Simple construction of spaces without the Hahn-Banach extension property

Jerzy Kakol*

Abstract. An elementary construction for an abundance of vector topologies ξ on a fixed infinite dimensional vector space E such that (E, ξ) has not the Hahn-Banach extension property but the topological dual $(E, \xi)'$ separates points of E from zero is given.

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A *topological vector space* (tvs) $E = (E, \tau)$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is said to have the Hahn-Banach extension property (HBEP) if for every subspace F of E and every continuous linear functional f on E there is a continuous linear extension of f to the whole space E. Then the topological dual E' of E is separating, i.e. E' separates points of E from zero. Clearly every locally convex space has HBEP. In [1] Duren, Romberg, Shields gave an example of an *F-space* E, i.e. a metrizable and complete tvs, such that E' is separating and E has a subspace M such that E/M has a closed weakly dense subspace; hence E has not HBEP. In [5] Shapiro showed that every sequence space ℓ^p, $0 < p < 1$, also contains such subspaces. In [2] Kalton developing basic sequence techniques to F-spaces proved that all F-spaces with HBEP are locally convex. (It is still unknown if the completeness in the above result is necessary.) Hence no non-locally convex F-space has HBEP.

This note gives an elementary construction for an abundance of vector topologies ξ on a fixed infinite dimensional vector space E such that (E, ξ) has not the HBEP, but $(E, \xi)'$ separates points of E from zero. We prove the following

Theorem. *Let E be an infinite dimensional vector space and $\triangle = \{(x, x) : x \in E\}$ the diagonal of $E \times E$. For every non-zero linear functional f on $\triangle$ there are two vector topologies ϑ_1, ϑ_2 on E such that f is $\vartheta_1 \times \vartheta_2 \,|\triangle$-continuous but f cannot be extended to a continuous linear functional on $(E, \vartheta_1) \times (E, \vartheta_2)$.*

We shall need the following simple

Lemma ([6, Example 2.5] or [4, Theorem 1.6])**.** *Let (E, F) be a dual pair and $\sigma(E, F)$ the weak topology on E. Let ξ be a Hausdorff nearly exotic vector topology on E, i.e. $(E, \xi)' = 0$. Then the topology $\inf\{\sigma(E, F), \xi\}$, i.e. the strongest vector topology on E weaker than ξ and $\sigma(E, F)$, is indiscrete.*

Note that every infinite dimensional vector space E admits a nearly exotic Hausdorff vector topology, cf. e.g. [4].

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Proof of Theorem: By Q we denote the map $x \to (x,x)$ of E onto $\triangle$. Choose on E a normed topology ξ such that $h := f \circ Q$ is ξ-discontinuous. Set $E' = (E,\xi)'$. Let τ_1 be a Hausdorff nearly exotic topology on E. Fix $y \in E$ such that $h(y) = 2$. Let $\tau_2 := T(\tau_1)$ be the image topology on E under the map $T(x) = x - h(x)y$. Since T is injective, τ_2 is Hausdorff and nearly exotic. Set $\vartheta_i = \sup\{\sigma(E,E'),\tau_i\}$, $i = 1,2$. By Lemma, $\inf\{\sigma(E,E'),\tau_i\}$, $i = 1,2$, is indiscrete, so $\triangle$ is dense in $(E,\sigma(E,E')\times(E,\tau_i))$, $i = 1,2$; clearly (E,ϑ_i) is isomorphic to $(\triangle, \sigma(E,E')\times\tau_i \,|\triangle)$, $i = 1,2$. This and the fact that τ_i is nearly exotic, $i = 1,2$, imply that ϑ_i and $\sigma(E,E')$ have the same continuous linear functionals. On the other hand h is continuous in $\sup\{\vartheta_1,\vartheta_2\}$ (since it is $\sup\{\tau_1,\tau_2\}$-continuous). Hence $h \circ Q^{-1} (= f)$ is $\vartheta_1 \times \vartheta_2 \,|\triangle$-continuous. If $h \circ Q^{-1}$ had a continuous linear extension to the whole space, then h would be ξ-continuous. This completes the proof. $\square$

Remark. It is known that every infinite dimensional separable normed space (E,ξ) admits a weaker metrizable nearly exotic vector topology, cf. e.g. [3, Lemma 2.1]. Let h be a ξ-discontinuous linear functional on E. Then (E,ϑ_1) (where ϑ_1 is defined as above) has HBEP. We do not know however if (E,ϑ_2) has also HBEP.

Problem. *Are there tvs E,F with HBEP whose product has not HBEP?*

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Institute of Mathematics, A. Mickiewicz University, 60-769 Poznań, Poland

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