Valuations of lines

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Abstract. We enlarge the problem of valuations of triads on so called lines. A line in an e-structure $\mathbb{A} = \langle A, F, E\rangle$ (it means that $\langle A, F\rangle$ is a semigroup and E is an automorphism or an antiautomorphism on $\langle A, F\rangle$ such that $E \circ E = \mathbf{Id} \restriction A$) is, generally, a sequence $\mathbb{A} \restriction B$, $\mathbb{A} \restriction U_c$, $c \in \mathbf{FZ}$ (where $\mathbf{FZ}$ is the class of finite integers) of substructures of $\mathbb{A}$ such that $B \subseteq U_c \subseteq U_d$ holds for each $c \leq d$. We denote this line as $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ and we say that a mapping H is a valuation of the line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ in a line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c\in\mathbf{FZ}}$ if it is, for each $c \in \mathbf{FZ}$, a valuation of the triad $\mathbb{A}(U_c, B)$ in $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})$. Some theorems on an existence of a valuation of a given line in another one are presented and some examples concerning equivalences and ideals are discussed. A generalization of the metrization theorem is presented, too.

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The problem of valuations concerns the question whether there exists a certain representation H of a given structure $\mathbb{A}$ in another one $\hat{\mathbb{A}}$. The structures in question are semigroups with a certain automorphism or antiautomorphism i.e. structures of the form $\langle A, F, E\rangle$ where $\langle A, F\rangle$ is a semigroup and E is an automorphism or an antiautomorphism of $\langle A, F\rangle$ such that $E \circ E$ is the identity on A. The described structure is called an e-structure. The mentioned representation preserves some structural properties of the relevant e-structures; it is called a valuation of $\mathbb{A}$ in $\hat{\mathbb{A}}$. (See [M2], [M3].) We specify, moreover, this general situation demanding to find a valuation of certain "descriptive" type in a hierarchy of classes. For example, having an e-structure $\mathbb{A} = \langle \mathcal{P}(a), \cup, \mathbf{Id}\rangle$, where a is a set, and the e-structure $\hat{\mathbb{A}} = \langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ (where $\mathbf{Id}$ is everywhere the identity on the relevant class and $\mathbf{Q}^+$ is the class of all non-negative rational numbers), we look for a set-valuation H of $\mathbb{A}$ in $\hat{\mathbb{A}}$. This means that we have $H : \mathcal{P}(a) \to \mathbf{Q}^+$ and H is a set-mapping such that $H(u \cup v) \leq H(u) + H(v)$. The relation $\leq$ is the so called canonical relation of the e-structure $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$; it is defined in this structure by the relation $x \leq y \Leftrightarrow (\exists z)(x + z = y)$. The solution of our task is trivial: Let $r \in \mathbf{Q}^+$ be fixed. Putting $H(u) = r$ for all $u \subseteq a$ we obtain a required mapping. Assume, moreover, that $U \subseteq \mathcal{P}(a)$ is a subclass closed under operation $\cup$, and put $[0]^+ = \{r \in \mathbf{Q}^+; r \doteq 0\}$. Then $[0]^+$ is a subclass of $\mathbf{Q}^+$ closed under the operation $+$. Now, the task to find a set-valuation as above with the additional property that $U = H^{-1\prime\prime}[0]^+$ and $\{\emptyset\} = H^{-1\prime\prime}\{0\}$ is more complicated. The required mapping is said to be a (set-) valuation of the triad $\mathbb{A}(U, \{\emptyset\})$ (i.e. of the triple $\mathbb{A}$, $\mathbb{A} \restriction U$, $\mathbb{A} \restriction \{\emptyset\}$ of substructures of the structure $\mathbb{A}$) in the triad $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle([0]^+, \{0\})$. The last triad is the so called canonical π-one; it is denoted as $\mathcal{T}_\pi$. A canonical σ-triad is the triad $\langle \mathbf{N}, +, \mathbf{Id}\rangle(\mathbf{FN}, \{0\})$; we denote this triad as $\mathcal{T}_\sigma$. It is known, for example, that every triad $\mathbb{A}(U, B)$, where $\mathbb{A}$, B are sets and U is a π-class (σ-class resp.) has a set-valuation in $\mathcal{T}_\pi$ ($\mathcal{T}_\sigma$-resp.).

In this article, we deal with so called lines. A line in a given e-structure $\mathbb{A}$ over B is a sequence $\mathbb{A} \restriction B$, $\mathbb{A} \restriction U_c, c \in \mathbf{FZ}$ of substructures of $\mathbb{A}$ such that $B \subseteq U_c \subseteq U_d$ holds for each $c \leq d$, $c, d \in \mathbf{FZ}$. We denote such a line as $\mathbb{A}(U_c, B)$. A mapping H is a valuation of a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ in a line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ if it is, for each $c \in \mathbf{FZ}$, a valuation of the triad $\mathbb{A}(U_c, B)$ in $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})$.

We shall study the so called $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-lines. It means that $\mathbb{A}$ and B belong to a so called standard universe $\mathfrak{S}$ of classes, the classes U_c with c odd are $\pi(\mathfrak{S})$-classes, i.e. classes of the form $\bigcap_{n \in \mathbf{FN}} X_n$, where $\{X_n\}_n \subseteq \mathfrak{S}$, and the classes U_c with c even are $\sigma(\mathfrak{S})$-classes, i.e. classes of the form $\bigcup_{n \in \mathbf{FN}} X_n$, where $\{X_n\}_n \subseteq \mathfrak{S}$. We shall look for a valuation $H \in \mathfrak{S}$ of such a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line in a canonical one. By a canonical line we mean a line $\mathcal{L}_{\pi-\sigma}(\zeta)$ which is defined in the section Lines and valuations; this line is a line in the e-structure $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ over $\{\emptyset\}$.

The system Sd_V of all set-theoretically definable classes is the basical example of the standard universe $\mathfrak{S}$ of classes. Note that, generally, we cannot find a valuation H of a given $(\pi(Sd_V) - \sigma(Sd_V))$-line in a canonical one such that H belongs to Sd_V; see Remarks in the paragraph Valuations of $(\pi - \sigma)$-Lines. We prove that our problem of valuations of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-lines is solvable in so called saturated standard universes of classes. Let us present that the revealment Sd_V of the system Sd_V is such a saturated standard universe of classes.

Let us introduce briefly two domains of applications of valuations of lines.

A sequence $\{\mathcal{E}_c\}_{c \in \mathbf{FZ}}$ of equivalences on a set z such that $\mathcal{E}_c \subseteq \mathcal{E}_{c+1}$ holds for each $c \in \mathbf{FZ}$, each $\mathcal{E}_c$ with c odd is a π-class and each $\mathcal{E}_c$ with c even is a σ-class, can be seen as a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line. This fact enables us to state a generalization of some known metrization theorems of equivalences ([M2],[Gui]).

Similarly, we can study "$(\pi - \sigma)$-lines" of ideals on a set a. Here, we must deal with so called monotonic valuations of lines, i.e. valuations which preserve canonical relations of relevant e-structures. We investigate this problems in the last paragraph.

§1 Lines and valuations

We work in the alternative set theory; we shall use usual notations of this theory. Recall that small Latin letters range over sets and i, j, k, l, m, n range over finite natural numbers. By a collection we mean a collection of classes which satisfy a given formula of the language $\mathbf{FL_V}$. The collection of all set-theoretically definable sets is denoted by $\mathbf{Sd}_V$; it is a codable system.

We say that a structure $\langle A, F, E\rangle$ is an *e-structure* if we have the following:

1) $\langle A, F\rangle$ is a semigroup (i.e. F is an associative operation on A),
2) $E \circ E$ is the identity on A,
3) we have either, for each x, $y \in A$, $F(E(x), E(y)) = E(F(x, y))$ or, for each x, $y \in A$, $F(E(x), E(y)) = E(F(y, x))$.

Let $\langle A, F, E\rangle$ be an e-structure. We define, on A, a *canonical relation* $\lhd_A \langle A, E\rangle$ by:

$$x \lhd_A y \Leftrightarrow (\exists z \in A)(F(x, z) = y).$$

Assume that $\mathbb{A} = \langle A, F, E\rangle, \hat{\mathbb{A}} = \langle \hat{A}, \hat{F}, \hat{E}\rangle$ are two e-structures. A mapping $H : A \to \hat{A}$ is said to be a *valuation of* $\mathbb{A}$ *in* $\hat{\mathbb{A}}$ if we have:

a) $H(F(x, y)) \lhd_{\hat{A}} \hat{F}(H(x, y))$ holds for each $x, y \in A$,

b) $H(E(x)) = \hat{E}(H(x))$ holds for each $x \in A$.

Let $\mathbb{A}$ be an e-structure. Then the triple $\langle \mathbb{A}, \mathbb{A} \upharpoonright U, \mathbb{A} \upharpoonright B\rangle$, where $B \subseteq U \subseteq A$ and $\mathbb{A} \upharpoonright B$, $\mathbb{A} \upharpoonright U$ are substructures of $\mathbb{A}$, is said to be a *triad over the e-structure* $\mathbb{A}$. We denote it as $\mathbb{A}(U, B)$. A mapping H is called a *valuation of the triad* $\langle A, F, E\rangle(U, B)$ *in the triad* $\langle \hat{A}, \hat{F}, \hat{E}\rangle(\hat{U}, \hat{B})$, if H is a valuation of $\langle A, F, E\rangle$ in $\langle \hat{A}, \hat{F}, \hat{E}\rangle$ and we have, moreover,

c) $H^{-1}{}'' \hat{U} = U$, $H^{-1}{}'' \hat{B} = B$.

We use the following notations: Let $r \in \mathbf{Q}^+$. Then $r \cdot [\mathbf{Q}^+] = \{r \cdot d;\, d \in \mathbf{Q}^+ \ \&\ d \doteq 0\}$, $r \cdot \mathbf{BQ}^+ = \{r \cdot b;\, b \in \mathbf{BQ}^+\}$. We have $r \cdot [\mathbf{Q}^+] = \{x \in \mathbf{Q}^+;\, (\forall n)(x \leq r \cdot n)\}$ and $r \cdot \mathbf{BQ}^+ = \{x \in \mathbf{Q}^+; (\exists n)(x \leq r \cdot n)\}$.

We can generalize the notion of the triad by the following way:

Let $\mathbb{A}$, $\mathbb{A} \upharpoonright B$ be two e-structures, $B \subseteq A$.

A *line in* $\mathbb{A}$ *over* B is a sequence $\{\mathbb{A}, \mathbb{A} \upharpoonright U_c, \mathbb{A} \upharpoonright B\}_{c \in \mathbf{FZ}}$ such that we have

1) $\mathbb{A} \upharpoonright U_c$ is an e-structure and $B \subseteq U_c$ holds for each $c \in \mathbf{FZ}$,

2) $c < d \Rightarrow U_c \subseteq U_d$.

Let us denote such a line by the symbol

$$\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}.$$

A *valuation of a line* $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ *in a line* $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ is a valuation H of the structure $\mathbb{A}$ in $\hat{\mathbb{A}}$ such that $H^{-1}{}''\hat{U}_c = U_c$ holds for each $c \in \mathbf{FZ}$ and $H^{-1}{}''\hat{B} = B$.

Example. Let $\zeta \in \mathbf{N} - \mathbf{FN}$. Put

$U_c(\zeta) = (2^\zeta)^c \cdot [0]^+$ whenever $c \in \mathbf{FZ}$ is odd,

$U_c(\zeta) = (2^\zeta)^{c-1} \cdot \mathbf{BQ}^+$ whenever $c \in \mathbf{FZ}$ is even.

Then

$$\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle(U_c, \{0\})$$

is a line in the e-structure $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ over $\{0\}$.

Let us present the following interpretation of a line. Let $\langle \mathcal{E}_1, \mathcal{E}_2\rangle$ be a couple of equivalences on a class $A \in \mathbf{Sd}_V$ such that $\mathcal{E}_1 \subseteq \mathcal{E}_2$ and $\mathcal{E}_1$ is a π-equivalence, $\mathcal{E}_2$ is a σ- equivalence. We can see this situation as a formalization of the phenomenon of two horizons; the horizon of "microcosmos" which is represented by the first of the equivalences in question and the horizon of "macrocosmos" represented by the second one. The couple presented is called biequivalence. The sequence $\{\mathcal{E}_c\}_{c \in \mathbf{FZ}}$, where each $\mathcal{E}_c$ is an equivalence on a fixed class from $\mathbf{Sd}_V$, $\mathcal{E}_c \subseteq \mathcal{E}_{c+1}$ and $\mathcal{E}_c$ is a π-class and $\mathcal{E}_{c+1}$ is a σ-class for each odd $c \in \mathbf{FZ}$, can be seen as a formalization of a "line of horizons".

Note that an equivalence $\mathcal{E}$ on a class B can be studied as an e-structure. Indeed, let $\mathbb{A} = \langle A, F, E\rangle$ be a structure defined as follows:

$A = B^2 \cup \{\emptyset\}$,

$F : A^2 \to A$ is a function defined by

$F(\langle x, y\rangle, \langle r, s\rangle) = \langle x, s\rangle$ (resp. $\emptyset$) $\Leftrightarrow y = r$ (resp. $y \neq r$)

$F(w, \emptyset) = F(\emptyset, w) = \emptyset$, whenever $w \in A$,

$E : A \to A$ is a function defined by

$$E(\langle x, y\rangle) = \langle y, x\rangle),\ E(\emptyset) = \emptyset.$$

We can see that $\mathbb{A}$ is an e-structure and $\mathcal{E}$ can be identified with the triad $\mathbb{A}(\mathcal{E} \cup \{\emptyset\}, \mathbf{Id} \restriction B \cup \{\emptyset\})$. Thus the sequence $\{\mathcal{E}_c\}_{c\in\mathbf{FZ}}$ induces the line $\mathbb{A}(\mathcal{E}_c \cup \{\emptyset\}, \mathbf{Id} \restriction B \cup \{\emptyset\})_{c\in\mathbf{FZ}}$; we denote it by the symbol $\mathcal{L}(\mathcal{E}_c)_{c\in\mathbf{FZ}}$.

We can see, moreover, that every valuation $H : A \to [0, 1]_{\mathbf{Q}}$ of the triad presented in $\langle\mathbf{Q}^+, +, \mathbf{Id}\rangle([0]^+, \{0\})$ induces a rational metric $D : B \times B \to [0, 1]_{\mathbf{Q}}$ given by the relation $D(x, y) = H(\langle x, y\rangle$. We have

$$\langle x, u\rangle \in \mathcal{E} \Leftrightarrow D(x, y) \doteq 0.$$

We conclude that a valuation H of the line $\mathcal{L}(\mathcal{E}_c)_{c\in\mathbf{FZ}}$ in the line from the previous example induces the mapping similarly as above, which can be seen as a "horizon-metric" for the line of equivalences in question. A proposition on the existence of such a valuation represents thus a generalization of the theorem on metrization (see [Gui],[M2]); such a proposition will be precisely formulated below.

We shall study, roughly speaking, "$(\pi - \sigma)$-lines", i.e. lines $\mathbb{A}(U_c, B)$ where $\mathbb{A}$, B belong to a collection $\mathfrak{S}$ and the classes U_c are $\pi(\mathfrak{S})$-classes whenever c is odd and $\sigma(\mathfrak{S})$-classes whenever c is even.

A class $\bigcap_n X_n$, where $\{X_n\} \subseteq \mathfrak{S}$, is called a $\pi(\mathfrak{S})$-*class* and a class $\bigcup_n X_n$, where $\{X_n\} \subseteq \mathfrak{S}$, is called a $\sigma(\mathfrak{S})$-*class*.

A line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is called a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-*line in* $\mathbb{A}$ *over* B, if $\mathbb{A}, B \in \mathfrak{S}$ and, for each $c \in \mathbf{FZ}$ odd, U_c is a $\pi(\mathfrak{S})$-class and U_{c+1} is a $\sigma(\mathfrak{S})$-class.

The collection $\mathfrak{S}$ plays a role of a universe of primary visual objects. The system $\mathbf{Sd}_V$ of all set-theoretically definable classes is a good example of such a collection; we can naturally see here σ-classes, π-classes, $\sigma\pi$-classes as secondary visual objects. We say that a collection of classes is *universe of classes* if it is closed under definition by normal formulas of the language $\mathbf{FL_V}$ with class-parameters from this collection. Thus, having a universe $\mathfrak{U}$ of classes and a normal formula $\varphi(x, X_1, X_2, \ldots, X_k)$ of the language $\mathbf{FL_V}$ such that the classes $X_1, X_2, \ldots, X_k$ belong to $\mathfrak{U}$, we see that the class $\{x; \varphi(x, X_1, X_2, \ldots, X_k)\}$ belongs to $\mathfrak{U}$, too. Note that every universe of classes contains all sets. More generally, every set-theoretically definable class belongs to each universe of classes. By a *standard universe of classes* we call each universe of classes which contains only such non-empty subclasses of the class of natural numbers which have the first element. We can see (see [M1]) that every standard universe of classes contains only the revealed classes and does not contain a proper semiset. It satisfies all axioms of Gödel-Bernays theory of finite sets.

A standard universe $\mathfrak{U}$ of classes is said to be *standard saturated universe of classes* if we have the following: Let $\{X_n\}_{n\in\mathbf{FN}}$ be a sequence of classes of this universe. Then there exists a relation R from $\mathfrak{U}$ such that

$$(\forall n)R''\{n\} = X_n.$$

Convention. Throughout this paper, let $\mathfrak{S}$ denote a standard universe of classes.

We see that the line from the first example is a $\pi(\mathbf{Sd}_V) - \sigma(\mathbf{Sd}_V)$)-line in $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ over $\{\emptyset\}$. We denote it by

$$\mathcal{L}_{\pi-\sigma}(\zeta).$$

Recall that it is defined for each $\zeta \in \mathbf{N} - \mathbf{FN}$.

§2 Valuations of $(\boldsymbol{\pi} - \boldsymbol{\sigma})$-lines

Theorem (on valuations of $(\pi(\mathfrak{S})-\sigma(\mathfrak{S}))$-lines). *Let $\mathfrak{S}$ be a codable saturated standard universe and let $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ be a $(\pi(\mathfrak{S})-\sigma(\mathfrak{S}))$-line in $\mathbb{A}$ over B. Then there exists a valuation of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to $\mathfrak{S}$.*

Let $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ be a $(\pi(\mathbf{Sd_V})-\sigma(\mathbf{Sd_V}))$-line in $\mathbb{A}$ over B and let A be a set. Then there exists a set-valuation of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$.

We give a proof of the theorem in the two following sections. At first, in the section Valuations of uniform $(\pi-\sigma)$-lines, we introduce a notion of a uniform $(\pi-\sigma)$-line in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$ and prove a proposition on an existence of a valuation $H \in \mathfrak{S}$ of such a line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$. Secondly, we prove that, in a codable saturated standard universe, the notion of the $(\pi(\mathfrak{S})-\sigma(\mathfrak{S}))$-line in $\mathbb{A}$ over B coincides with this one of the uniform $(\pi-\sigma)$-line in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$. This second step is done in the section $(\pi-\sigma)$-uniform lines w.r.t. a saturated standard universe. Finally, our theorem is an easy consequence of the mentioned assertions. This is presented at the end of this section.

Valuations of uniform $(\boldsymbol{\pi} - \boldsymbol{\sigma})$-lines.

A relation R is called a *σ-string*, if we have

$$dom(R) \in \mathbf{N} \ \ \& \ \ (\forall \alpha \in dom(R)-1)(R''\{\alpha\} \subseteq R''\{\alpha+1\}).$$

Let $\langle A, F, E\rangle(B, B)$ be a triad. We say that a σ-string R is a *σ-string in $\langle A, F, E\rangle$ over B* if the following items hold:

1) $R(0) = B,\ R(\theta) = A$, where $\theta + 1 = dom(R)$,
2) $F''R^2(\alpha) \subseteq R(\alpha+1)$, $F_3''R^3(\alpha) \subseteq R(\alpha+1)$ is true for each $\alpha \in \theta$; we denote $F_3 : A^3 \to A$ the function defined by $F_3(x, y, z) = F(F(x, y), z)$,
3) $E''R(\alpha) \subseteq R(\alpha)$ holds for each $\alpha \leq \theta$.

Recall the lemma on a valuation over a σ-string which is proved in [M2].

Lemma (on a valuation over a σ-string). *Let $\mathfrak{S}$ be a standard universe. Let $\mathbb{A}(B,B) \in \mathfrak{S}$ be a triad and assume that $R \in \mathfrak{S}$ is a σ-string in $\mathbb{A}$ over B. Suppose, moreover, that $\theta + 1 = dom(R)$.*

Then there exists a valuation $H \in \mathfrak{S}$ of a triad $\mathbb{A}(B,B)$ in $\langle \mathbf{N}, +, \mathbf{Id}\rangle(\{0\},\{0\})$ such that, for each $\alpha \in \theta$, we have

$$R(\alpha) \subseteq \{x \in A; H(x) \leq 2^{\alpha}\} \subseteq R(\alpha+1).$$

Natural number $\zeta \notin \mathbf{FN}$ is called a *distance in a σ-string* R, whenever there exists a number $\eta \in \mathbf{N} - \mathbf{FN}$ so that we have $dom(R) - 1 = 2\zeta \cdot \eta$. Suppose that $\zeta \notin \mathbf{FN}$ is a distance in a σ-string R and let $\eta \in \mathbf{N} - \mathbf{FN}$ be a number such that we have $dom(R) - 1 = 2\zeta \cdot \eta$. We define the function $z_{R,\zeta} = z$ on $[-\eta,\eta]$ with values in $dom(R)$ as follows: Let us denote $\theta = dom(R) - 1$. Then $z(0) = \frac{\theta}{2}$, $z(c) = z(0) + c \cdot \zeta$ whenever $c \in [-\eta,\eta] - \{0\}$ holds. Assuming R is a σ-string in $\mathbb{A}$ over B, we can see that $R(z(-\eta)) = B$, $R(z(\eta)) = A$ hold.

Let ζ be a distance in a σ-string R in $\mathbb{A}$ over B.

We say that a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a $\langle R,\zeta\rangle$*-line in* $\mathbb{A}$ *over* B if we have:

$U_c = \bigcap_n R''\{z(c) - n\}$, whenever $c \in \mathbf{FZ}$ is odd,
$U_c = \bigcup_n R''\{z(c-1) + n\}$, whenever $c \in \mathbf{FZ}$ is even.

Let ζ be a distance in a σ-string R in $\mathbb{A}$ over B and let $R \in \mathfrak{S}$, $dom(R) = \theta + 1$, $\mathbb{A}$, $B \in \mathfrak{S}$. Then there exists a valuation H over a σ-string R, $H \in \mathfrak{S}$. We have $(\forall x \in B) H(x) \leq 2^{-\theta}$. We have, moreover, for each $c \in \mathbf{FZ}$ odd,

$x \in U_c \Leftrightarrow (\forall n) H(x) \leq 2^{z(c)-n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^{\zeta})^c \cdot [0]^+$,
$x \in U_{c+1} \Leftrightarrow (\exists n) H(x) \leq 2^{z(c)+n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^{\zeta})^c \cdot \mathbf{BQ}^+$.

We say that a line $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a *uniform* $(\pi - \sigma)$*-line in* $\mathbb{A}$ *over* B *w.r.t.* $\mathfrak{S}$, if there are $\mathbb{A}$, $B \in \mathfrak{S}$ and a σ-string $R \in \mathfrak{S}$ in $\mathbb{A}$ over B and a distance ζ in this string such that $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is an $\langle R,\zeta\rangle$-line in $\mathbb{A}$ over B. Saying a uniform $(\pi - \sigma)$-line w.r.t. $\mathfrak{S}$ we mean a uniform $(\pi - \delta)$-line in an $\mathbb{A}$ over a B w.r.t. $\mathfrak{S}$.

We can see that the line $\mathcal{L}_{\pi-\sigma}(\zeta)$ is a $(\pi - \sigma)$-line in $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ over $\{0\}$ w.r.t. $\mathbf{Sd_V}$. Indeed, let $\eta \in \mathbf{N} - \mathbf{FN}$. We define the relation R on $\theta + 1 = 2\zeta\eta + 1$ by $R(\alpha) = \{x \in \mathbf{Q}^+; x \leq 2^{\alpha - \zeta\eta}\}$ whenever $0 < \alpha\theta$ and we put in addition $R(0) = \{0\}$, $R(\theta) = \mathbf{Q}^+$. Then $R \in \mathfrak{S}$ and R is a σ-string in $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ over $\{0\}$. Let $z : [-\eta,\eta] \to \theta$ be a function defined by the relation $z(0) = \theta/2$, $z(\gamma) = z(0) + \gamma\zeta = \zeta\eta + \gamma\zeta$ for $\gamma \in [-\eta,\eta] - \{0\}$. We have, for each $c \in \mathbf{FZ}$ odd and $x \in \mathbf{Q}^+$, the following:

$x \in \bigcap_n R(z(c) - n) \Leftrightarrow (\forall n)(x \leq 2^{z(c)-n-\zeta\eta}) \Leftrightarrow (\forall n)(x \leq 2^{c\zeta - n}) \Leftrightarrow x \in 2^{c\zeta} \cdot [0]^+$,
$x \in \bigcup_n R(z(c) + n) \Leftrightarrow (\exists n)(x \leq 2^{z(c)+n-\zeta\eta}) \Leftrightarrow (\exists n)(x \leq 2^{c\zeta + n}) \Leftrightarrow x \in 2^{c\zeta} \cdot \mathbf{BQ}^+$.

Thus $\mathcal{L}_{\pi-\sigma}(\zeta)$ is an $\langle R,\zeta\rangle$-line in $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle$ over $\{0\}$. We can deduce from the fact above that the following theorem holds.

Theorem (on a valuation of a uniform $(\pi - \sigma)$-line). *Let $\mathfrak{S}$ be a standard universe. Then every uniform $(\pi - \sigma)$-line w.r.t. $\mathfrak{S}$ has a valuation in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to $\mathfrak{S}$.*

Let us clarify some questions about uniform lines.

Lemma. *Let $\mathfrak{S}$ be a standard universe and let $\mathcal{L}$ be a line in a structure from $\mathfrak{S}$. Let $H \in \mathfrak{S}$ be a valuation of $\mathcal{L}$ in a uniform $(\pi - \sigma)$-line w.r.t. $\mathfrak{S}$. Then $\mathcal{L}$ is a uniform $(\pi - \sigma)$-line w.r.t. $\mathfrak{S}$.*

Proof: Suppose that $\mathcal{L} = \mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ has the valuation $H \in \mathfrak{S}$ in $\hat{\mathcal{L}} = \hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ and let $\hat{\mathcal{L}}$ be an $\langle R, \zeta \rangle$-uniform line in $\hat{\mathbb{A}}$ over $\hat{B}$, $R \in \mathfrak{S}$. Put $z = z_{R,\zeta}$ as above. Let us define the string S on $\theta + 1 = 2\eta\zeta + 1 = dom(R)$ by the relation $S(\alpha) = H^{-1}{}'' R(\alpha)$. We can see that $S \in \mathfrak{S}$ and that it is a σ-string in $\mathbb{A}$ over B. Indeed, assuming $\mathbb{A} = \langle A, F, E \rangle$, we have $x, y \in S(\alpha) \Rightarrow H(x), H(y) \in R(\alpha)$ whenever $\alpha + 1 \in dom(S)$. Further, the relation $E'' S(\alpha) \subseteq S(\alpha)$ is easy.

We have, for each $c \in \mathbf{FZ}$ odd: $U_c = H^{-1}{}'' \hat{U}_c = \bigcap_n H^{-1}{}'' R(z(c) - n) = \bigcap_n S(z(c) - n)$, $U_{c-1} = H^{-1}{}'' \hat{U}_{c-1} = H^{-1}{}'' \bigcup_n R(z(c) + n) = \bigcup_n H^{-1}{}'' R(z(c) + n) = \bigcup_n S(z(c) + n)$. Thus, $\mathcal{L}$ is an $\langle S, \zeta \rangle$-line in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$. □

Lemma. *Let $\mathcal{L}$ be an $\langle R, \zeta \rangle$-line in $\mathbb{A}$ over B, $\widetilde{\zeta} \in \zeta - \mathbf{FN}$. Then there is an $\widetilde{R}$ such that $\mathcal{L}$ is an $\langle \widetilde{R}, \widetilde{\zeta} \rangle$-line in $\mathbb{A}$ over B. If $R \in \mathfrak{S}$ then $\widetilde{R}$ can be chosen from $\mathfrak{S}$.*

Proof: Let $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c \in \mathbf{FZ}}$ be an $\langle R, \zeta \rangle$-uniform line and assume that $R \in \mathfrak{S}$. Let $z = z_{R,\zeta}$ be as above and let $\widetilde{\eta} \in \eta - \mathbf{FN}$. We define a function $\widetilde{z} : [-\widetilde{\eta}, \widetilde{\eta}] \to [0, \theta]$, where $\widetilde{\theta} = 2 \cdot \widetilde{\eta} \cdot \widetilde{\zeta}$, by $\widetilde{z}(\gamma) = \widetilde{\eta} \cdot \widetilde{\zeta} + \gamma \cdot \widetilde{\zeta}$. We define the relation $\widetilde{R}$ with $dom(\widetilde{R}) = [0, \widetilde{\theta}]$ as follows: Put $\widetilde{R}(0) = B$, $\widetilde{R}(\widetilde{\theta}) = A$. Put, for each $\gamma \in [-\widetilde{\theta} + 1, \widetilde{\theta} - 1]$ and $\alpha \in [-\widetilde{\zeta}, \widetilde{\zeta} - 1]$ such that $\widetilde{\zeta}(\gamma) + \alpha > 0$, $\widetilde{R}(\widetilde{z}(\gamma) + \alpha) = R(z(\gamma) + \alpha)$. We wish to prove that $\widetilde{R}$ and $\widetilde{\zeta}$ have the required properties.

Let us prove, at first, that $\widetilde{R}$ is a σ-string in $\mathbb{A}$ over B. Assume that $\mathbb{A} = \langle A, F, E \rangle$. It is easy that $E'' \widetilde{R}(\delta) \subseteq \widetilde{R}(\delta)$ holds for each $\delta \leq \widetilde{\theta}$. Assume that $\delta = \widetilde{z}(\gamma) + \alpha$ holds for some $\gamma \in [-\widetilde{\theta} + 1, \widetilde{\theta} - 1]$, $\alpha \in [-\widetilde{\zeta}, \widetilde{\zeta} - 1]$. We have, for each $x, y \in \widetilde{R}(\delta)$, $F(x, y) \in R(z(\gamma) + \alpha + 1)$. Thus $F(x, y) \in \widetilde{R}(\delta + 1)$ holds. We can similarly prove that $x, y, z \in \widetilde{R}(\delta) \Rightarrow F_3(x, y, z) \in \widetilde{R}(\delta + 1)$.

It remains to prove that, for each $c \in \mathbf{FZ}$ odd, $U_c = \bigcap_n \widetilde{R}(\widetilde{z}(c) - n)$ and $U_{c-1} = \bigcup_n \widetilde{R}(\widetilde{z}(c) + n)$ hold. But it follows immediately from the definition of $\widetilde{R}$. Finally, assuming $R \in \mathfrak{S}$, we can see that $\widetilde{R} \in \mathfrak{S}$. □

The following proposition is an easy consequence of the above results.

Proposition (on valuation in $\mathcal{L}_{\pi-\sigma}(\zeta)$). *Let $\mathfrak{S}$ be a standard universe and let $\mathcal{L}$ be a line in an $\mathbb{A}$ which belongs to $\mathfrak{S}$. Then $\mathcal{L}$ has a valuation from $\mathfrak{S}$ in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ iff $\mathcal{L}$ is a uniform $(\pi - \sigma)$-line w.r.t. $\mathfrak{S}$.*

$(\boldsymbol{\pi} - \boldsymbol{\sigma})$-uniform lines w.r.t. a saturated standard universe.

We give a criterion of the uniformity of lines.

We shall use the following notation: Let $S \subseteq \mathbf{V}^3$. We denote, for $\langle x, y \rangle \in dom(S)$, the class $S''\{\langle x, y \rangle\}$ by $S(x, y)$.

Lemma (on uniform lines). *Let $\mathfrak{S}$ be a saturated standard universe and suppose that $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ is a line, $\mathbb{A}, B \in \mathfrak{S}$. Let S be a relation and $\xi \in \mathbf{N} - \mathbf{FN}$ be such that we have*

1) $dom(S) = \mathbf{FZ}$,

2) $S''\{c\}$ *is, for each* $c \in \mathbf{FZ}$, *a* σ*-string in* $\mathbb{A}$ *over* B, $dom(S''\{c\}) = \xi + 1$ *and* $S''\{c\} \in \mathfrak{S}$,
3) *the relations* $U_c = \bigcap_n S(c, \xi - n)$, $U_{c+1} = \bigcup_n S(c, n)$ *hold for each* $c \in \mathbf{FZ}$ *odd.*

Then $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ *is a uniform* $(\pi - \sigma)$*-line in* $\mathbb{A}$ *over* B *w.r.t.* $\mathfrak{S}$.

PROOF: We are looking for a σ-string $R \in \mathfrak{S}$ and a number β such that the line in question is an $\langle R, \beta\rangle$-line in $\mathbb{A}$ over B. We shall say that a σ-string R is a *weak* σ*-string in* $\mathbb{A}$ *over* B if there exists a relation R on natural numbers with values in A such that the extension of each number from its domain contains B as a subclass and

a) $F''R^2(\alpha) \subseteq R(\alpha + 1)$, $F_3''R^3(\alpha) \subseteq R(\alpha + 1)$ holds for each $\alpha \in \theta$, where $F_3 : A^3 \to A$ is a function defined by the relation $F_3(x, y, z) = F(F(x, y), z)$.
b) $E''R(\alpha) \subseteq R(\alpha)$ holds for each $\alpha \in \theta + 1$.

We say sometimes that the domain of a σ-string is the *length of the string* in question.

The classes U_c, $c \in \mathbf{FZ}$, are linearly ordered by the inclusion. We deduce from this and by using the prolongation axiom that there exists a number $\delta \in \mathbf{N} - \mathbf{FN}$ and numbers

$$l_c \in \mathbf{FN},\ c \in \mathbf{FZ} \text{ odd},$$
$$n_c \in \mathbf{FN},\ c \in \mathbf{FZ} \text{ even},$$

so that we have for each $c \in \mathbf{FZ}$ odd

$$S(c, \xi - (n_c + \delta)) \subseteq S(c, \xi - n_c) \subseteq S(c+1, l_c) \subseteq S(c+1, l_c + \delta) \subseteq S(c+2, \xi - (n_{c+2} + \delta)).$$

We have, for each $c \in \mathbf{FZ}$ odd:

$$F_3''(S(c, \xi - n_c - 1))^3 \subseteq S(c, \xi - n_c) \subseteq S(c + 1, l_c + 1)$$

and

$$F_3''(S(c + 1, l_c + \delta - 1))^3 \subseteq S(c + 1, l_c + \delta) \subseteq S(c + 2, \xi - (n_{c+2} + \delta)) \subseteq S(c + 2, \xi - (n_{c+2} + \delta - 1)).$$

We denote, for each $c \in \mathbf{FZ}$ odd, $I_c = \{\alpha; \xi - (n_c + \delta) < \alpha < \xi - n_c\}$ and $I_{c+1} = \{\alpha; l_c < \alpha < l_c + \delta\}$. Every set I_c has $\delta - 1$ elements. Let $\widetilde{S}_c$ be the relation which is obtained from $S(c) \upharpoonright I_c$ by the natural renumbering of its domain I_c (starting from the number 0). We see that $\widetilde{S}_c$ is a weak σ-string in $\mathbb{A}$ over B of the length $\delta - 1$, which belongs, moreover, to $\mathfrak{S}$. We can see that, for each $c < d$ from $\mathbf{FZ}$ and $\alpha, \beta \leq \delta - 1$, the relation $F_3''(\widetilde{S}_c(\alpha))^3 \subseteq \widetilde{S}_d(\beta)$ holds. Put $\beta = \delta - 1$. We have, for $c \in \mathbf{FZ}$ odd,

$$U_c = \bigcap_n \widetilde{S}_c(\beta - n), \quad U_{c+1} = \bigcup_n \widetilde{S}_c(n).$$

There exists a relation $\widetilde{S}$ on $\mathbf{FZ}$ such that we have for each $c \in \mathbf{FZ}$: $\widetilde{S}(c) = \widetilde{S}_c$. There exists, moreover, a number $\gamma \in \mathbf{N} - \mathbf{FN}$ and a relation $T \in \mathfrak{S}$ such that

1) $dom(T) = [-\gamma, \gamma]$,

2) $\widetilde{S} \subseteq T$,
3) $c \in dom(T) \Rightarrow T''\{c\}$ is a weak σ-string in $\mathbb{A}$ over B of the length β,
4) $c, d \in dom(T)$ & $c < d \Rightarrow (\alpha, \alpha' \in \beta \Rightarrow T(c,\alpha) \subseteq T(d,\alpha')$ & $F_3''(T(c,\alpha))^3 \subseteq T(d,\alpha')$ & $F''(T(c,\alpha))^2 \subseteq T(d,\alpha'))$.

Especially, we have for each $c \in \mathbf{FZ}$ odd:

$$U_c = \bigcap_n T(c, \beta - n), \quad U_{c+1} = \bigcup_n T(c, n).$$

Let us define the function $z : [-\gamma, \gamma] \to [0, 2\beta\gamma]$ by $z(c) = \beta\gamma + c\gamma$. Let R be the relation with domain $[0, 2\beta\gamma]$ such that $R \in \mathfrak{S}$, $R(0) = B$, $R(2\beta\gamma) = A$ and, for each $c \in [-\gamma, \gamma - 1]$ and $0 \le \alpha \le \beta - 1$, holds

$$R(z(c) + \alpha) = T(c, \alpha).$$

Thus R is a σ-string in $\mathbb{A}$ over B and β is a distance in R. We have for each $c \in \mathbf{FZ}$: $1 \le \alpha \le \beta \Rightarrow R(z(c) - \alpha) = R(z(c-1) + \beta - 1)$. Thus, for $c \in \mathbf{FZ}$ odd, it holds

$$\bigcap_n R(z(c) - n) = \bigcap_n R(z(c-1) + \beta - n) = \bigcap_n T(c, \beta - n) = U_c,$$
$$\bigcup_n R(z(c) + n) = \bigcup_n T(c, n) = U_{c+1}.$$

The proof is finished. □

Lemma. *Let $\mathfrak{S}$ be a codable saturated standard universe and let $\mathbb{A}(U_c, B)_{c \in \mathbf{FZ}}$ be a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line in $\mathbb{A}$ over B. Then there exist a relation S and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that the items 1) – 3) from the lemma on uniform lines hold.*

Proof: Let U be a relation such that $dom(U) = \mathbf{FZ}$ and, for each $c \in \mathbf{FZ}$, $U_c = U''\{c\}$ holds. Let $\langle T, K \rangle$ be a codable pair for $\mathfrak{S}$, i.e. we assume that $\mathfrak{S} = \{T''\{z\}; z \in K\}$ holds. Take $\xi \in \mathbf{N} - \mathbf{FN}$. Let L be the relation on $\mathbf{FZ}$ defined by the formula

$$\begin{aligned} L''\{c\} = \{z; z \text{ is a code of a } \sigma\text{-string } R \in \mathfrak{S} \text{ in } \mathbb{A} \text{ over } B \text{ of the length } \xi \notin \mathbf{FN} \text{ and} \\ \textstyle\bigcap_n R''\{\xi - n\} = U''\{c\} \text{ holds whenever } c \text{ is odd and} \\ \textstyle\bigcup_n R''\{n\} = U''\{c\} \text{ holds whenever } c \text{ is even}\}. \end{aligned}$$

Note that the existence of the relation R, mentioned in the definition of L, follows from the proposition on σ-string in e-structure (see [M2]) and from our assumption that $\mathfrak{S}$ is a saturated standard universe.

Let G be such a function on $\mathbf{FZ}$ which satisfies: $c \in \mathbf{FZ} \Rightarrow G(c) \in L''\{c\}$. Let S be the relation with domain $\mathbf{FZ}$, such that we have for each $c \in \mathbf{FZ}$:

$$S''\{c\} = T''G(c).$$

Then S has the required properties. □

We deduce from this lemma and by using the lemma on uniform lines that the following proposition holds.

Proposition. *Let $\mathfrak{S}$ be a codable saturated standard universe. Then the collection of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-lines in $\mathbb{A}$ over B is equal to this one of uniform $(\pi - \sigma)$-lines in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$.*

Now, we can finish the proof of the theorem on valuations of $(\pi(\mathfrak{S})-\sigma(\mathfrak{S}))$-lines. The first part is an immediate consequence of the previous results and of the lemma on a valuation of uniform $(\pi - \sigma)$-line. The second part follows from the first one. It is because the line in question is a $(\pi(\mathbf{Sd}_{\mathbf{V}^*}) - \sigma(\mathbf{Sd}_{\mathbf{V}^*}))$-line in $\mathbb{A}$ over B and that $\mathbf{Sd}_{\mathbf{V}^*}$ is a codable saturated standard universe. Thus there exists a valuation from $Sd_{\mathbf{V}^*}$ of the line in question in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which has the domain equal to the set A. Consequently, this valuation is a set. $\square$

Remarks.

1. We cannot omit, in the theorem on valuations of $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-lines, the assumption, that $\mathfrak{S}$ is saturated. Indeed, the equivalence $\overset{\circ}{=}$ defined on $\mathbf{V}$ by the relation $x \overset{\circ}{=} y \Leftrightarrow (\varphi(x) \Leftrightarrow \varphi(y))$ *holds for each set-formula* φ *of the language* $\mathbf{FL}_\emptyset$) is a $\pi(\mathbf{Sd}_\mathbf{V})$-equivalence which is no $\pi^\mathbf{V}$-class. (See [M1] .) Thus there is no valuation $H \in \mathbf{Sd}_\mathbf{V}$ of $\mathcal{L}(\overset{\circ}{=})$ in an $\mathcal{L}_{\pi-\sigma}(\zeta, 1)$.

2. By a *finite* $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$*-line* we mean a finite sequence $\{\mathbb{A}, \mathbb{A} \upharpoonright U_n, \mathbb{A} \upharpoonright B\}_{n\in[1,l]}$ of triads (where $1 \le l$) such that $1 \le m < n \le l \Rightarrow B \subseteq U_m \subseteq U_n$ and, for each $1 \le m \le l$ odd, U_m is a $\pi(\mathfrak{S})$-class, U_{m+1} is a $\sigma(\mathfrak{S})$-class (whenever $m + 1 \le 1$) and $\mathbb{A}$, $B \in \mathfrak{S}$. We denote this line as $\mathbb{A}(U_c, B)_{c\le l}$; l is called the *length* of the presented line. The notion of a valuation of a finite line of the length l in another one of the length l is defined naturally. Put, for each $c \in \mathbf{FZ}$, $c \le 0$, $U_c = B$ and, for each $c > l$, $U_c = A$. Then $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line. Having such a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ and assuming that $\mathfrak{S}$ is a standard saturated universe we can see that there exists a relation $S \in \mathfrak{S}$ and $\zeta \in \mathbf{N} - \mathbf{FN}$ such that the items 1) - 3) from the lemma on uniform lines hold. We deduce from this that there exists a valuation $H \in \mathfrak{S}$ of $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ in $\mathcal{L}_{\pi-\sigma}(\zeta)$.

Denoting $\mathcal{L}_{\pi-\sigma}(\zeta, l)$, for $l \ge 1$, the finite line $\langle \mathbf{Q}^+, +, \mathbf{Id}\rangle(U_n(\zeta), \{0\})_{n\in[1,l]}$, we can formulate the following proposition.

Proposition. *Let $\mathfrak{S}$ be a standard saturated universe. Assume that $\mathcal{L}$ is a finite $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line of the length l. Then there exists a valuation $H \in \mathfrak{S}$ of $\mathcal{L}$ in an $\mathcal{L}_{\pi-\sigma}(\zeta, l)$.*

Metrization.

Let $\{\mathcal{E}_c\}_{c\in I}$, where $I = \mathbf{FZ}$ or $I = [1, l]$ for some $l \ge 1$, be a sequence of equivalences on a class $Z \in \mathfrak{S}$. Suppose that, for each $c < d$, $c, d \in I$, the relation $\mathcal{E}_c \subseteq \mathcal{E}_d$ holds and we have, for each $c \in I$ odd (even, resp.) that U_c is a $\pi(\mathfrak{S})$-class ($\sigma(\mathfrak{S})$-class resp.). We say that $\{\mathcal{E}_c\}_{c\in I}$ is a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$*-line of equivalences.*

Theorem (on a metrization of a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line of equivalences). *Let $\{\mathcal{E}_c\}_{c\in I}$ be a $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line of equivalences on a class $Z \in \mathfrak{S}$, where $\mathfrak{S}$ is a standard saturated universe.*

1) *Assume that $I = \mathbf{FZ}$ and let $\mathfrak{S}$ be a codable system. Then there exist*

a rational metric $D : Z^2 \to \mathbf{Q}^+$ and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that we have, for each $c \in \mathbf{FZ}$ odd:

$$(*)\quad \langle x, y\rangle \in \mathcal{E}_c \Leftrightarrow D(x,y) \in (2^{c\cdot\zeta})\cdot[0]^+,\quad \langle x,y\rangle \in \mathcal{E}_{c+1} \Leftrightarrow D(x,y) \in (2^{c\cdot\zeta})\cdot \mathbf{BQ}^+.$$

2) *Assume that $I = [1, l]$. Then there exist a rational metric $D : Z^2 \to \mathbf{Q}^+$ and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that $(*)$ holds for each c odd, provided $c \in [1, l]$.*

Proof: 1) Let $\mathcal{L}(\mathcal{E}_c)_{c\in\mathbf{FZ}}$ be the $(\pi-\sigma)$-line described above. We deduce from the last theorem that there exists a valuation H in $\mathfrak{S}$ of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$. The metric in question can be defined on Z^2 by $D(x,y) = H(\langle x,y\rangle)$. 2) can be proved quite analogously by using the last proposition. $\square$

§3 Monotonic valuations of $(\boldsymbol{\pi} - \boldsymbol{\sigma})$-lines

A valuation H of an e-structure $\mathbb{A}$ in $\hat{\mathbb{A}}$ is called a *monotonic valuation* if we have $(x, y \in A \;\&\; x \lhd_A y) \Rightarrow (H(x) \lhd_{\hat{A}} H(y))$. By a *monotonic valuation* of a line in another one we mean a valuation of the first line in the second one such that it is a monotonic valuation of the relevant e-structures. More explicitly, a mapping H is said to be a monotonic valuation of the line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ in the line $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c\in\mathbf{FZ}}$ if H is a valuation of $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ in $\hat{\mathbb{A}}(\hat{U}_c, \hat{B})_{c\in\mathbf{FZ}}$ such that H is a monotonic valuation of $\mathbb{A}$ in $\hat{\mathbb{A}}$.

By a *closed line in* $\mathbb{A}$ *over* B we mean a line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ such that all U_c and B are closed under $\lhd$. We can naturally modify the definitions above to obtain the relevant ones which are connected with the notion of a closed line. A σ-string S is said to be a *closed σ-string in* $\mathbb{A}$ *over* B if it is a σ-string in $\mathbb{A}$ over B and we have for each $\alpha \in dom(S)$: $\lhd\,''S(\alpha) \subseteq S(\alpha)$. By an *$\langle R,\zeta\rangle$-closed line in* $\mathbb{A}$ *over* B we mean an $\langle R,\zeta\rangle$-line in $\mathbb{A}$ over B such that R is a closed σ-string in $\mathbb{A}$ over B. $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is said to be a *uniform $(\pi-\sigma)$-closed line in* $\mathbb{A}$ *over B w.r.t.* $\mathfrak{S}$ if there exist an $R \in \mathfrak{S}$ and $\zeta \in \mathbf{N} - \mathbf{FN}$ such that the line in question is an $\langle R,\zeta\rangle$-closed line in $\mathbb{A}$ over B and $\mathbb{A}, B \in \mathfrak{S}$.

Recall (see [M3]) that an e-structure $\langle A, F, E\rangle \in \mathfrak{S}$ *has a u-expansion in* $\mathfrak{S}$ if there exists a mapping $G : A^2 \to A$, $G \in \mathfrak{S}$ such that 1) $x \lhd_A y \Rightarrow G(y,x) = x$ and 2) $G(x,y) \lhd_A x$ holds for each $x, y \in A$ and that it is *commutative* if F is a commutative mapping on A.

Theorem (on monotonic valuation of uniform $(\pi-\sigma)$-closed lines). *Let $\mathfrak{S}$ be a standard universe. Assume that $\mathcal{L} = \mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$, $\mathbb{A}$ is commutative and $\mathbb{A}$ has a u-expansion in $\mathfrak{S}$. Then every uniform $(\pi-\sigma)$-closed line w.r.t. $\mathfrak{S}$ has a monotonic valuation in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to $\mathfrak{S}$.*

Proof: We can state, observing the proof of the theorem on monotonic valuation of $\sigma^{\mathfrak{S}}$- and $\pi^{\mathfrak{S}}$-triads in [M3], p.383-384, that there exists a normal formula $\Phi(x, y, X, Y)$ of the language $\mathbf{FL_V}$ such that we have: Let $\mathbb{A}(B, B) \in \mathfrak{S}$ be a closed commutative triad and suppose that $\mathbb{A}$ has a u-expansion in $\mathfrak{S}$. Let S be a closed σ-string in $\mathbb{A}$ over B. Then the mapping $H = \{\langle x,y\rangle;\ \Phi(x, y, \mathbb{A}, S)\}$ is a monotonic valuation of $\mathbb{A}(B, B)$ in $\langle \mathbf{N}, +, \mathbf{Id}\rangle(\{0\}, \{0\})$ such that, for each $\alpha + 1 \in dom(S)$, the relation $S(\alpha) \subseteq \{x;\ H(x) \leq 2^{\alpha}\} \subseteq S(\alpha + 1)$ holds.

Let $R \in \mathfrak{S}$ be a closed π-string in $\mathbb{A}$ over B and let $\zeta \in \mathbf{N} - \mathbf{FN}$ be such that the line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ in question is an $\langle R, \zeta\rangle$-closed line in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$. Let H be defined as above. Assuming that $dom(S) = \theta + 1$ we have, for each $c \in \mathbf{FZ}$ odd, similarly as it is mentioned before the theorem on valuation of uniform $(\pi - \sigma)$-lines: a) $x \in U_c \Leftrightarrow (\forall n) H(x) \leq 2^{z(c)-n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^{\zeta})^c \cdot [0]^+$, b) $x \in U_{c+1} \Leftrightarrow (\exists n) H(x) \leq 2^{z(c)+n} \Leftrightarrow 2^{-z(0)} \cdot H(x) \in (2^{\zeta})^c \cdot \mathbf{BQ}^+$. Thus the mapping H is a monotonic valuation of the line in question in $\mathcal{L}_{\pi-\sigma}(\zeta)$. $\square$

We shall clarify a question what kinds of lines are closed uniform ones. Replacing, in the lemma on uniform lines, the assumption "$\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is a line, $\mathbb{A}, B \in \mathfrak{S}$" by "$\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is a closed line, $\mathbb{A}, B \in \mathfrak{S}$" and the assumption "$S(c)$ is a σ-string in $\mathbb{A}$ over B" by "$S(c)$ is a closed σ-string in $\mathbb{A}$ over B" we obtain, replacing the conclusion by "$\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is a closed uniform $(\pi - \sigma)$-line in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$", a true proposition. We define similarly as above that a line $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ is a *closed* $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$*-line in* $\mathbb{A}$ *over* B, if $\mathbb{A}$, $B \in \mathfrak{S}$ and such that, for each $c \in \mathbf{FZ}$ odd, U_c is a $\pi(\mathfrak{S})$-class and U_{c+1} is a $\sigma(\mathfrak{S})$-class and, moreover, each U_c is closed under the canonical relation of $\mathbb{A}$. We can see, analyzing the proof of the lemma on monotonic valuations of $\sigma^{\mathfrak{S}}$- and $\pi^{\mathfrak{S}}$-triads in [M3] that the following holds:

Let $\mathfrak{S}$ be a standard universe. Let $\mathbb{A}(B, B) \in \mathfrak{S}$ be a closed triad and suppose that $S \in \mathfrak{S}$ is a σ-string in $\mathbb{A}$ over B. Then there exists a closed σ-string R in $\mathbb{A}$ over B such that $\bigcup_n R(n) = \bigcup_n S(n)$.

We can prove, by using this assertion, quite analogously as above that the next proposition holds:

Let $\mathfrak{S}$ be a codable saturated standard universe. Then the collection of closed $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-lines in $\mathbb{A}$ over B is equal to this one of closed uniform $(\pi - \sigma)$-lines in $\mathbb{A}$ over B w.r.t. $\mathfrak{S}$.

We obtain immediately as a consequence:

Theorem (on monotonic valuations of $(\pi - \sigma)$-lines). *Let $\mathfrak{S}$ be a codable saturated standard universe and assume that $\mathbb{A} \in \mathfrak{S}$ is a commutative e-structure which has a u-expansion in $\mathfrak{S}$. Let $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ be a closed $(\pi(\mathfrak{S}) - \sigma(\mathfrak{S}))$-line in $\mathbb{A}$ over B. Then there exists a monotonic valuation of this line in an $\mathcal{L}_{\pi-\sigma}(\zeta)$ which belongs to $\mathfrak{S}$.*

Let $\mathbb{A}(U_c, B)_{c\in\mathbf{FZ}}$ be a closed $(\pi(\mathbf{Sd_V}) - \sigma(\mathbf{Sd_V}))$-line in $\mathbb{A}$ over B and let A be a set. Then there exists a monotonic set-valuation of this triad in an $\mathcal{L}_{\pi-\sigma}(\zeta)$.

Let us present an application of the last theorem.

Theorem. *Let A be a set and let $\{J_c\}_{c\in\mathbf{FZ}}$ be a class of ideals on A such that $J_{c+1} \subseteq J_c$ holds for each $c \in \mathbf{FZ}$ and, for each $c \in \mathbf{FZ}$ odd, J_c is a π-class and J_{c+1} is a σ-class. Then there exist a monotone and subadditive set-mapping $h : \mathcal{P}(A) \to \mathbf{Q}^+$ and a number $\zeta \in \mathbf{N} - \mathbf{FN}$ such that we have $h^{-1\prime\prime}\{0\} = \{\emptyset\}$ and, for each $c \in \mathbf{FZ}$ odd, $U_c = h^{-1\prime\prime}(2^{\zeta\cdot c}) \cdot [0]^+$, $U_{c-1} = h^{-1\prime\prime}(2^{\zeta\ \cdot c}) \cdot \mathbf{BQ}^+$.*

A proof of this theorem follows from the fact that $\langle \mathcal{P}(A), \cup, \mathbf{Id}\rangle (J_c, \{\emptyset\})_{c \in \mathbf{FZ}}$ is a closed $(\pi^{\mathbf{V}} - \sigma^{\mathbf{V}})$-line and the structure $\langle \mathcal{P}(A), \cup, \mathbf{Id}\rangle$ is a commutative e-structure which has a u-expansion in $\mathbf{Sd}_{\mathbf{V}}$.

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