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On a weak Freudenthal spectral theorem

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Abstract. Let X be an Archimedean Riesz space and $\mathcal{P}(X)$ its Boolean algebra of all band projections, and put $\mathcal{P}_e = \{Pe : P \in \mathcal{P}(X)\}$ and $\mathcal{B}_e = \{x \in X : x \wedge (e - x) = 0\}$, $e \in X^+$. X is said to have Weak Freudenthal Property (WFP) provided that for every $e \in X^+$ the lattice $lin\,\mathcal{P}_e$ is order dense in the principal band e^{dd}. This notion is compared with strong and weak forms of Freudenthal spectral theorem in Archimedean Riesz spaces, studied by Veksler and Lavrič, respectively. WFP is equivalent to X^+-denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ for every $e \in X^+$, and every Riesz space with sufficiently many projections has WFP (THEOREM).

Keywords: Freudenthal spectral theorem, band, band projection, Boolean algebra, disjointness

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0. Introduction.

The classical Freudenthal spectral theorem states that if a Riesz space X has the principal projection property (PPP, in short) then every positive element of the principal ideal A_e (for a given $e \in X^+$) is e-uniformly approximated by linear combinations of $\mathcal{P}_e := \{Pe : P \in \mathcal{P}(X)\}$, where $\mathcal{P}(X)$ denotes the Boolean algebra of all projection bands of X.[1] Put $\mathcal{B}_e := \{x \in X : x \wedge (e - x) = 0\}$. The set $\mathcal{B}_e$ is the Boolean algebra of all components of e with operations $\wedge$ and $\vee$ restricted from X and the complementation $x' = e - x$, and $\mathcal{P}_e$ is its Boolean subalgebra. It is easy to check that both $lin\,\mathcal{P}_e$ and $lin\,\mathcal{B}_e$ are sublattices of A_e (with $lin\,\mathcal{P}_e \subset lin\,\mathcal{B}_e$). Veksler [8] and Lavrič [5] considered independently the following notions of weak and strong forms of Freudenthal's theorem (the symbol $()_V$, resp. $()_L$, denotes that the notion goes back to Veksler, resp. Lavrič).

$(\mathrm{WF})_V$ *Weak form. For every $e \in X^+$, $lin\,\mathcal{B}_e$ is order dense in e^{dd}.*

$(\mathrm{SF})_L$ *Strong form. For every $e \in X^+$, each $x \in A_e$ is e-uniformly approximated by elements of $lin\,\mathcal{P}_e$.*

The notion of PPP can be generalized in two ways:

$(*)$ *every nontrivial band of X contains a nontrivial projection band,*

$(**)$ *every pair of disjoint elements of X is contained in a disjoint pair of projection bands of X.*

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[1] We consider, for simplicity, Archimedean Riesz spaces only and use the terminology and notations of [6] to which the reader is referred; some of our results are valid in the non-Archimedean case where bands should be replaced by disjoint complements.

Riesz spaces fulfilling $(*)$ are said to have *sufficiently many projections* (SMP, in short; see [6]) or the property (P_2) ([8, p. 10]), while $(**)$ is called *the property* δ ([5, p. 418]) or *the property* (P_3) ([8, p. 10]). The above forms of Freudenthal's theorem are characterized by Veksler and Lavrič, respectively, as follows:

Proposition 0.1 (See [8], Theorem 2.5, and [5], Proposition 3.3 and Theorem 3.8). $(\mathrm{WF})_V$ *holds for* X *iff every principal ideal of* X *has* SMP.
$(\mathrm{SF})_L$ *holds for* X *iff* X *has the property* δ *iff* $\mathcal{P}_e = \mathcal{B}_e$ *for every* $e \in X^+$.

(Note that the notions of $(\mathrm{SF})_V$ and $(\mathrm{WF})_L$, concerned with e-uniform approximation of elements of A_e by elements of $\mathit{lin}\, \mathcal{B}_e$, coincide but have been characterized by each of the authors in somewhat different way: *every* (*Riesz*) *homomorphic image of every principal ideal of* X *has the property* (P_3) ([8, Theorem 2.8]), and *every principal ideal of* X *is zero-dimensional in the sense of Lavrič* ([5, Corollary 2.8]).)

The present paper is concerned with a characterization of the following notion of *the weak Freudenthal property* (WFP, in short):

For every $e \in X^+$, *the lattice* $\mathit{lin}\, \mathcal{P}_e$ *is order dense in* e^{dd}.

Thus we replace the lattice $\mathit{lin}\, \mathcal{B}_e$ occurring in $(\mathrm{WF})_V$ by the lattice $\mathit{lin}\, \mathcal{P}_e$ (as in $(\mathrm{SF})_L$) which is more convenient in applications. Obviously, $(\mathrm{SF})_L \Rightarrow \mathrm{WFP} \Rightarrow (\mathrm{WF})_V$, thus WFP is (essentially, see Section 3) stronger than $(\mathrm{WF})_V$ but still "weak". Our main result (THEOREM), presented in Section 1, states that WFP is equivalent to X^+-denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ for every $e \in X^+$, and that every Riesz space with SMP has WFP. Such a characterization of WFP is similar to that of δ given by Lavrič (see Proposition 0.1), thus these two properties are very close in terms of $\mathcal{P}_e - \mathcal{B}_e$. The reader should note that THEOREM includes also an interesting connection between natural Boolean algebras occurring in the theory of Riesz spaces: *Boolean denseness of* $\mathcal{B}(X)$ *in* $\mathcal{A}(X)$ (= the Boolean algebras of all projection bands, and bands of X, respectively) *implies* X^+*-denseness of* $\mathcal{P}_e$ *in* $\mathcal{B}_e$ *for every* $e \in X^+$ *and these conditions are equivalent provided* X *has a weak order unit.*

The paper is organized as follows. In Section 1 we present preliminary facts, notions and our main result, Section 2 is devoted to the study of Riesz spaces with sufficiently many projections, Section 3 is concerned with the weak Freudenthal property, and Section 4 includes some examples.

Problems presented in this paper have appeared in a natural way when the author tried to verify whether the order convergence of orthomorphisms on Riesz spaces with SMP is pointwise. This conjecture is confirmed in [3, Corollary 2.5] by the method of extended orthomorphisms, but an alternative proof can be given with the help of order density of $\mathit{lin}\, \mathcal{P}_e$ in e^{dd} for every $e \in X^+$.

1. Preliminaries.

Let X be an Archimedean Riesz space. $\mathcal{A}(X)$, $\mathcal{B}(X)$, and $\mathcal{P}(X)$, respectively, denote the Boolean algebras of all bands, projection bands, and band projections of X, respectively. If $P \in \mathcal{P}(X)$ [resp. $A \in \mathcal{A}(X)$] then $P^c := I - P$ [resp. $A^d := \{x \in X : |x| \wedge |a| = 0 \text{ for all } a \in A\}$]. $\mathcal{P}(X)$ and $\mathcal{B}(X)$ are Boolean isomorphic via the mapping $P \to PX$ with $P^c \to (PX)^d = P^cX$. A_x and x^{dd}, respectively, stand for the principal ideal and band, respectively, generated by a given $x \in X$. Notice that *a Riesz space X has sufficiently many projections* (see $(*)$, Section 0) *iff $\mathcal{B}(X)$ is Boolean dense in $\mathcal{A}(X)$*. For fundamental information concerning SMP (as well as $\mathcal{A}(X)$, $\mathcal{B}(X)$and $\mathcal{P}(X)$), the reader is referred to the monograph [6, Section 30], where it is proved, for instance, that X is Archimedean whenever X has SMP, and that the class of all Riesz spaces with PPP is properly included in the class of all Riesz spaces possessing SMP (pp. 174–175) (see also Section 4).

We shall now introduce a very useful tool in descriptions both of Riesz spaces with SMP and WFP. Let A be an order ideal of X. It is easy to check that the class $\mathcal{J}(A) := \{B \in \mathcal{B}(X) : B \subset A\}$ is an ideal of $\mathcal{B}(X)$; $\mathcal{J}_{\mathcal{P}}(A)$ denotes the ideal $\{P \in \mathcal{P}(X) : PX \subset A\}$ of $\mathcal{P}(X)$ corresponding to $\mathcal{J}(A)$. Put $[A] = \bigcup \mathcal{J}(A)$ and notice that *$[A]$ is always a linear subspace and a solid subset of X, thus an order ideal of X included in A*. Now let A be a band of X. We have always $[A]^{dd} \subset A$. In the sequel we shall be interested in characterization of those bands for which $[A]^{dd} = A$.

Definition 1.1. Let A be a band of X. A is said to be *a generative band of X* provided that the ideal $[A]$ is order dense in A or, equivalently, $[A]^{dd} = A$. In other words, $A = \sup \mathcal{J}(A)$ where the "sup" is taken in $\mathcal{A}(X)$.

The first lemma summarizes fundamental properties of the ideal $[A]$. The routine proof is omitted.

Lemma 1.2. *Let A, A_1, A_2 be order ideals of X.*

(i) *We have always $[A] \subset A$ and $A^d \subset [A]^d$.*
(ii) *If $A_1 \subset A_2$ then $\mathcal{J}(A_1) \subset \mathcal{J}(A_2)$ (and $\mathcal{J}_{\mathcal{P}}(A_1) \subset \mathcal{J}_{\mathcal{P}}(A_2)$) whence $[A_1] \subset [A_2]$ and $[A_1]^{dd} \subset [A_2]^{dd}$.*
(iii) *An element x is a member of $[A]$ iff for every $P \in \mathcal{J}(A)$ we have $Px = x$.*
(iv) *An element y is a member of $[A]^d$ iff for every $P \in \mathcal{J}(A)$ we have $Py = 0$.*

Moreover, if A is a band then

(v) *A is generative iff there is a nonempty class $\mathcal{V}$ of projection bands of X with $A = (\bigcup \mathcal{V})^{dd}$; in particular, for every ideal A of X the band $[A]^{dd}$ is generative;*
(vi) *$[A]^{dd}$ is the greatest generative band included in A; in particular, $[x^{dd}]^{dd}$ is the greatest generative band included in x^{dd}.*

Corollary 1.3. *Let A be an order ideal of X. Then for every $y \in A^d$ and every $\mathcal{P} \in \mathcal{J}_{\mathcal{P}}(A)$ we have $Py = 0$. In particular, for every $e \in X^+$ and any $u \in \mathcal{P}_e$ we have $\mathcal{J}_{\mathcal{P}}(u^{dd}) \subset \{P \in \mathcal{P}(X) : Pu = Pe\} = \{P \in \mathcal{P}(X) : Pe \leq u\}$.*

Proof: The first part is implied by Lemma 1.2, (i) and (iv). The inclusion of the second part is followed by $e - u \in u^d = (u^{dd})^d$ and the first part. To prove the equality notice first that we have always the inclusion $\subset$. On the other hand, if $Pe \leq u$ then $Pe = P^2e \leq Pu$ and so $0 \leq P(e-u) \leq Pu - Pu = 0$; this yields the required equality. $\square$

In the next lemma, we give equivalent conditions for a band A to be generative.

Lemma 1.4. *Let A be a band of X. Then the following conditions are equivalent.*

(i) *A is generative.*
(ii) *$A = (\bigcup \mathcal{V})^{dd}$ for some nonempty $\mathcal{V} \subset \mathcal{B}(X)$.*
(iii) *$A^d = \bigcap_{B \in \mathcal{V}} B^d$ for some nonempty $\mathcal{V} \subset \mathcal{B}(X)$.*
(iv) *$A^d = \bigcap_{B \in \mathcal{J}(A)} B^d = \bigcap_{P \in \mathcal{J}_{\mathcal{P}}(A)} P^c X$.*
(v) *There is a nonempty $\mathcal{W} \subset \mathcal{P}(X)$ such that: $Px = 0$ for every $P \in \mathcal{W}$ iff $x \in A^d$.*
(vi) *If $Px = 0$ for every $P \in \mathcal{J}_{\mathcal{P}}(A)$ then $x \in A^d$.*
(vii) *An element x is a member of A^d iff $Px = 0$ for every $P \in \mathcal{J}_{\mathcal{P}}(A)$. In particular, the intersection of a generative band in X with an order ideal Y of X is a generative band in Y.*

Proof: The equivalence of (i)–(vii) is followed by the parts (i), (iv) and (v) of Lemma 1.2 and the correspondence between projection bands and band projections described above. We shall now prove the second part of the lemma. Let Y be an order ideal of X, and let for any nonempty subset V of Y the symbol V^D denote the disjoint complement of V in Y. Since the intersection of any [projection] band of X with the ideal Y is a [projection] band of Y, we have $A_Y := A \cap Y \in \mathcal{A}(Y)$ and $B_Y := B \cap Y \in \mathcal{B}(Y)$ for all $B \in \mathcal{J}(A)$. By the definition of a disjoint complement we have both $A_Y^D = A^d \cap Y$ and $B_Y^D = B^d \cap Y$, thus the result is followed by the part (iii) of our lemma. □

Let $\mathcal{G}(X)$ denote the set of all generative bands of X. We have $\mathcal{B}(X) \subset \mathcal{G}(X) \subset \mathcal{A}(X)$ and $\mathcal{B}(X)$ is always dense in $\mathcal{G}(X)$. In the lemma below we discuss the definition of SMP in terms of elements of $\mathcal{G}(X)$. A detailed description of SMP will be given by Proposition 2.1.

Lemma 1.5. *The following conditions are equivalent:*

(i) *X has the* SMP *(i.e. $\mathcal{B}(X)$ is Boolean dense in $\mathcal{A}(X)$).*
(ii) *Every band of X is generative (i.e. $\mathcal{G}(X) = \mathcal{A}(X)$).*
(iii) *For every nonempty subset V of X we have $[V^{dd}]^{dd} = V^{dd}$.*
(iv) *Every principal band of X is generative (i.e. for every $x \in X$ we have $[x^{dd}]^{dd} = x^{dd}$).*
(v) *Every nontrivial principal band of X contains a nontrivial projection band.*

Proof: Obviously, (i), (ii) and (iii) are equivalent by definition, (i) implies (v), and (iii) implies (iv). Since for every band A of X we have $A = \bigcup_{a \in A} a^{dd}$, the class $\mathcal{V} = \{B \in \mathcal{B}(X) : B \in \mathcal{J}(a^{dd})$ for some $a \in A\}$ fulfils $(\bigcup \mathcal{V})^{dd} = A$ whenever every principal band of X is generative; thus, by Lemma 1.2 (v), (iv) implies (ii). Similarly, (v) implies (ii). □

Riesz spaces with SMP will be characterized in Section 2 by the following notion of *g-disjoint elements of X* (a similar concept, of completely disjoint elements, which characterizes the property δ, can be found in [5, p. 418]).

Definition 1.6. Two elements $u, v \in X$ are said to be *generatively disjoint* (*g-disjoint*, in short) if there is a generative band A in X with: $u \in A$ and $v \in A^d$.

Obviously, g-disjoint elements are disjoint and these notions coincide whenever X has WFP (see THEOREM).

Let $e \in X^+$. Since $\mathcal{P}_e$ is a Boolean subalgebra of $\mathcal{B}_e$ and both $\mathcal{P}_e$ and $\mathcal{B}_e$ are sublattices of the lattice X^+ (as well as A_e^+), we may conjecture that there is some kind of denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ provided X has WFP. We may think of Boolean or X^+-denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ (i.e., for every $u \in \mathcal{B}_e$, $u = \sup\{v \in \mathcal{P}_e : v \leq u\}$, where the "sup" is taken in X^+). It occurs that there is a strict connection between the notions of WFP, g-disjointness and X^+-denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ (see THEOREM). We shall now present a somewhat general result which links X^+-denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ with g-disjointness of elements of $\mathcal{B}_e$. This will be used in the proof of Proposition 2.3.

Lemma 1.7 (cf. [5, Proposition 3.2])**.** *Let $e \in X^+$. Then the following are equivalent.*

(i) *$\mathcal{P}_e$ is X^+-dense in $\mathcal{B}_e$ (i.e. for every $u \in \mathcal{B}_e$ there is a net $(\mathcal{P}_\alpha) \subset \mathcal{P}(X)$ with $u = \sup \mathcal{P}_\alpha e$ where the* "sup" *is taken in X^+).*

(ii) *Every pair of disjoint components of e is g-disjoint.*

Proof: (i) *implies* (ii). Let $u, v \in \mathcal{B}_e$ with $u \wedge v = 0$. Denote $\mathcal{W} = \{P \in \mathcal{P}(X) : Pe \leq u\}$. By assumption, $\mathcal{W}$ is nonempty; thus, by Lemma 1.4, the band $A = (\bigcup_{P \in \mathcal{W}} PX)^{dd}$ is generative. We shall now prove that $u \in A$ and $v \in A^d$, i.e. u and v are g-disjoint. Since $Pe \in A$ for all $P \in \mathcal{W}$, $u = \sup Pe \in A$ (by X^+-denseness). On the other hand, $u + v = u \vee v \leq e$, and so $Pu + Pv \leq u$ for all $P \in \mathcal{W}$. It follows that $Pv \leq P^c u$ and, consequently, $Pv = 0$. Thus, by Lemma 1.4 (v), $v \in A^d$.

(ii) *implies* (i). Let $u \in \mathcal{B}_e$. We have to show that $u = \sup P_\alpha e$ (in X^+) for some net $(P_\alpha) \subset \mathcal{P}(X)$. By assumption, there exists a generative band A with

$$u \in A \quad \text{and} \quad e - u \in A^d. \tag{1}$$

Since $[A]$ is order dense in A, there is a net $(P_\alpha) \subset \mathcal{J}_{\mathcal{P}}(A)$ with $u = \sup P_\alpha x_\alpha$ for some $(x_\alpha) \subset [A]$. By (1) and Lemma 1.4 (vii), $P_\alpha e = P_\alpha u$, whence $u \geq P_\alpha e \geq P_\alpha x_\alpha$. Consequently, $u = \sup P_\alpha e$ (in X^+), as claimed. □

We are now in a position to formulate our main result.

THEOREM. *Consider the following four conditions.*

(i) *X has* SMP.
(ii) *X has* WFP.
(iii) *For every $e \in X^+$, $\mathcal{P}_e$ is X^+-dense in $\mathcal{B}_e$.*
(iv) *Every pair of disjoint elements of X is g-disjoint.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). *If, moreover, X has a weak order unit then all four conditions are equivalent.*

The proof follows immediately from Proposition 2.2 and Proposition 3.2. For the proof of the second part of THEOREM (via Proposition 2.2) we need the following lemma on nontrivial complementations of positive elements to weak order units (the constructive proof presented here has been communicated to the author by Professor Z. Lipecki).

Lemma 1.8. *Let $\varepsilon(X)$ denote the set of all weak order units of X. If $\varepsilon(X) \neq \emptyset$ then for every $x > 0$ there exists $y \geq 0$ with $y \notin \varepsilon(X)$ such that $x \vee y \in \varepsilon(X)$ (equivalently, $x^{dd} \supset y^d$).*

Proof: Assume that $x \notin \varepsilon(X)$ and let $0 < e \in \varepsilon(X)$. It follows that for all $n \in \mathbb{N}$ we have $(e - nx)^+ > 0$. Since X is Archimedean, we have $(e - mx)^- > 0$ for some $m \in \mathbb{N}$; in particular, $y := (e - mx)^+$ is not a weak order unit of X. We claim that $x \vee y \in \varepsilon(X)$. Indeed, if $(x \vee y) \wedge z = 0$ for some $z \in X$ then $mx \wedge z = 0$ and therefore $(y + mx) \wedge z = 0$; consequently, $e \wedge z = 0$, whence $z = 0$. □

Remark 1.9. If X is a separable Banach lattice then we have a stronger form of the above result: *For every positive x which is not a weak order unit of X there exists a weak order unit $e > 0$ with $x \wedge (e - x) = 0$* (i.e. $x \in \mathcal{B}_e$). To prove this use the fact that any maximal disjoint system in X (containing x, for example) is countable.

2. Riesz spaces with sufficiently many projections.

In the first theorem of this section we describe Riesz spaces with SMP in terms of ideals of $\mathcal{P}(X)$ introduced before Definition 1.1. Such a description enables us to link SMP both with g-disjointness of elements of X and X^+-denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$, as presented in the second theorem. As before, if required, X is Archimedean. Our first result follows, among others, that $\mathcal{P}(X)$ is sufficiently rich to determine x^+ by x provided X has SMP (part (iv)). Observe that in the case of X with PPP, the ideal $\mathcal{J}_\mathcal{P}(x^{dd})$ contains the greatest element P_x (= the band projection onto x^{dd}), thus the statement (ii)–(v) hold automatically.

Proposition 2.1 (cf. [5, Proposition 3.3]). *The following conditions are equivalent.* (*The* "sup" *in conditions* (ii)–(v) *is taken in X^+.*)

(i) *X has* SMP (*thus, X satisfies the equivalent assertions of Lemma* 1.5).
(ii) *For every band A of X and any $x \in A^+$ we have $x = \sup\{Px : P \in \mathcal{J}_\mathcal{P}(A)\}$.*
(iii) *For every $e \in X^+$ and any $u \in \mathcal{B}_e$ we have $u = \sup\{Pe : P \in \mathcal{J}_\mathcal{P}(u^{dd})\}$.*
(iv) *For every $x \in X$ we have $x^+ = \sup\{Px : P \in \mathcal{J}_\mathcal{P}((x^+)^{dd})\}$.*
(v) *For every $x \in X^+$ we have $x = \sup\{Px : P \in \mathcal{J}_\mathcal{P}(x^{dd})\}$.*

Proof: We shall prove that the following implications hold:
(i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii).

(i) *implies* (ii). Let $y \in X^+$ and $x - Px \geq y$ for every $P \in \mathcal{J}_\mathcal{P}(A)$. We shall show that $y = 0$, i.e. $\inf\{x - Px : P \in \mathcal{J}_\mathcal{P}(A)\} = 0$. We have $x - Px \in P^cX$ for all $P \in \mathcal{J}_{\mathcal{P}}(A)$, whence, by assumption and Lemma 1.4 (iv), $y \in A^d$. On the other hand, since $x \in A^+$, we have $y \in A$; consequently, $y \in A \cap A^d = \{0\}$.

(ii) *implies* (i). Let A be a band of X. We shall prove that (ii) follows $A \cap [A]^d = \{0\}$ and therefore (as $A^d \subseteq [A]^d$, see Lemma 1.2) we have $A = [A]^{dd}$; in other words, A is generative. If $0 \leq x \in A \cap [A]^d$ then, by Lemma 1.2 (i), $Px = 0$ for every $P \in \mathcal{J}_{\mathcal{P}}(A)$; now (ii) yields $x = 0$.

(iii) *implies* (ii). It follows by Lemma 1.2 (ii), as $x \in A$ implies $x^{dd} \subset A$.

(ii) *implies* (iii). By (ii), $u = \sup\{Pu : P \in \mathcal{J}_{\mathcal{P}}(u^{dd})\}$, thus Corollary 1.3 implies (iii).

(iii) *implies* (iv). Let $e = |x| = x^+ + x^-$. We have $x^+, x^- \in \mathcal{B}_e$ and $x^- \in (x^+)^d$. It follows, by Corollary 1.3, that for every $P \in \mathcal{J}_{\mathcal{P}}((x^+)^{dd})$ we have $Px = 0$, whence $Px^+ = Pe = Px$; now (iii) implies (iv).

(iv) *implies* (v). Obvious.

(v) *implies* (ii). It follows by Lemma 1.2 (ii), as $x \in A$ implies $x^{dd} \subseteq A$. □

The rest of this section is mainly devoted to the study of SMP in Riesz spaces with weak order units.

Proposition 2.2. *Consider the following conditions.*

(i) *X has* SMP.
(ii) *For every $e \in X^+$, $\mathcal{P}_e$ is X^+-dense in $\mathcal{B}_e$.*
(iii) *Every pair of disjoint elements of X is g-disjoint.*

We have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). *Moreover, if X has a weak order unit then all three conditions are equivalent.*

Proof: By Proposition 2.1 (iii), (i) implies (ii). The equivalence of (ii) and (iii) is followed by Lemma 1.7. In fact, the implication (iii) $\Rightarrow$ (ii) is obvious, and putting $e = |x| + |y|$ whenever $|x| \wedge |y| = 0$ we obtain that (ii) implies (iii). Now let X have a weak order unit. To prove that in this case (iii) implies (i) it suffices to show, by Lemma 1.5, that any nontrivial principal band contains a nontrivial projection band. Let $x \in X^+ \setminus \{0\}$ and consider the principal band x^{dd}. We may assume that x is not a weak order unit of X. By Lemma 1.8, there exist two strictly positive elements y, z in X with $x^{dd} \supset y^d \ni z$. By (iii), there is a nontrivial (containing z) generative band A with $y \in A^d$. Thus, $x^{dd} \supset y^d \supset A \supset B$ for some nontrivial projection band B. □

Combining the above result with Proposition 0.1 we get

Corollary 2.3 ([8, Theorem 1.6.2]). *In Riesz spaces with weak order units,* $(\mathrm{SF})_L$ *implies* SMP.

Remark 2.4. The second part of Proposition 2.2 (and its proof) cannot be reduced to the case where e is a weak order unit; in other words, *the denseness of $\mathcal{P}_e$ in $\mathcal{B}_e$ for some weak order unit e does not imply the denseness of $\mathcal{P}(X)$ in $\mathcal{B}(X)$.* Indeed, $\mathcal{B}_e$ may even be trivial and equal to $\mathcal{P}_e$, as in the case of the Riesz space $C[0,1]$, which obviously has no SMP, and $e = \mathbf{1}_{[0,1]}$ (the constant-one function on $[0,1]$).

The following observation explains the role of the above example. It is readily seen that if e is a weak order unit then the mapping $h : \mathcal{B}_e \to \mathcal{A}(X)$ of the form $h(u) = u^{dd}$ is a Boolean isomorphism into. Since $(Px)^{dd} = x^{dd} \cap PX$ holds for any $P \in \mathcal{P}(X)$ and $x \in X^+$, h restricted to $\mathcal{P}_e$ is *onto* $\mathcal{B}(X)$. Thus, *if* $\mathcal{P}_e$ *is dense in* $\mathcal{B}_e$ *then* $\mathcal{B}(X)$ *is dense in* $h(\mathcal{B}_e)$; however, *in general* $h(\mathcal{B}_e)$ *is not a dense subalgebra of* $\mathcal{A}(X)$, as the example given in Remark 2.4 shows. Nevertheless, by the above observation, Proposition 0.1, Proposition 2.2 and Corollary 2.3, we have

Corollary 2.5. *Let e be a weak order unit of X. If X has* SMP *then $h(\mathcal{B}_e)$ is a Boolean dense subalgebra of $\mathcal{A}(X)$. Moreover, if X is a δ-space then we have additionally $h(\mathcal{B}_e) = \mathcal{B}(X)$.*

Recall (see [9, Chapter 20]) that an order bounded operator T on X is called *an orthomorphism* provided T is band preserving or, equivalently, $x \wedge y = 0$ implies $|Tx| \wedge y = 0$. The set $Orth\,(X)$ of all orthomorphisms on X forms an Archimedean Riesz space with

$$(2) \qquad (T_1 \vee T_2)x = T_1x \vee T_2x \quad \text{and} \quad (T_1 \wedge T_2)x = T_1x \wedge T_2x \quad \text{for all} \quad x \in X^+,$$

and I (= the identity on X) is a weak order unit of $Orth\,(X)$. An orthomorphism T is called *central* if $|T| \le \lambda \cdot I$ for some $\lambda \ge 0$, and the set $Z(X)$ of all central orthomorphisms on X is an order ideal of $Orth\,(X)$. It is readily seen that if X is (σ-)Dedekind complete then $Orth\,(X)$ is (σ-)Dedekind complete as well, and it is proved in [9, Exercises 140.12 and 140.13 (v)] that both uniform completeness and projection property are heredited by $Orth\,(X)$. In the last theorem of this section we shall prove that the same holds for SMP.

Proposition 2.6. *If X has* SMP *then both every order ideal of X and $Orth\,(X)$ (as well as $Z(X)$) have* SMP.

Proof: Let Y be an order ideal of X and let $A \in \mathcal{A}(Y)$. By assumption, the band A^{dd} of X contains a nontrivial projection band B. We have $B \cap A \neq \{0\}$ (if not, then by [6, Theorem 19.3 (iv)], $B \cap A^{dd} = \{0\}$, a contradiction), and so $B_1 := B \cap Y$ is a nontrivial projection band in Y (see the proof of Lemma 1.4). We claim that $B_1 \subset A$, equivalently $B_1^D \supset A^D$, but this is evident, as $B \cap Y^{dd} = B$ implies (by [6, Theorem 19.3 (iv)]) $B_1^D = (B \cap Y^{dd})^d \cap Y = B^d \cap Y \supset A^d \cap Y = A^D$. Thus we have proved that $\mathcal{B}(Y)$ is Boolean dense in $\mathcal{A}(Y)$, i.e. Y has SMP.

In proof of the second part of our theorem we shall employ Lemma 1.8. Let $T \in Orth\,(X)$. By Lemma 1.5 (v), we have to show that T^{dd} contains a nontrivial projection band. Without loss of generality we may assume that T is not a weak order unit of $Orth\,(X)$. By Lemma 1.8, there exists $T_1 > 0$ which is not a weak order unit of $Orth\,(X)$ with

$$(3) \qquad T^{dd} \supset T_1^d.$$

By assumption, there is a band projection P on X with $ker\,P \subset ker\,T_1$ (as, by [9, Theorem 140.5 (i)], $ker\,T_1$ is a band of X) which, by (2), implies that

$$(4) \qquad T_1^d \supset P^d.$$

Now observe that P^d is a projection band of $Orth\,(X)$, since $P^d = \{S \in Orth\,(X) : PS = SP = 0\} = \{S : \hat{P}(S) = 0\} = ker\,\hat{P}$, where $\hat{P}$ is the band projection of $Orth\,(X)$ defined by the formula $\hat{P}(S) = PS$. Thus, by (3) and (4), T^{dd} contains a nontrivial projection band, as required. By the first part of our theorem, $Z(X)$ has SMP as well. □

3. Weak Freudenthal property.

In this section we study connections between WFP and $(\mathrm{WF})_V$. We start with a lemma which will simplify the proof of the next theorem.

Lemma 3.1. (a) *The property given in the part* (iii) (*or equivalently,* (ii)) *of Proposition* 2.2 *is heredited by order ideals.*

(b) *The following conditions are equivalent:*

- (i) $(\mathrm{WF})_V$ *holds for* X.
- (ii) *For every* $e \in X^+$, *every pair of disjoint elements of the principal ideal* A_e *is* g*-disjoint* (*in* A_e).

It follows that if $\mathcal{P}_e$ *is* X^+*-dense in* $\mathcal{B}_e$ *for every* $e \in X^+$ *then* $(\mathrm{WF})_V$ *holds for* X.

Proof: (a) Assume that Y is an order ideal of X and that any two disjoint elements of X are g-disjoint. Let y_1 and y_2 be disjoint elements of Y. By assumption, there exists a generative band A in X with $y_1 \in A$ and $y_2 \in A^d$. Putting $C = A \cap Y$ we see that, by Lemma 1.4, C is a nonempty generative band of Y with $y_1 \in C$ and $y_2 \in C^D$; thus, y_1 and y_2 are g-disjoint in Y.

(b) This part is followed by Proposition 0.1, Proposition 2.2 and part (a). □

We shall now present the main result of this section, which is a counterpart of Proposition 3.3 and Theorem 3.8 of [5]. The equivalence of the parts (i) and (iii) gives a full description of the relations between WFP and $(\mathrm{WF})_V$.

Proposition 3.2. *The following conditions are equivalent.*

- (i) X *has* WFP.
- (ii) $\mathcal{P}_e$ *is* X^+*-dense in* $\mathcal{B}_e$ *for every* $e \in X^+$.
- (iii) $(WF)_V$ *holds for* X *and* $\mathcal{P}_e$ *is Boolean dense in* $\mathcal{B}_e$ *for every* $e \in X^+$.
- (iv) *Every pair of disjoint elements of* X *is* g*-disjoint.*
- (v) *For every* $x \in X$ *we have* $x^+ = \sup\{Px : Px^- = 0$ *and* $P \in \mathcal{P}(X)\}$.

Proof: We shall prove that the following relations hold:
(i) ⇒ (ii) ⇒ (iii) ⇒ (i) and (iv) ⇔ (ii) ⇔ (v).

(i) *implies* (ii). Let $u \in \mathcal{B}_e$ for some $e \in X^+$ and put $E = [0, u] \cap lin\,\mathcal{P}_e$. Let $lin^{\,+}\mathcal{P}_e$ denote the set of all linear and positive combinations of orthogonal elements of $\mathcal{P}_e$. It is easy to check that $(lin\,\mathcal{P}_e)^+ = lin^{\,+}\mathcal{P}_e$, whence $E = [0, u] \cap lin^{\,+}\mathcal{P}_e$. Now let y be a fixed element of E; thus $y = \sum_{i=1}^n \lambda_i P_i e$ for some $\lambda_i > 0$, $i = 1, 2, \ldots, n$, and $P_i e \wedge P_j e = 0$ for $i \neq j$. Note that without loss of generality we may assume that $P_i \perp P_j$, i.e. $P_j P_i = 0$, which follows $Q := \sum_{i=1}^n P_i \in \mathcal{P}(X)$. Since $e \geq u \geq$

$\lambda_i P_i e = (\lambda_i P_i(e-u)) \vee (\lambda_i P_i u)$, we have $\lambda_i \leq 1$ and $0 = u \wedge (e-u) \geq \lambda_i P_i(e-u) \geq 0$, whence

$$P_i e = P_i u \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{5}$$

By (5), we have $u \geq Qu = Qe \geq \sum_{i=1}^n \lambda_i P_i e = y$, thus, by part (i), $u = \sup_{X^+} E = \sup_{X^+}\{Pe : Pe \leq u\}$. In other words, $\mathcal{P}_e$ is X^+-dense in $\mathcal{B}_e$, as claimed.

(ii) *implies* (iii). Obviously (ii) implies the second part of (iii), and the first one is followed by Lemma 3.1 (b).

(iii) *implies* (i). We have to show that $lin\,\mathcal{P}_e$ is order dense in A_e. An inspection of the proof of ([8, Lemma 2.1]) shows that a somewhat general result than given there is true: *Let $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{B}_e$. Then $lin\,\mathcal{B}_e$ is an order dense sublattice of A_e iff for every $x \in A_e^+$ there are $u \in \mathcal{B}$ and $\lambda > 0$ with $\lambda \cdot u \leq x$.* Now this result and part (iii) easily imply part (i).

The parts (ii) *and* (iv) *are equivalent* by Proposition 2.2.

(ii) *and* (v) *are equivalent.* Fix $x \in X$ and put $e = |x|$, $u = x^+$. By Corollary 1.3, we have

$$\{P \in \mathcal{P}(X) : Px = 0\} = \{P \in \mathcal{P}(X) : P|x| \leq x^+\}, \tag{6}$$

thus (ii) and (6) imply (v). If $u \in \mathcal{B}(X)$ then for $x = 2u - e = u - (e-u)$ we have $x^+ = u$ and $x^- = e - u$, and so $|x| = e$; now (6) and (v) imply (ii). □

Proof of THEOREM: It follows by Propositions 2.2 and 3.2. □

By Lemma 3.1 (a) and Proposition 3.2, we obtain

Corollary 3.3. WFP *is heredited by order ideals.*

Theorem 2.3 of [8] states that if a Riesz space X has a strong order unit then $(\mathrm{WF})_V$ holds for X iff X has SMP; thus, by THEOREM, we have

Corollary 3.4. *If X has a strong order unit then* $(\mathrm{WF})_V$ *and* WFP *coincide* (*via* SMP).

Corollary 3.5. *If X has* SMP *then for every $T \in Orth\,(X)$ we have $T^+ = \sup\{PT : PT^- = 0 \;\&\; P \in \mathcal{P}(X)\}$.*

Proof: By Proposition 2.6 and THEOREM, WFP holds for $Orth\,(X)$, thus the result is implied by Proposition 3.2 (iv) and (unique) representation of elements $\mathcal{P}(Orth\,(X))$ by elements of $\mathcal{P}(X)$ (this is followed by the relation $Orth\,(Orth\,(X)) = Orth\,(X)$; see [9, Theorems 140.9 and 141.1]). □

Remark 3.6. Using some representation theorems Veksler has constructed two Riesz spaces X and Y with the following properties:

(A) X has a weak (but not strong) order unit e and $(\mathrm{WF})_V$ holds for $X = e^{dd}$, but X fails to have SMP ([8, Example 3.3]);

(B) Y has no weak order unit, possesses the property δ (equivalently, by Proposition 0.1, $(\mathrm{SF})_L$ holds for Y) and fails to have SMP ([8, Example 3.4]).

(A) proves that

(i) WFP *is essentially stronger than* $(\mathrm{WF})_V$ (in fact, by THEOREM, X fails to have WFP as well, and obviously WFP implies $(\mathrm{WF})_V$ in general),

(ii) WFP *cannot be lifted from principal ideals to principal bands* (this is a simple consequence of Proposition 0.1 and Corollary 3.4 applied to A_e and $e^{dd} = X$), and, by Proposition 3.2,

(iii) *in contrast to* WFP, $(\mathrm{WF})_V$ *for* X *does not imply that for every* $e \in X^+$, $\mathcal{P}_e$ *is Boolean dense in* $\mathcal{B}_e$.

(B) shows that

(iv) WFP *is essentially stronger than* SMP,

(v) (together with Corollary 3.4 and Example 4.2 (a) (or 4.3)) *there are no connections between* $(\mathrm{SF})_L$ *and* SMP, *in general* (cf. Corollary 2.3).

In the next theorem we collect main results of the paper (see $(*)$ and $(**)$ in Section 0, THEOREM, Corollary 2.3, Proposition 3.2, and Corollary 3.4). For this purpose let (BD) (= Boolean denseness) denote the property $\mathcal{P}_e$ *is Boolean dense in* $\mathcal{B}_e$ *for every* $e \in X^+$ (see Proposition 3.2 (iii)).

Theorem 3.7. *The following relations hold, in general:*

$$\mathrm{PPP} \begin{array}{ccc} \Rightarrow & \mathrm{SMP} & \Rightarrow \\ \Rightarrow & (\mathrm{SF})_L & \Rightarrow \end{array} \mathrm{WFP} \Leftrightarrow (\mathrm{WF})_V \ \&\ (\mathrm{BD}) \Rightarrow (\mathrm{WF})_V.$$

If X *has a weak order unit then, additionally,*

$$(\mathrm{SF})_L \Rightarrow \mathrm{SMP} \Leftrightarrow \mathrm{WFP},$$

and if X *has a strong order unit then* SMP $\Leftrightarrow$ WFP $\Leftrightarrow$ $(\mathrm{WF})_V$. *By Remark* 3.6, *all implications are strict.*

4. Examples.

In this section we give several examples of Riesz spaces which have SMP and fail to have $(\mathrm{SF})_L$ or PPP (see the diagram in Theorem 3.7). All examples are dependent upon the results collected in the lemma below. Recall that a compact Hausdorff space Ω is [quasi-] Stonean if every open [open and F_σ] subset of Ω has an open closure; it is an F-space if every two disjoint and open F_σ subsets of Ω have disjoint closures. Ω is zero-dimensional if it possesses a base of closed-open subsets. We have: Stonean $\Rightarrow$ quasi-Stonean $\Rightarrow$ F-space (see [7, p. 432]). Recall also that for $C(\Omega)$-spaces the notions of PPP and σ-Dedekind completeness (as well as SMP and WFP, see Theorem 3.7) coincide ([6, Theorem 43.2]).

Lemma 4.1. (i) *Every discrete* (*i.e. possessing a maximal disjoint system consisting of discrete elements*) *Archimedean Riesz space has* SMP.

(ii) $C(\Omega)$ *is Dedekind complete* [*has* PPP, $(\mathrm{SF})_L$ *holds for* $C(\Omega)$, $C(\Omega)$ *has* WFP] *iff* Ω *is Stonean* [*quasi-Stonean, zero-dimensional* F*-space, zero-dimensional*].

(iii) *The Cartesian product of two infinite compact spaces cannot be an* F*-space.*

(iv) *Any closed subspace of an F-space is an F-space.*

PROOF: Part (i) is followed by [1, Theorem 2.16], while (ii) is included in Theorem 1.3 of [8](the parts (5)–(8)) (cf. [6, Section 43]). The part (iii) goes back to W. Rudin ([2, p. 50]). For (iv) see [7, Proposition 24.2.5]. □

In the first example, discrete Archimedean Riesz spaces which, by Lemma 4.1 (i), have SMP (and therefore have WFP also), are given. Call X to have *strict* WFP if X has WFP and $(\mathrm{SF})_L$ does not hold for X.

Example 4.2. *Let A be an infinite set. Then for every [finite] compactification r of the discrete space A, the Riesz space $C(rA)$ has* SMP [*strict* WFP].

Parts (ii)–(iv) of Lemma 4.1, and THEOREM yield the following

Example 4.3. (a) *Let D denote the discrete space consisting of two distinct points, and let $\mathfrak{m}$ be a cardinal number. Then for every $\mathfrak{m} \geq \aleph_0$, $C(D^{\mathfrak{m}})$ has strict* WFP*; in particular, $C(\Delta)$ is of this type, where Δ is the Cantor set.*

(b) *Let A be as in Example* 4.2*, and let βA denote its Stone-Čech compactification.Then for every closed infinite subspace K of βA, $C(K)$ has* $(\mathrm{SF})_L$*, while for $\mathfrak{m} \geq 2$, $C(K^{\mathfrak{m}})$ has strict* WFP (notice that $C(\beta A)$ $(= l_\infty(A))$ is Dedekind complete and, by [7, Propositions 16.5.6 and 24.2.4], $C(\beta A \setminus A)$ fails to have PPP).

Ω is said to be *dyadic* if there is a continuous mapping from $D^{\mathfrak{m}}$ onto Ω for some $\mathfrak{m} \geq \aleph_0$. It is well known that every metric compact space is dyadic. Efimov and Engelking have proved that *infinite dyadic spaces are not quasi-Stonean* and that *every compact metrizable subspace X of $D^{\mathfrak{m}}$ is dyadic* ([4, Theorem 12 and Corollary 8, respectively]). The last example is based on these results (see Lemma 4.1 (ii)).

Example 4.4. *Let Ω be infinite, zero-dimensional and dyadic. Then $C(\Omega)$ has* WFP *and fails to have* PPP*; in particular, every closed metrizable subspace Ω of $D^{\mathfrak{m}}$ is of this type.*

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