Multivalued pseudo-contractive mappings defined on unbounded sets in Banach spaces

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Abstract. Let X be a real Banach space. A multivalued operator T from K into 2^X is said to be pseudo-contractive if for every x, y in K, $u \in T(x)$, $v \in T(y)$ and all $r > 0$, $\|x-y\| \leq \|(1+r)(x-y)-r(u-v)\|$. Denote by $G(z,w)$ the set $\{u \in K : \|u-w\| \leq \|u-z\|\}$. Suppose every bounded closed and convex subset of X has the fixed point property with respect to nonexpansive selfmappings. Now if T is a Lipschitzian and pseudo-contractive mapping from K into the family of closed and bounded subsets of K so that the set $G(z,w)$ is bounded for some $z \in K$ and some $w \in T(z)$, then T has a fixed point in K.

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Let X be a Banach space, K a nonempty subset of X, and for $M = K$ or X let $\mathbf{K}(M)$ denote the family of all nonempty compact subsets of M which is equipped with the Hausdorff metric. Let $\mathbf{B}(M)$ denote the family of all nonempty closed and bounded subsets of M. A mapping $T : K \to \mathbf{K}(M)$ is said to be **Lipschitzian** if there exists $L > 0$ such that for each $x, y \in K$

$$H(T(x), T(y)) \leq L\|x - y\|.$$

For $L < 1$ ($L = 1$) such mappings are said to be contractive (respectively, nonexpansive). A point $x \in K$ is called a fixed point of T if $x \in T(x)$.

An operator $T : K \to 2^X$ is said to be ***k*-pseudo-contractive** ($k > 0$) (see [7]) if for each $x, y \in K$, $u \in T(x)$, $v \in T(y)$ and $\lambda > k$

$$(1) \qquad (\lambda - k)\|x - y\| \leq \|\lambda(x - y) - (u - v)\|.$$

For $k = 1$ ($k < 1$) such mappings are said to be **pseudo-contractive** (respectively, **strongly pseudo-contractive**). By letting $r = 1/(\lambda - 1)$ and $k = 1$ in (1), we derive the original definition of pseudo-contractive mappings, due to Browder [1], as follows:

$$(2) \qquad \|x - y\| \leq \|(1 + r)(x - y) - r(u - v)\|$$

holds for all x and y in K, $u \in T(x)$, $v \in T(y)$ and all $r > 0$. However, by taking a semi-inner approach (see also Kato [5]) we may describe (2) by

$$\langle u - v, j\rangle \leq \|x - y\|^2$$

for some $j \in J(x-y)$. The mapping $J : X \to 2^{X^*}$ is called the normalized duality mapping which is defined by

$$J(x) = \{j \in X^* : \langle x, j\rangle = \|x\|^2, \|j\| = \|x\|\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. As we mentioned in [8], this latter family of mappings is intimately related to the so-called accretive operators, which play an important role in the theory of evolution equations.

The main purpose of this paper is to examine the behavior of multivalued mappings in the case the domain K is unbounded. Certainly, under this constraint we fail to obtain fixed points even in the single-valued case (see [11]). Nevertheless, we are able to show that a mild boundedness condition on T (the condition (4) below) is sufficient to guarantee the existence of a fixed point. In this context we prove that Lipschitzian and pseudo-contractive multivalued mappings satisfying an "inwardness" condition have fixed points. We should mention that this result extends, in various directions, one of the main results of Canetti et al. [2]. We also include an application where the mild boundedness condition (4) holds for the so-called acute cones.

Throughout the paper we will assume that the domain K of the operator T is closed and convex and also the notion that T satisfies the weakly inward condition may be described by

$$\lim_{h\to 0} h^{-1} \operatorname{dist}((1-h)x + hy, K) = 0 \tag{3}$$

for each $x \in K$ and $y \in T(x)$. In addition, we observe that the boundedness condition (introduced in [4]), which has been mentioned earlier, asserts that for z and w in X

$$G(z, w) = \{u \in K : \|u - w\| \leq \|u - z\|\}. \tag{4}$$

is a bounded subset of K.

While we may observe that there is a firm connection between the fixed point theory for pseudo-contractive and nonexpansive mappings in the single-valued case, this is not so for multivalued operators. However, additional conditions on the mapping T will allow us to derive some relationships between the single and multivalued cases. We begin studying some results for single-valued operators. Our first theorem is a slight extension of Theorem 5 of [9], whose proof is included for the sake of completeness.

Theorem 1. *Let X be a Banach space whose bounded closed and convex subsets have the fixed point property with respect to nonexpansive selfmappings. Let K be a close convex subset of X (with $0 \in K$) and let $T : K \to X$ be a continuous pseudo-contractive mapping. Suppose T holds the weakly inward condition for each $x \in K$. If the set*

$$E = \{x \in K : T(x) = \lambda x \quad \text{for some} \quad \lambda > 1\} \tag{5}$$

is bounded, then T has a fixed point in K.

PROOF: Let $f_r = (1+r)I - rT$ for $r > 0$. Then, as in the proof of Theorem 5 of [9], the mapping $g : K \to K$ defined by $g = f_r^{-1}$ is nonexpansive. Also the set $F = \{y \in K : g(y) = \mu y$ for some $\mu > 1\}$ is bounded. To complete the proof, we select a sequence $\{y_n\}$ in F such that $g(y_n) = \mu_n y_n$ with $\mu_n \to 1$. Then if $\lim_{h\to\infty} \sup \|y_n\| = L$, a standard argument shows that the set

$$C = \{y \in K : \lim_{n\to\infty} \sup \|y_n - y\| \leq L\}$$

is closed bounded convex and invariant under g. Therefore, by assumption, g has a fixed point in K, which is also a fixed point for T. □

Corollary 1. *Let X be as in Theorem 1 and let K be a closed and convex subset of X (with $0 \in K$). Suppose T is a continuous pseudo-contractive mapping from K to K. Also, suppose the set E defined by (5) is bounded. Then T has a fixed point in K.*

Now, as a consequence of Theorem 1, we improve Theorem 1 of Ray [12]. We should mention that his proof relies on the intrinsic properties of uniformly convex spaces, contrary to our approach that appears to be less involved.

Theorem 2 (cf. Theorem 6 of [9]). *Let X be as in Theorem 1 and let K be a closed and convex subset of X. Let $T : K \to X$ be a continuous pseudo-contractive mapping. Suppose there exists $z \in K$ for which $T(z) \in K$ and the set*

$$G(z, T(z)) = \{u \in K : \|u - T(z)\| \leq \|u - z\|\} \tag{6}$$

is bounded. Suppose also that T holds the weakly inward condition for each $x \in K$. Then T has a fixed point in K.

PROOF: We may assume without loss of generality that $z = 0$. We shall show that the set E, defined by (5), is bounded. To see that, let $T(x) = \lambda x$ for $x \in K$ and $\lambda > 1$. Since T is pseudo-contractive we have

$$\|x\| \leq \|(1 + r - r\lambda)x + rT(0)\|. \tag{7}$$

By choosing $r = 1/(\lambda - 1)$ and $\lambda \geq \lambda_0$ for some $\lambda_0 \in (1, 2)$, we conclude that $\|x\| \leq \|T(0)\|/(\lambda_0 - 1)$. Select now $r = 1$ (in (7)) and $\lambda \leq \lambda_0$. Then we have

$$(2 - \lambda)\|x\| \leq \|x\| \leq \|(2 - \lambda)x + T(0)\|,$$

yielding $(2 - \lambda)x \in G(-T(0), 0)$. Since $G(0, T(0))$ is bounded, so is $G(-T(0), 0)$. Therefore E is bounded and Theorem 1 completes the proof. □

As a result of Theorem 2 we may derive some known corollaries (see [6], [12], [9]).

Corollary 2. *Let X and K be as in Theorem 2. Let $T : K \to K$ be a continuous pseudo-contractive mapping. Suppose for some $z \in K$ the set $G(z, T(z))$ is bounded. Then T has a fixed point in K.*

In view of the fact that every nonexpansive mapping is continuous and pseudo-contractive, we easily derive Theorem 1.3 of [6] as a consequence of Corollary 2.

Corollary 3. *Let X and K be as in Theorem 2. Let $T : K \to K$ be a nonexpansive mapping. Suppose there exists $z \in K$ for which the set $G(z, T(z))$ is bounded. Then T has a fixed point in K.*

Next we prove our main theorem for multivalued mappings. This result represents a significant extension of Theorem 9 of [2].

Theorem 3. *Let X be a Banach space whose bounded closed convex subsets have the fixed point property with respect to single-valued nonexpansive selfmappings. Let K be closed convex subset of X and let $T : K \to B(K)$ be a Lipschitzian pseudo-contractive mapping. Suppose that for some $z \in K$ the set $G(z, w)$ is bounded for some $w \in T(z)$. Then T has a fixed point in K.*

Proof: We may assume without loss of generality that $z = 0$. Let L be a Lipschitz constant of T and choose $0 < \alpha < \min\{1, 1/L\}$. For each $y \in K$, the mapping $T_y : K \to \mathbf{B}(K)$ defined by $T_y(x) = (1-\alpha)y + \alpha T(x)$ is a multivalued contraction. Since K is complete and $T_y(x) \subset K$, Theorem 5 of Nadler [10] implies that T_y has a fixed point $F_\alpha(y)$ in K. This means

$$F_\alpha(y) \in (1-\alpha)y + \alpha T(F_\alpha(y)).$$

Thus, if $x, y \in K$ there exist $u \in T(F_\alpha(x))$ and $v \in T(F_\alpha(y))$ such that $F_\alpha(x) = (1-\alpha)x + \alpha u$ and $F_\alpha(y) = (1-\alpha)y + \alpha v$. Since T is pseudo-contractive, there exists $j \in J(F_\alpha(x) - F_\alpha(y))$ so that

$$\langle F_\alpha(x) - F_\alpha(y), j\rangle = (1-\alpha)\langle x - y, j\rangle + \alpha\langle u - v, j\rangle.$$

This implies that

$$\|F_\alpha(x) - F_\alpha(y)\|^2 \le (1-\alpha)\|x-y\|\|F_\alpha(x) - F_\alpha(y)\| + \alpha\|F_\alpha(x) - F_\alpha(y)\|^2,$$

and thus

$$\|F_\alpha(x) - F_\alpha(y)\| \le \|x - y\|.$$

Therefore F_α is a single-valued nonexpansive mapping of K into K. Since T and F_α have the same fixed points, it is sufficient to show that F_α has a fixed point. However, in view of Corollary 1 we just need to show that the set

$$E = \{x \in K : F_\alpha(x) = \lambda x \ \text{ for some } \lambda > 1\}$$

is bounded. To see that, let $F_\alpha(x) = \lambda x$ for $x \in K$ and $\lambda > 1$. Then $\lambda x \in (1-\alpha)x + \alpha T(\lambda x)$, which implies that $[\alpha^{-1}(\lambda - 1) + 1]x \in T(\lambda x)$. Since T is pseudo-contractive, it follows that

$$\|\lambda x\| \le \|(1+r)\lambda x - r[(\alpha^{-1}(\lambda - 1) + 1)x - w]\| \tag{8}$$

for some $w \in T(0)$. First of all, we select $\lambda_0 \in (1, 2)$ so that $\alpha^{-1} < 1 + \lambda_0(\lambda_0 - 1)^{-1}$ and then we consider two cases. For $\lambda \geq \lambda_0$ we choose $r = \alpha\lambda/(1-\alpha)(\lambda - 1)$ in (8), and this yields

$$\|x\| \leq \alpha\|w\|/(1-\alpha)(\lambda_0 - 1).$$

Now, suppose $\lambda < \lambda_0$. Then choose $r = 1$ (in (8)) and we obtain

$$\|\lambda x\| \leq \|[(\alpha^{-1} - 1)(1 - \lambda) + \lambda]x + w\|.$$

By writing $\mu = (\alpha^{-1} - 1)(1 - \lambda) + \lambda$, we may choose $\mu_0 > 0$ so that $0 < \mu_0 \leq \mu < \lambda$ for all $\lambda \in (1, \lambda_0)$. This means $\mu x \in K$ and thus $\mu x \in G(-w, 0)$. Since, by assumption, $G(w, 0)$ is bounded, so is $G(-w, 0)$. Therefore the set E is bounded. This completes the proof. □

In Theorem 3, the condition on T of mapping K into itself can be relaxed by imposing a compactness assumption on $T(x)$.

Theorem 4. *Let X be as in Theorem 3 and let K be a closed convex subset of X. Let $T : K \to \mathbf{K}(X)$ be a Lipschitzian pseudo-contractive mapping. Suppose for some $z \in K$ for which $T(z) \in K$ the set $G(z, w)$ is bounded for some $w \in T(z)$, suppose also that (3) holds for each $x \in K$. Then T has a fixed point in K.*

Proof: As in the proof of Theorem 3, the mapping $T_y : K \to \mathbf{K}(X)$ defined by $T_y(x) = (1 - \alpha)y + \alpha T(x)$ is a multivalued contraction. Since T_y also satisfies the weakly inward condition on K (see for example Lemma 1 of [9]), Corollary 2 of [3] implies that T_y has a fixed point $F_\alpha(y)$ in K. These conclusions will then enable us to complete the proof following the argument of Theorem 3. □

We will discuss now an application of Theorem 4 to mappings defined on cones. Following [6] we define the notion of a sufficiently "sharp" cone.

Let X be a normed linear space. A cone K in X with vertex 0 is said to be **acute** if, for each x in K, the set $G(x, 0)$ is bounded. For some specific classes of spaces, this notion can be formulated in more tangible terms. For instance, if X is either l^p-space or the L^p-space (with $1 < p < \infty$), then acute cones are described as those for which

$$\operatorname{Sup}\{D_z(-x) : z \in K \text{ and } \|z\| = 1\} < 0$$

for each nonzero x in K, where D_z is the derivative of the norm of X at z. However, in Hilbert spaces this becomes

$$\inf\{\langle x, y\rangle : y \in K,\ \|y\| = 1\} > 0$$

for each nonzero x in K. This latter characterization corresponds to the well known description of acute cones in Euclidean geometry.

Theorem 5. *Let X be a uniformly convex Banach space and let K be an acute cone of X. Let $T : K \to \mathbf{K}(X)$ be a Lipschitzian pseudo-contractive mapping that satisfies (3) for each $x \in K$. If T has an eigenvalue in $[0, 1)$, then T has a fixed point in K.*

PROOF: Suppose $tz \in T(z)$ for some $t \in [0, 1)$. Since K is an acute cone, the set $G(z, 0)$ is bounded, and thus by Lemma 3.3 of [6] the set $G(z, tz)$ is bounded. Therefore Theorem 4 completes the proof. □

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