{V_i}^{\,c}$ and $g \in \pi_i^{-1}(\overline{V_i}^{\,c})$, while $\pi_i(\overline{V_i}^{\,c}) \cap F = \emptyset$), but the point is that F^* is closed too. That is an immediate consequence of the following lemma:

Lemma 3.3. *If the* $(\Lambda\Psi)$*-condition holds, then* $\{i \in I \mid A_i \textit{ is closed}\} \in \Psi$ *implies* $(\prod A_i)^*$ *is closed.*

Proof: Let $A = \{i \in I \mid A_i \text{ is closed}\} \in \Psi$ and $f \notin (\prod A_i)^*$. Since $B = \{i \in I \mid f(i) \in A_i\} \notin \Psi$ (1.3 (c)) there exists some $L \in \Lambda$ such that $L \subseteq A \cap B^c$ and $L^c \notin \Psi$. For any $i \in L$, $f(i) \in A_i^c$ and A_i^c is open. It follows: $f \in \bigcap_{i\in L} \pi_i^{-1}(A_i^c) \in \mathcal{O}^\Lambda$ and $\bigcap_{i\in L} \pi_i^{-1}(A_i^c) \cap (\prod A_i)^* = \emptyset$. □

($3\frac{1}{2}$) In order to simplify the notation we will immediately assume that X_i is a $T_{3\frac{1}{2}}$-space for each $i \in I$. There is no restriction in it — one just should recall Lemma 2.6 and note that if $A \in \Psi$ and $(\Lambda\Psi)$ holds, then $(\Lambda_A\Psi_A)$ holds as well. Of course, this assumption could have been used in all previous cases.

$\mathbf{X}_i$ being a completely regular space, topology $\mathcal{O}_i$ is uniform for some uniformity $\mathcal{U}_i$ for X_i, generated with the base $\mathcal{B}_i = \{B_i \in \mathcal{U}_i \mid B_i \text{ is open}\}$.

Definition 3.4. *For* $L \in \Lambda$, $A \in \Psi$ *and family* $\{U_i \in \mathcal{U}_i \mid i \in L\}$, *let* $R_A^L U_i$ *be the binary relation on* $\prod X_i/\sim_\Psi$ *defined by:*

$([f],[g]) \in R_A^L U_i$ *if and only if* $\forall\, i \in L \cap A\ (f(i), g(i)) \in U_i$.

$R_\Psi^L U_i = \bigcup_{A\in\Psi} R_A^L U_i$.

Lemma 3.5. (a) *If* $\forall\, i \in L$, $U_i \subseteq V_i$, *then* $R^L_\Psi U_i \subseteq R^L_\Psi V_i$;

(b) $(R^L_\Psi U_i)^{-1} = R^L_\Psi U_i^{-1}$;

(c) $R^L_\Psi (U_i \circ U_i) = R^L_\Psi U_i \circ R^L_\Psi U_i$

(*here* $\circ$ *is standard product of relations:* $(x,y) \in U \circ V$ *iff* $\exists\, z((x,z) \in V \wedge (z,y) \in U)$);

(d) $R^{L_1}_\Psi U_i \cap R^{L_2}_\Psi V_i = R^{L_1 \cup L_2}_\Psi W_i$ *where*

$$W_i = \begin{cases} U_i & i \in L_1 - L_2 \\ U_i \cap V_i & i \in L_1 \cap L_2 \\ V_i & i \in L_2 - L_1. \end{cases}$$

Proof: (c) $([f],[g]) \in R^L_\Psi (U_i \circ U_i)$
iff

$$\exists\, A \in \Psi \forall\, i \in L \cap A\ (f(i), g(i)) \in U_i \circ U_i$$

iff

$$\exists\, A \in \Psi \forall\, i \in L \cap A \exists\, a_i \subset X_i\ ((f(i), a_i) \in U_i \wedge (a_i, g(i)) \in U_i)$$

iff

$$\exists\, A \in \Psi \exists\, h \in \prod X_i\ (([f],[h]) \in R^L_A U_i \wedge ([h],[g]) \in R^L_A U_i)$$

iff

$$\exists\, h \in \prod X_i\ (([f],[h]) \in R^L_\Psi U_i \wedge ([h],[g]) \in R^L_\Psi U_i)$$

iff

$$([f],[g]) \in R^L_\Psi U_i \circ R^L_\Psi U_i.$$

(d)

$$([f],[g]) \in R^{L_1}_\Psi U_i \cap R^{L_2}_\Psi V_i$$

iff

$$\exists\, A_1, A_2 \in \Psi (\forall\, i \in L_1 \cap A_1\ (f(i), g(i)) \in U_i \wedge \forall\, i \in L_2 \cap A_2\ (f(i), g(i)) \in V_i)$$

iff

$$\exists\, A \in \Psi \forall\, i \in (L_1 \cup L_2) \cap A\ (f(i), g(i)) \in W_i.$$

The last equivalence is checked directly, when the implication ($\Leftarrow$) is in question, use $A_1 = A_2 = A$. □

Definition 3.6. $\mathcal{B} = \{R^L_\Psi B_i \mid L \in \Lambda,\ B_i \in \mathcal{B}_i\}$.

Lemma 3.7. $\mathcal{B}$ *is a base for some uniformity* $\mathcal{U}$ *for* $\prod X_i / \sim_\Psi$.

Proof: Let us fix two elements $R^L_\Psi U_i$ and $R^{L_1}_\Psi V_i$ from $\mathcal{B}$ $(U_i, V_i \in \mathcal{B}_i)$. Obviously, $\forall\, f \in \prod X_i$ $([f],[f]) \in R^L U_i$ $(\forall\, i \Delta_{X_i} \subseteq U_i)$. Let, for $i \in L$, $B_i \in \mathcal{B}_i$ be such that $B_i \circ B_i \subseteq U_i \cap U_i^{-1}$. Then:

$$R^L_\Psi B_i \subseteq R^L U_i^{-1} = (R^L_\Psi U_i)^{-1}$$

$$(R^L_\Psi B_i) \circ (R^L_\Psi B_i) = R^L_\Psi (B_i \circ B_i) \subseteq R^L_\Psi U_i.$$

Let, for $i \in L \cap L_1$, $W_i \in \mathcal{B}_i$ be a subset of $U_i \cap V_i$ and for $i \in L - L_1$ let $W_i = U_i$, for $i \in L_1 - L$ let $W_i = V_i$. Then (by 3.5 (d)) $R^{L \cup L_1}_\Psi W_i \subseteq R^L_\Psi U_i \cap R^{L_1}_\Psi V_i$. □

Lemma 3.8. $q^{-1}((R^L_\Psi B_i)[[f]]) = \bigcup_{A\in\Psi}\bigcap_{i\in L\cap A}\pi_i^{-1}(B_i[f(i)])$.

PROOF: $g \in q^{-1}((R^L_\Psi B_i)[[f]])$ iff

$$[g] \in (R^L_\Psi B_i)[[f]]$$

iff

$$([f],[g]) \in R^L_\Psi B_i$$

iff

$$\exists\, A \in \Psi \forall\, i \in L\cap A\ (f(i), g(i)) \in B_i$$

iff

$$\exists\, A \in \Psi\ g \in \bigcap_{i\in L\cap A}\pi_i^{-1}(B_i[f(i)]).$$

Let us note that, since B_i is open (in $X_i \times X_i$ with Tychonoff product topology), for (any) $a \in X_i$ $B_i[a]$ is open, too. Thus, $q^{-1}((R^L_\Psi B_i)[[f]]) \in \mathcal{O}^\Lambda$. □

Lemma 3.9. *Uniform topology $\mathcal{O}_\mathcal{U}$ of the uniformity $\mathcal{U}$ is equal to topology $\mathcal{O}^\Lambda_\Psi$.*

PROOF: Let $O \in \mathcal{O}_\mathcal{U}$ and $f \in q^{-1}(O)$. Then for some $R^L_\Psi B_i \in \mathcal{B}$, $(R^L_\Psi B_i)[[f]] \subseteq O$ whence

$$f \in [f] = q^{-1}(\{[f]\}) \subseteq q^{-1}((R^L_\Psi B_i)[[f]]) \subseteq q^{-1}(O),$$

and by the previous remark $q^{-1}(O) \in \mathcal{O}^\Lambda$, that is $O \in \mathcal{O}^\Lambda_\Psi$.

Suppose now $O \in \mathcal{O}^\Lambda_\Psi$ and $[f] \in O$. It follows $f \in \bigcap_{i\in L}\pi_i^{-1}(O_i) \subseteq q^{-1}(O)$ for some $L \in \Lambda$ and some family $\{O_i \in \mathcal{O} \mid i \in L\}$. Since for $i \in L$, $f(i) \in O_i$, there exists some $B_i \in \mathcal{B}_i$ such that $f(i) \in B_i[f(i)] \subseteq O_i$. Therefore $f \in \bigcap_{i\in L}\pi_i^{-1}(B_i[f(i)])$ and since (by 1.3, 1.5 and 3.8)

$$q^{-1}((R^L_\Psi B_i)[f]) = q^{-1}((R^L_\Psi B_i)[f])^* = (\bigcup_{A\in\Psi}\bigcap_{i\in L\cap A}\pi_i^{-1}(B_i[f(i)]))^* =$$

$$= \bigcup_{A\in\Psi}(\bigcap_{i\in L\cap A}\pi_i^{-1}(B_i[f(i)]))^* = (\bigcap_{i\in L}\pi_i^{-1}(B_i[f(i)]))^* \subseteq (\bigcap_{i\in L}\pi_i^{-1}(O_i))^* \subseteq q^{-1}(O),$$

it follows

$$[f] \in q(\bigcap_{i\in L}\pi_i^{-1}(B_i[f(i)])^*) = (R^L_\Psi B_i)[f] \subseteq O.$$

Thus $O \in \mathcal{O}_\mathcal{U}$.

We conclude: $\mathcal{O}^\Lambda_\Psi$ is uniform topology for uniformity $\mathcal{U}$, thus $\prod^\Lambda_\Psi X_i$ is completely regular space. □

4. Discussion about the $(\Lambda\Psi)$-condition.

In the end let us examine some of the cases when the $(\Lambda\Psi)$-condition is satisfied. In the sequence we will be using the following notation (see 2.2): for $A \subseteq I$, $A\uparrow$ and $A\downarrow$ are, respectively, filter and ideal generated by A.

Lemma 4.1. *Let $\Psi = A\uparrow$. Then the $(\Lambda\Psi)$-condition holds if and only if Λ contains all finite subsets of A and under assumption that $(\Lambda\Psi)$ holds, $\prod_\Psi^\Lambda \mathbf{X}_i$ is homeomorphic to $(\prod_{i\in A} X_i, \mathcal{O}^{\Lambda_A})$.*

Proof: Let $\Psi = A\uparrow$. In order to avoid trivialities we assume that $A \neq \emptyset$. Then if $a \in A$, $A - \{a\} \notin \Psi$ whence for some $L \in \Lambda$, $L \subseteq A \cap (A - \{a\})^c = \{a\}$. L must be $\{a\}$, otherwise it would be $\emptyset^c \notin \Psi$. On the other hand, if Λ contains all finite subsets of A and $B \notin \Psi$ (i.e. $A \nsubseteq B$), then $\emptyset \neq A \cap B^c$ and if L is a finite nonempty subset of $A\cap B^c$, we have $L \in \Lambda$ and $L^c \notin \Psi$.

As for the second part of the lemma, it is an immediate consequence of 2.6: $\prod_\Psi^\Lambda \mathbf{X}_i$ is homeomorphic to $\prod_{\Psi_A}^{\Lambda_A} \mathbf{X}_i$ and since $\Psi_A = \{A\}$, $\sim_{\Psi_A}$ is the identity relation on A. □

Let us note (if necessary at all) that Tychonoff product is obtained when $\Psi = \{I\} = I\uparrow$ and Λ is the set of all finite subsets ("Fréchet ideal").

Lemma 4.2. *Let $\Lambda = L\downarrow$. Then the $(\Lambda\Psi)$-condition holds if and only if $L \in \Psi$. If the condition $(\Lambda\Psi)$ is satisfied, $\prod_\Psi^\Lambda \mathbf{X}_i$ is homeomorphic to the quotient space $(\prod X_i, \mathcal{O}^{P(L)})/\sim_{\Psi_L}$.*

Proof: Let $\Lambda = L\downarrow$. If it were: $(\Lambda\Psi)$ holds but $L \notin \Psi$, then for some nonempty set L_1 from Λ it would be $L_1 \subseteq L^c$, a contradiction.

Let us suppose now $L, A \in \Psi$, $B \notin \Psi$. Then $L \cap A \cap B^c$ is a nonempty element of Λ and, certainly, $(L \cap A \cap B^c)^c \notin \Psi$. ($(L\cap A\cap B^c)^c = (L\cap A)^c \cup B \in \Psi$ would imply $((L\cap A)^c \cup B) \cap (L \cap A) = L \cap A \cap B \in \Psi$, a contradiction to $B \notin \Psi$.)

If $(\Lambda\Psi)$ holds, i.e. $L \in \Psi$, $\prod_\Psi^\Lambda \mathbf{X}_i$ is homeomorphic to $\prod_{\Psi_L}^{\Lambda_L} \mathbf{X}_i$, but $\Lambda_L = P(L)$. □

Thus, this time, we obtain a quotient of the box product (as it is defined in [4]), that is, just the box product when $\Psi = \{I\}$ (and $\Lambda = P(I)$). This case includes ultraproduct spaces ([1]) as well: Let L be I and Ψ a non-principal ultrafilter. Let us still remark, maybe superfluously, that the last part of the lemma is independent of whether Λ is principal or not.

As we see from 4.1 and 4.2, the situation is completely clear (from the viewpoint of the $(\Lambda\Psi)$-condition) when either ideal Λ or filter Ψ is principal. The fact that neither of them is principal, however, significantly interferes with our work. Thus, at least for this time, only a few observations will be made.

We have already shown (see the proof of 4.2) that if $(\Lambda\Psi)$ holds then $\cup\Lambda \in \Psi$. Because of it, 4.2 and the last remark following 4.2, we can as well assume that $\cup\Lambda = I$ and $\Lambda \cap \Psi = \emptyset$.

Lemma 4.3. *If Λ, Ψ are nonprincipal, $\cup\Lambda = I$, $\Lambda\cap\Psi = \emptyset$ and $A \in \Psi$ then Λ_A, Ψ_A are nonprincipal, $\cup\Lambda_A = A$ and $\Lambda_A \cap \Psi_A = \emptyset$.*

Proof: Clearly, $\cup\Lambda_A = A$, and if it were that Λ_A is principal, it would follow $A \in \Lambda\cap\Psi$. If Ψ_A were principal generated by $A_0 (\subseteq A)$, Ψ itself would be generated by the same set. Finally, it is obvious that if $B_0 \in \Lambda_A \cap \Psi_A$ then also $B_0 \in \Lambda \cap \Psi$. □

At this place let us recall that it has been noted that if $(\Lambda\Psi)$ holds then $(\Lambda_A\Psi_A)$ holds too. Therefore one consequence of Lemma 4.3 is that under given assumptions there is no use of 2.6 in clarifying what $\prod_{\Psi}^{\Lambda}\mathbf{X}_i$ space looks like.

We end with a comment that evidently the $(\Lambda\Psi)$-condition is not sufficient for preserving T_4-property and

Lemma 4.4. *Let $|I| = \lambda \geq \mu > \tau \geq \aleph_0$ and let $\Lambda = \{L \subseteq I \mid |L| < \mu\}$, $\Psi = \{A \subseteq I \mid |A^c| < \tau\}$. Then the $(\Lambda\Psi)$-condition holds and $\prod_{\Psi}^{\Lambda}\mathbf{X}_i$ is exactly the box product space from* [5].

PROOF: Let $A \in \Psi$ and $B \notin \Psi$. Then $|A^c| < \tau$ (thus $|A| = \lambda$ and $\Lambda \cap \Psi = \emptyset$) and $|B^c| \geq \tau$. From $B^c = (B^c \cap A) \cup (B^c \cap A^c)$ it follows $|B^c \cap A| \geq \tau$ and any subset of $B^c \cap A$ of cardinality τ belongs to Λ, while its complement does not belong to Ψ. □

References

[1] Bankston P., *Ultraproducts in topology*, Gen. Topology Appl. **7** (1977), 283–308.
[2] van Douwen E.K., *The box product of countably many metrizable spaces need not be normal*, Fundamenta Mathematicae **LXXXVIII.2** (1975), 127–132.
[3] Engelking R., *General Topology*, PWN – Polish Scientific Publishers, 1977.
[4] Kelly J.L., *General Topology*, Springer Verlag, Graduate Texts in Mathematics 27.
[5] Knight C.J., *Box topologies*, Quart. J. Math. Oxford **15** (1964), 41–54.

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