# Quasilinear degenerate elliptic variational inequalities with discontinuous coefficients

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*Abstract.* We obtain an existence theorem for the problem (0.1) where the coefficients $a_{ij}(x,s)$ satisfy a degenerate ellipticity condition and hypotheses weaker than the continuity with respect to the variable $s$.

*Keywords:* degenerate elliptic variational inequalities, pseudomonotone operators

*Classification:* 35J85, 35J70

## 0. Introduction.

In this paper we consider the variational inequalities (the obstacle problem)

$$(0.1) \qquad u \in K \qquad \int_\Omega \sum_1^m {}_{ij} a_{ij}(x,u) \frac{\partial u}{\partial x_j} \frac{\partial (v-u)}{\partial x_i}\, dx \geq \langle f, v-u \rangle, \ \forall\, v \in K$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^m$, $f \in H^{-1}(\nu^{-1},\Omega)$, and the coefficients $a_{ij}(x,s)$ satisfy the ellipticity condition

$$\sum_1^m {}_{ij} a_{ij}(x,s)\xi_i\xi_j \geq \nu(x) \sum_1^m {}_i \xi_i^2 \quad \text{a.e.} \quad (x,s) \in \Omega \times \mathbb{R}, \ \forall\, \xi \in \mathbb{R}^m$$

with $\nu(x)$, $\nu^{-1}(x)$ having integrability hypotheses Murty-Stampacchia's kind (see e.g. [9]). Existence results for variational inequality (0.1) have been established in [7] and in [8] but all the results require that the coefficients $a_{ij}(x,s)$ be Carathéodory's functions. The aim of this paper is to prove an existence theorem for the (0.1) (see n. 3) working on the coefficients hypotheses weaker than the continuity with respect to the variable $s$; as a particular case we obtain existence results for the quasilinear elliptic degenerate equations of the type:

$$(0.2) \qquad \begin{cases} -\sum_1^m {}_{ij} \frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f \ \text{ in } \ \Omega \\ u \in H_0^1(\nu,\Omega)\,. \end{cases}$$

We can find similar results, in non-degenerate case, in [10] and in [2], while we get an existence result, concerning the problem (0.2), in [5] where the coefficients $a_{ij}(x,s)$ are Carathéodory's functions and $f$ has a polynomial growth.

## 1. Function spaces.

Let $\mathbb{R}^m$ be the Euclidean $m$-space with generic point $x = (x_1, x_2, \ldots, x_m)$, $\Omega$ a bounded open subset of $\mathbb{R}^m$. The notation $\mathrm{mis}_x$ will indicate in the sequel the Lebesgue measure $m$-dimensional.

**Hypothesis 1.1.** *Let $\nu(x)$ be a positive function defined on $\Omega$; there exist two real numbers $\sigma \in ]0,1[$ and $g > \frac{m}{2}$ such that:*

$$\nu(x) \in L^{1+\sigma}(\Omega), \quad \frac{1}{\nu(x)} \in L^g(\Omega).$$

$H^1(\nu,\Omega)$ denotes the space of the functions $u \in L^2(\Omega)$, which derivatives (in distributional sense on $\Omega$) $\frac{\partial u}{\partial x_i}$ $(i = 1, 2, \ldots, m)$ are functions such that $\sqrt{\nu}\frac{\partial u}{\partial x_i}$ belongs to $L^2(\Omega)$.

$H^1(\nu,\Omega)$ is a Hilbert space with respect to the norm:

$$\|u\|_1 = \Big(\int_\Omega \big(|u|^2 + \sum_1^m {}_i \nu(x)\big|\frac{\partial u}{\partial x_i}\big|^2\big)\, dx\Big)^{1/2}.$$

$H_0^1(\nu,\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\nu,\Omega)$.

If $\gamma(x)$ is a positive measurable function in $\Omega$ and $p$ is a real number greater than or equal to 1, $L^p_\gamma(\Omega)$ denotes the space of the measurable functions $\omega(x)$ in $\Omega$ such that:

$$\|\omega(x)\|_{p,\gamma,\Omega} = \Big(\int_\Omega \gamma(x)|\omega(x)|^p\, dx\Big)^{1/2} < +\infty.$$

$H^{-1}(\nu^{-1},\Omega)$ denotes, finally, the dual space of $H_0^1(\nu,\Omega)$.

## 2. Hypotheses, problems and results.

**Hypothesis 2.1.** *The coefficients $a_{ij}(x,s)$ $(i = 1, 2, \ldots, m)$ are functions defined and measurable in $\Omega \times \mathbb{R}$ and one has:*

$$\frac{a_{ij}(x,s)}{\nu(x)} \in L^\infty(\Omega \times \mathbb{R}) \quad (i,j = 1, \ldots, m).$$

**Hypothesis 2.2.** *For almost every $(x,s)$ in $\Omega \times \mathbb{R}$, it results:*

$$\sum_1^m {}_{ij} a_{ij}(x,s)\xi_i\xi_j \ge \nu(x)\sum_1^m {}_i \xi_i^2 \quad \forall\, \xi \in \mathbb{R}^m.$$

**Hypothesis 2.3.** *For every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \Omega$ with $\text{mis}_x(\Omega \setminus K_\varepsilon) < \varepsilon$, such that for every $r > 0$ the functions of the family $\{a_{ij}(\cdot, s)\}_{|s| \leq r; i,j=1,\ldots,m}$ are equicontinuous on $K_\varepsilon$.*

Let $K$ be a closed, convex and non empty subset of $H_0^1(\nu, \Omega)$(1), $f \in H^{-1}(\nu^{-1}, \Omega)$. Holding the hypotheses(1.1), (2.1), we will consider the following:

**Problem 2.1.** Find a function $u(x) \in K$ such that

$$\int_\Omega \sum_1^m {}_{ij} a_{ij}(x, u) \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_i}(u - v)\, dx \leq \langle f, u - v \rangle \text{ (2)}$$

for any $v(x) \in K$.

In Section 3 we will show the following:

**Theorem 2.1.** *If the hypotheses* (1.1), (2.1), (2.2), (2.3), *hold, the problem* (2.1) *admits solutions.*

## 3. Proof of Theorem 2.1.

We can assume $a_{ij}(x, s) \in \mathscr{B}(\Omega) \otimes \mathscr{L}_1$(3) $(i, j = 1, \ldots, m)$, in fact it is possible to get a real function $b_{ij}(x, s) \in \mathscr{B}(\Omega) \otimes \mathscr{L}_1$ and a set $E_{ij} \in \mathscr{B}(\Omega)$ with $\text{mis}_x(E_{ij}) = 0$ such that $a_{ij}(x, s) = b_{ij}(x, s)$ for every $(x, s) \in (\Omega \setminus E_{ij}) \times \mathbb{R}$(4).

Then, there results a well defined operator

$$A : H_0^1(\nu, \Omega) \to H^{-1}(\nu^{-1}, \Omega) \text{ such that}$$

$$\langle Au, v \rangle = \int_\Omega \sum_1^m {}_{ij} a_{ij}(x, u(x)) \frac{\partial u(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_i}\, dx \text{ (5)}.$$

The operator $A$ is bounded, coercive and pseudomonotone.

To achieve the pseudomonotonicity it is enough to show that:

(i) the operator $A$ is sequentially weakly continuous, i.e. for every $w \in H_0^1(\nu, \Omega)$

$$\lim_{n \to \infty} \langle Au_n, w \rangle = \langle Au, w \rangle \text{ if } u_n \rightharpoonup u \text{ in } H_0^1(\nu, \Omega);$$

(ii) the functional $u \to \langle Au, u \rangle$ is sequentially weakly lower semicontinuous, i.e.

$$\liminf_{n \to \infty} \langle Au_n, u_n \rangle \geq \langle Au, u \rangle \text{ if } u_n \rightharpoonup u \text{ in } H_0^1(\nu, \Omega).$$

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(1) We will take in $H_0^1(\nu, \Omega)$ the following equivalent norm:

$$\|u\|_{1,0} = \left( \int_\Omega \nu(x) \sum_1^m {}_i \left| \frac{\partial u}{\partial x_i} \right|^2 dx \right)^{1/2}, \text{ see [9, Corollary 3.5].}$$

(2) We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H_0^1(\nu, \Omega)$ and $H^{-1}(\nu^{-1}, \Omega)$.

(3) $\mathscr{B}(\Omega)$ denotes the Borel $\sigma$-algebra, $\mathscr{L}_1$ denotes the Lebesgue $\sigma$-algebra on $\mathbb{R}$.

(4) See [6, Theorem 6.1].

(5) We observe that the functions $a_{ij}(x, u(x))$ are Lebesgue-measurable in $\Omega$.

Fix $i, j = 1, \dots, m$, set $B : H_0^1(\nu, \Omega) \to L^2_{1/\nu}(\Omega)$ such that

$$Bu = a_{ij}(x,u)\frac{\partial u}{\partial x_j}\,.$$

The operator $B$ is sequentially weakly continuous between $H_0^1(\nu,\Omega)$ and $L^2_{1/\nu}(\Omega)$, i.e.

$$Bu_n \rightharpoonup Bu \ \text{ in } \ L^2_{1/\nu}(\Omega) \ \text{ when } \ u_n \rightharpoonup u \ \text{ in } \ H_0^1(\nu,\Omega).$$

Indeed, if $T$ is a linear and continuous functional on $L^2_{1/\nu}(\Omega)$, there exists $v \in L^2_\nu(\Omega)$ such that

$$T(w) = \int_\Omega vw\,dx \qquad \text{for every} \qquad w \in L^2_{1/\nu}(\Omega) \ ({}^6);$$

therefore it remains to show that if $u_n \rightharpoonup u$ in $H_0^1(\nu,\Omega)$, for every $v \in L^2_\nu(\Omega)$ one has:

$$\lim_{n\to\infty}\int_\Omega v(x)a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}\,dx = \int_\Omega v(x)a_{ij}(x,u(x))\frac{\partial u}{\partial x_j}\,dx. \tag{3.1}$$

$\frac{a_{ij}(x,s)}{\nu(x)}$ being a bounded function in $\Omega \times \mathbb{R}$ and $C_0^\infty(\Omega)$ dense in $L^2_\nu(\Omega)({}^7)$, we may restrict ourselves to the case $v \in C_0^\infty(\Omega)$.

In fact, let $v \in L^2_\nu(\Omega)$ be, then there exists a sequence $\{\varphi_h\} \subset C_0^\infty(\Omega)$ going to $v$ in $L^2_\nu(\Omega)$.

Now, one has:

$$\begin{aligned}
&\left|\int_\Omega a_{ij}(x,u_n(x))\frac{\partial u_n}{\partial x_j}v(x)\,dx - \int_\Omega a_{ij}(x,u(x))\frac{\partial u}{\partial x_j}v(x)\,dx\right| \le \\
&\le \left|\int_\Omega a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}v(x)\,dx - \int_\Omega a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}\varphi_h(x)\,dx\right| + \\
&+ \left|\int_\Omega a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}\varphi_h(x)\,dx - \int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_j}\varphi_h(x)\,dx\right| + \\
&+ \left|\int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_j}v(x)\,dx - \int_\Omega a_{ij}(x,u)\frac{\partial u}{\partial x_j}\varphi_h(x)\,dx\right|
\end{aligned} \tag{3.2}$$

for every $n$ and $h$.

(6) See [9, p. 7, Proposition 2.3].

(7) In fact, if $f \in L^2_{1/\nu}(\Omega)$ and $\int_\Omega f(x)\varphi(x)\,dx = 0 \ \ \forall \varphi \in C_0^\infty(\Omega)$ then $f(x) = 0$ a.e. in $\Omega$.

Denoting by $E$ a measurable subset of $\Omega$, it results:

$$\begin{aligned}
&\int_E \Big| a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}\varphi_h(x)\Big|\,dx = \\
&= \int_E \Big|\frac{a_{ij}(x,u_n)}{\nu(x)}\Big|\sqrt{\nu(x)}\Big|\frac{\partial u_n}{\partial x_j}\Big|\sqrt{\nu(x)}|\varphi_h(x)|\,dx \le \\
&\le M\int_E \sqrt{\nu(x)}\Big|\frac{\partial u}{\partial x_j}\Big|\sqrt{\nu(x)}|\varphi_h(x)|\,dx \le M\|u_n\|_{1,0}\|\sqrt{\nu(x)}\varphi_h(x)\|_{2,1,E} \le \\
&\le M\cdot R\cdot \|\varphi_h(x)\|_{2,\nu,E}\ (^8) \quad \forall n\in\mathbb{N}.
\end{aligned}$$

Moreover, we observe that:

$$\lim_{h\to\infty} a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}\varphi_h(x) = a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}v(x) \ \text{ a.e. in } \ \Omega,$$
$$\lim_{h\to\infty}\varphi_h(x) = v(x) \ \text{ in } \ L^2_\nu(\Omega).$$

Consequently the first term of the second member of (3.2) goes to zero letting $h\to+\infty$, for any $n\in\mathbb{N}$.

In the same way one proves that the last term of the second member of (3.2) goes to zero.

More, changing $a_{ij}(x,s)$ into $-a_{ij}(x,s)$, it is enough to prove that for every $v\in C_0^\infty(\Omega)$, the functional

$$I(u,\Omega) = \int_\Omega v(x)a_{ij}(x,u)\frac{\partial u}{\partial x_j}\,dx$$

is sequentially weakly lower semicontinuous on $H_0^1(\nu,\Omega)$.

Fix $v(x)\in C_0^\infty(\Omega)$ and set for every $m\in\mathbb{N}$:

$$f_m(x,s,z) = \max[(\varphi(x)a_{ij}(x,s)z_j),(-m)];$$

by Theorem 4.15 of [1] one has that

$$I_m(u,\Omega) = \int_\Omega f_m(x,u,Du)\,dx\ (^9)$$

is sequentially weakly lower semicontinuous on $W^{1,1}(\Omega)$(10). Now, taking into account Lemma 4.3 of [4] and the fact that $u_n \rightharpoonup u$ in $H_0^1(\nu,\Omega)$ we achieve that

(8) $M = \max_{i,j=1,\dots,m}\sup_{\Omega\times\mathbb{R}}\frac{a_{ij}(x,s)}{\nu(x)}$; since $u_n \rightharpoonup u$ in $H_0^1(\nu,\Omega)$, by Banach-Steinhaus's principle, the number $R=\sup_n\|u_n\|_{1,0}$ is finite.

(9) $Du = \left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)$.

(10) $W^{1,1}(\Omega)$ is the space of functions $u\in L^1(\Omega)$ such that $\frac{\partial u}{\partial x_i}$ (in distributional sense) belongs to $L^1(\Omega)$ $(i=1,\dots,m)$.

$Du_n \rightharpoonup Du$ in $L^1(\Omega;\mathbb{R}^n)$; moreover $u_n \to u$ in $L^1(\Omega)$. Consequently $u_n \rightharpoonup u$ in $W^{1,1}(\Omega)$(11) and, then

$$\liminf_{n\to\infty} I_m(u_n,\Omega) \geq I_m(u,\Omega) \quad \forall\, m \in \mathbb{N}. \tag{3.3}$$

It results:

$$\begin{aligned} I_m(u_n,\Omega) &= \int_\Omega f_m(x,u_n,Du_n)\,dx \leq \int_{\Omega_{m,n}} v(x)a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j}\,dx \leq \\ &\leq I(u_n,\Omega) + \int_{\Omega\setminus\Omega_{m,n}} |v(x)||a_{ij}(x,u_n)|\Big|\frac{\partial u_n}{\partial x_j}\Big|\,dx \leq \\ &\leq I(u_n,\Omega) + c\int_{\Omega\setminus\Omega_{m,n}} \sqrt{\nu(x)}\cdot\sqrt{\nu(x)}\Big|\frac{\partial u_n}{\partial x_j}\Big|\,dx \leq \\ &\leq I(u_n,\Omega) + c\Big[\mathrm{mis}_x(\Omega\setminus\Omega_{m,n})\Big]^{\frac{\sigma}{2(1+\sigma)}} \|\nu(x)\|^{1/2}_{1+\sigma,1,\Omega}\cdot\|u_n\|_{1,0}\,, \end{aligned}$$

where

$$\Omega_{m,n} = \Big\{x\in\Omega : v(x)a_{ij}(x,u_n)\frac{\partial u_n}{\partial x_j} \geq -m\Big\}.$$

Being $\Omega\setminus\Omega_{m,n} \subseteq \Omega'_{m,n} = \Big\{x\in\Omega : c\nu(x)\big|\frac{\partial u_n}{\partial x_j}\big| > m\Big\}$, one has:

$$\begin{aligned} I_m(u_n,\Omega) &\leq I(u_n,\Omega) + c\Big[\mathrm{mis}_x(\Omega'_{m,n})\Big]^{\frac{\sigma}{2(1+\sigma)}} \|\nu(x)\|^{1/2}_{1+\sigma,1,\Omega}\cdot\|u_n\|_{1,0} \leq \\ &\leq I(u_n,\Omega) + \frac{\tilde c}{m^{\frac{\sigma}{2(1+\sigma)}}} \end{aligned} \tag{3.4}$$

where $\tilde c$ is a constant dependent on $(\mathrm{mis}_x\,\Omega, \|\nu(x)\|^{1/2}_{1+\sigma,1,\Omega}, R, M, \max_\Omega|\varphi|)$.

By (3.3) and (3.4) we obtain:

$$\begin{aligned} I(u,\Omega) \leq I_m(u,\Omega) &\leq \liminf_{n\to\infty} I_m(u_n,\Omega) \leq \\ &\leq \liminf_{n\to\infty} I(u_n,\Omega) + \frac{\tilde c}{m^{\frac{\sigma}{2(1+\sigma)}}} \quad \forall\, m\in\mathbb{N}, \end{aligned}$$

and for $m$ going to infinity:

$$I(u,\Omega) \leq \liminf_{n\to\infty} I(u_n,\Omega) \ \text{ if } \ u_n \rightharpoonup u \ \text{ in } \ H^1_0(\nu,\Omega).$$

Hence to achieve (i), fix $w \in H^1_0(\nu,\Omega)$, it will be enough to take in (3.1) $v = \frac{\partial w}{\partial x_i}$ $(i = 1,\dots,m)$.

Now, we observe that if $u_n \rightharpoonup u$ in $H^1_0(\nu,\Omega)$, $u_n \rightharpoonup u$ in $W^{1,1}(\Omega)$; by Theorem 4.7 of [1] we obtain (ii).

(11) See p. 3 of [1] for the weak convergence definition in $W^{1,1}(\Omega)$.

Finally, if $u_n \rightharpoonup u$ in $H_0^1(\nu,\Omega)$ and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \le 0$, it follows:

$$\begin{aligned}\liminf_{n\to\infty}\langle Au_n, u_n - w\rangle &= \liminf_{n\to\infty}\{\langle Au_n, u_n\rangle - \langle Au_n, w\rangle\} =\\ &= \liminf_{n\to\infty}\langle Au_n, u_n\rangle - \langle Au, w\rangle \ge \langle Au, u\rangle - \langle Au, w\rangle =\\ &= \langle Au, u - w\rangle \quad \text{for any} \quad w \in H_0^1(\nu,\Omega).\end{aligned}$$

Hence the operator $A$ is pseudomonotone. Next, by hypotheses (2.1) and (2.2), for every $u, w \in H_0^1(\nu,\Omega)$, it results

$$\langle Au, u - w\rangle \ge \|u\|_{1,0}^2 - Mm^2\|u\|_{1,0}\|w\|_{1,0}\,.$$

The existence of a solution is a consequence of well known theorems (see e.g. Theorem 4.17 of [11] with the Proposition 3.1 of [3]). □

**Remark.**

Taking $K = H_0^1(\nu,\Omega)$, we obtain an existence theorem for the problem

$$\tag{4.1} \begin{cases} -\sum_1^m{}_{ij}\frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f \quad \text{in} \quad \Omega \quad (f \in H^{-1}(\nu^{-1},\Omega))\\ u \in H_0^1(\nu,\Omega)\end{cases}$$

moreover, if $\nu(x) \in L^\infty(\Omega)$, $\frac{1}{\nu(x)} \in L^g(\Omega)$ $(g > m)$, for the solutions of the problem (4.1) the maximum principle shown in [9, Theorem 7.2] holds.

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