Ostrowski-Kantorovich theorem and S-order of convergence of Halley method in Banach spaces

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Abstract. Ostrowski-Kantorovich theorem of Halley method for solving nonlinear operator equations in Banach spaces is presented. The complete expression of an upper bound for the method is given based on the initial information. Also some properties of S-order of convergence and sufficient asymptotic error bound will be discussed.

Keywords: nonlinear operator equations, Banach spaces, Halley type method, Ostrowski-Kantorovich convergence theorem, Ostrowski-Kantorovich assumptions, optimal error bound, S-order of convergence, sufficient asymptotic error bound

Classification: 65H10, 65J15, 47H17

1. Introduction.

In this paper, we are concerned with the numerical solution of the nonlinear operator equation

$$P(X) = 0 \tag{1.1}$$

in Banach spaces setting. In recent years, V. Candela and A. Marquina [Computing, **44** (1990), 169–194; Computing, **45** (1990), 355–367] presented a typical Ostrowski-Kantorovich convergence theorem for Halley method in Banach spaces. They also convinced that Halley method is applicable by providing many numerical examples. In this paper, we reconsider the same problem by employing classical Ostrowski-Kantorovich analysis techniques [3], [4]. Under similar assumptions of Newton-Kantorovich theorem for Newton method, we can establish the sufficient conditions and give a complete expression of the error bound for Halley method. And we also continue to discuss how we can find (sufficient conditions) order of convergence without assuming the existence of the solution X^* of equations in Banach spaces.

2. Basic iterative relations.

First we redefine the equivalent iteration form for Halley method. The original Halley method for general one dimensional function $F(x)$,

$$\begin{aligned} H(x_n) &= \frac{F(x_n)F''(x_n)}{F'(x_n)F'(x_n)}\,, \\ x_{n+1} &= x_n - \frac{F(x_n)/F'(x_n)}{\left[1 - \frac{1}{2}H(x_n)\right]}\,. \end{aligned} \tag{2.1}$$

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can be rewritten in operator form in Banach spaces as follows.

$$\begin{aligned} &Y_n = X_n - P'(X_n)^{-1}P(X_n),\\ &H(X_n,Y_n) = -P'(X_n)^{-1}P''(X_n)(Y_n - X_n),\\ &X_{n+1} = Y_n - \frac{1}{2}P'(X_n)^{-1}\big[I - \frac{1}{2}H(X_n,Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2. \end{aligned} \tag{2.2}$$

Now we try to find an equivalent expression of $P(X_{n+1})$ related with the $g(t_{n+1})$ so that $P(X_{n+1})$ can be estimated by $g(t_{n+1})$, where $g(t)$ is defined in Section 3.

Lemma 2.3.

$$\begin{aligned} P(X_{n+1}) = &\int_0^1 P''[Y_n + t(X_{n+1} - Y_n)](1-t)\,dt\ (X_{n+1} - Y_n)^2\\ &- \frac{1}{2}\int_0^1 P''[X_n + t(Y_n - X_n)]\,dt\ (Y_n - X_n)P'(X_n)^{-1}\\ &\cdot \big[I - \frac{1}{2}H(X_n,Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2\\ &+ \frac{1}{2}\big[I - \frac{1}{2}H(X_n,Y_n)\big]^{-1}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\\ &\cdot \int_0^1 P''[X_n + t(Y_n - X_n)](1-t)\,dt\ (Y_n - X_n)^2\\ &+ \big[I - \frac{1}{2}H(X_n,Y_n)\big]^{-1}\int_0^1 [P''[X_n + t(Y_n - X_n)](1-t)\\ &- \frac{1}{2}P''(X_n)]\,dt\ (Y_n - X_n)^2. \end{aligned} \tag{2.4}$$

Proof: We start with the identity

$$\begin{aligned} P(X_{n+1}) &= P(X_{n+1}) - P(Y_n) - P'(Y_n)(X_{n+1} - Y_n)\\ &\quad + P(Y_n) + P'(Y_n)(X_{n+1} - Y_n)\\ &= \int_0^1 P''[Y_n + t(X_{n+1} - Y_n)](1-t)\,dt\ (X_{n+1} - Y_n)^2\\ &\quad + P(Y_n) + P'(Y_n)(X_{n+1} - Y_n), \end{aligned} \tag{2.5}$$

observe that from (2.2), we have

$$X_{n+1} - Y_n = -\frac{1}{2}P'(X_n)^{-1}\big[I - \frac{1}{2}H(X_n,Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2.$$

So
(2.6)

$$
\begin{aligned}
&P(Y_n) + P'(Y_n)(X_{n+1} - Y_n) \\
&= P(Y_n) - \frac{1}{2}P'(Y_n)P'(X_n)^{-1}\big[I - \frac{1}{2}H(X_n, Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2 \\
&= P(Y_n) - \frac{1}{2}[P'(Y_n) - P'(X_n)]P'(X_n)^{-1}\big[I - \frac{1}{2}H(X_n, Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2 \\
&\quad - \frac{1}{2}P'(X_n)P'(X_n)^{-1}\big[I - \frac{1}{2}H(X_n, Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2 \\
&= P(Y_n) - \frac{1}{2}\big[I - \frac{1}{2}H(X_n, Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2 \\
&\quad - \frac{1}{2}\int_0^1 P''[X_n + t(Y_n - X_n)]\,dt\ (Y_n - X_n)P'(X_n)^{-1} \\
&\quad \cdot \big[I - \frac{1}{2}H(X_n, Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2.
\end{aligned}
$$

If we denote $\Delta = I - \frac{1}{2}H(X_n, Y_n)$, then (2.6) becomes

$$
\begin{aligned}
&P(Y_n) + P'(Y_n)(X_{n+1} - Y_n) = \Delta^{-1}\big[I + \frac{1}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\big]P(Y_n) \\
&\quad - \frac{1}{2}\Delta^{-1}P''(X_n)(Y_n - X_n)^2 \\
&\quad - \frac{1}{2}\int_0^1 P''[X_n + t(Y_n - X_n)]\,dt\ (Y_n - X_n)P'(X_n)^{-1}\Delta^{-1}P''(X_n)(Y_n - X_n)^2 \\
&= \Delta^{-1}\big[P(Y_n) - \frac{1}{2}P''(X_n)(Y_n - X_n)^2\big] + \frac{1}{2}\Delta^{-1}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)P(Y_n) \\
&\quad - \frac{1}{2}\int_0^1 P''[X_n + t(Y_n - X_n)]\,dt\ (Y_n - X_n)P'(X_n)^{-1}\Delta^{-1}P''(X_n)(Y_n - X_n)^2.
\end{aligned}
$$

Notice the fact that

$$
P(Y_n) = \int_0^1 P''[X_n + t(Y_n - X_n)](1 - t)\,dt\ (Y_n - X_n)^2,
$$

thus we should have

$$
\begin{aligned}
&P(Y_n) + P'(Y_n)(X_{n+1} - Y_n) \\
&= \Delta^{-1}\int_0^1 \big[P''[X_n + t(Y_n - X_n)](1 - t) - \frac{1}{2}P''(X_n)\big]\,dt\ (Y_n - X_n)^2 \\
&+ \frac{1}{2}\Delta^{-1}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\int_0^1 P''[X_n + t(Y_n - X_n)](1 - t)\,dt\ (Y_n - X_n)^2 \\
&- \frac{1}{2}P''[X_n + t(Y_n - X_n)]\,dt\ (Y_n - X_n)P'(X_n)^{-1}\Delta^{-1}P''(X_n)(Y_n - X_n)^2.
\end{aligned}
$$

Substituting in (2.5) we have (2.4). □

3. Some important inequalities.

Lemma 3.1. *Suppose that*

$$\text{(A1)} \qquad \|Y_n - X_n\| \le s_n - t_n,$$

$$\text{(A2)} \qquad \|P'(X_n)^{-1}\| \le -g'(t_n)^{-1},$$

$$\text{(A3)} \qquad \|P''(X_n)\| \le g''(t_n),$$

where $g(t) = \frac{K}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}$, *where* K, β, η, M, *and* N *should satisfy the conditions in Theorem* 4.1. *Then*

$$\text{(C1)} \qquad \big\|\big[I + \frac{1}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\big]^{-1}\big\| \le \big[1 + \frac{1}{2}g'(t_n)^{-1}g''(t_n)(s_n - t_n)\big]^{-1},$$

$$\text{(C2)} \qquad \|X_{n+1} - Y_n\| \le t_{n+1} - s_n,$$

$$\text{(C3)} \qquad \Big\|\int_0^1 \big[2P''[X_n + t(Y_n - X_n)](1-t) - P''(X_n)\big]\,dt\Big\| \le \frac{N}{3}\|Y_n - X_n\|,$$

$$\text{(C4)} \qquad \|P(X_{n+1})\| \le g(t_{n+1}),$$

$$\text{(C5)} \qquad \|Y_{n+1} - X_{n+1}\| \le s_{n+1} - t_{n+1},$$

where $t_0 = 0$, t_n *and* s_n *are defined as follows*

$$\begin{aligned} s_n &= t_n - \frac{g(t_n)}{g'(t_n)}, \\ h_g(t_n, s_n) &= -g'(t_n)^{-1}g''(t_n)(s_n - t_n), \qquad (3.2)\\ t_{n+1} &= s_n - \frac{1}{2}(s_n - t_n)^2 \frac{g'(t_n)^{-1}g''(t_n)}{1 - \frac{1}{2}h_g(t_n, s_n)}\,. \end{aligned}$$

PROOF: (C1): Note that

$$\begin{aligned} \big\|\frac{1}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\big\| &\le \frac{1}{2}\|P'(X_n)^{-1}\|\|P''(X_n)\|\|Y_n - X_n\| \\ &\le -\frac{1}{2}g'(T-n)^{-1}g''(t_n)(s_n - t_n) = \frac{1}{2}\frac{K(s_n - t_n)}{\frac{1}{\beta} - Kt_n} < \frac{1}{2}, \end{aligned}$$

so $\big[I + \frac{1}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\big]^{-1}$ exists and

$$\begin{aligned} &\big\|\big[I + \frac{1}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\big]^{-1}\big\| \\ &\qquad \le \frac{1}{1 - \frac{1}{2}\|P'(X_n)^{-1}\|\|P''(X_n)\|\|Y_n - X_n\|} \\ &\qquad = \big[1 + \frac{1}{2}g'(t_n)^{-1}g''(t_n)(s_n - t_n)\big]^{-1}. \end{aligned}$$

(C2): From (2.2),

$$X_{n+1} - Y_n = -\frac{1}{2}P'(X_n)^{-1}\big[I - \frac{1}{2}H(X_n, Y_n)\big]^{-1}P''(X_n)(Y_n - X_n)^2,$$

then we estimate both sides, we have

$$\begin{aligned}\|X_{n+1} - Y_n\| &\le \frac{1}{2}\|P'(X_n)^{-1}\|\big[1 - \frac{1}{2}\|H(X_n, Y_n)\|\big]^{-1}\|P''(X_n)\|\|Y_n - X_n\|^2\\ &\le -\frac{1}{2}g'(t_n)^{-1}\big[1 - \frac{1}{2}h_g(t_n, s_n)\big]^{-1}g''(t_n)(s_n - t_n)^2 = t_{n+1} - s_n\,.\end{aligned}$$

(C3):

$$\begin{aligned}&\Big\|\int_0^1 [2P''[X_n + t(Y_n - X_n)](1-t) - P''(X_n)]\,dt\Big\|\\ &\qquad\le 2\int_0^1 \|P''[X_n + t(Y_n - X_n)] - P''(X_n)\|(1-t)\,dt = \frac{N}{3}\|Y_n - X_n\|.\end{aligned}$$

(C4): From Lemma 2.3, we should have

$$\begin{aligned}\|P(X_{n+1})\| &\le \frac{M}{2}\|X_{n+1} - Y_n\|^2 + \frac{\frac{\left[\frac{3M^2}{4} + \frac{N}{6\beta}\right]}{\frac{1}{\beta} - M\|X_n - X_0\|}\cdot\|Y_n - X_n\|^3}{1 - \frac{1}{2}\frac{M\|Y_n - X_n\|}{\frac{1}{\beta} - M\|X_n - X_0\|}}\\ &\le \frac{K}{2}(t_{n+1} - s_n)^2 + \frac{K^2}{4}\frac{\frac{(s_n - t_n)^3}{\frac{1}{\beta} - Kt_n}}{1 - \frac{1}{2}\frac{K(s_n - t_n)}{\frac{1}{\beta} - Kt_n}} = g(t_{n+1}).\end{aligned}$$

(C5):

$$\begin{aligned}\|Y_{n+1} - X_{n+1}\| &= \|-P'(X_{n+1})^{-1}P(X_{n+1})\|\\ \le \|P'(X_{n+1})^{-1}\|\|P(X_{n+1})\| &\le -g'(t_{n+1})^{-1}g(t_{n+1}) = s_{n+1} - t_{n+1}.\end{aligned}$$

□

4. The Ostrowski-Kantorovich theorem.

Theorem 4.1. *Let $P(X) : D_0 \subset X_B \to Y_B$, X_B, Y_B are Banach spaces, real or complex and D_0 is an open convex domain. Assume that P has 2nd order continuous Fréchet derivatives on D_0 and for a given initial value X_0 in D_0 it satisfies the following conditions:*

(4.2) $\|P''(X_n)\| \le M$, $\|P''(X_n) - P''(Y)\| \le N\|X - Y\|$, *for all* $X, Y \in D_0$

(4.3) $$\|P'(X_0)^{-1}\| \le \beta, \quad \|Y_0 - X_0\| \le \eta.$$

Set

$$\sqrt{3}M\big[1+\frac{2N}{9M^2\beta}\big]^{\frac{1}{2}} \le K, \tag{4.4}$$

$$h = K\beta\eta \le \frac{1}{2}, \tag{4.5}$$

$$\overline{S(X_0, r_1)} \subset D_0, \tag{4.6}$$

where $\overline{S(x,r)} = \{x' \in X \mid \|x'-x\| \le r\}$ *and*

$$g(t) = \frac{1}{2}Kt^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}, \tag{4.7}$$

$$r_1 = \frac{1-\sqrt{1-2h}}{h}\eta, \tag{4.8}$$

$$\theta = \frac{1-\sqrt{1-2h}}{1+\sqrt{1-2h}}, \tag{4.9}$$

where r_1 *is the smallest room of the equation* (4.7). *Then the Halley procedure* (2.2) *is convergent. Also* $X_n, Y_n \in \overline{S(X_0, r_1)}$, *for all* $n \in N_0$. *The limit* X^* *is a solution of the equation* (1.1) $P(X) = 0$, *we have the following error estimates and the optimal error constant.*

$$\big\|X_n - X^*\big\| \le r_1 - t_n, \qquad \text{for all } n \ge 0, \tag{4.10}$$

$$\big\|Y_n - X^*\big\| \le r_1 - s_n, \qquad \text{for all } n \ge 0, \tag{4.11}$$

$$r_1 - t_n = \frac{(1-\theta^2)\eta}{1-\theta^{3^n}}\theta^{3^n-1}. \tag{4.12}$$

Proof: It suffices to show that the following items are true for all n by mathematical induction.

$$X_n \in \overline{S(X_0, t_n)}; \tag{I_n}$$

$$\big\|Y_n - X_n\big\| \le (s_n - t_n); \tag{II_n}$$

$$Y_n \in \overline{S(X_0, s_n)}; \tag{III_n}$$

$$\big\|P'(X_n)^{-1}\big\| \le -g'(t_n)^{-1}; \tag{IV_n}$$

$$\begin{aligned}&\big\|\big[I + \frac{1}{2}P'(X_n)^{-1}P''(X_n)(Y_n - X_n)\big]^{-1}\big\| \\ &\le \big[1 + \frac{1}{2}g'(t_n)^{-1}g''(t_n)(s_n - t_n)\big]^{-1};\end{aligned} \tag{V_n}$$

$$\big\|X_{n+1} - Y_n\big\| \le t_{n+1} - s_n. \tag{VI_n}$$

It is easy to check in the case of $n = 0$ by initial conditions. Now assume that (I_n)–(VI_n) is true for a fixed $n \ge 1$. Then

(I_{n+1}):

$$\begin{aligned}\|X_{n+1}-X_0\| &\le \|X_{n+1}-Y_n\|+\|Y_n-X_n\|+\|X_n-X_0\|\\ &\le (t_{n+1}-s_n)+(s_n-t_n)+(t_n-t_0)\\ &= t_{n+1}.\end{aligned}$$

(II_{n+1}): From (C5) we have

$$\|Y_{n+1}-X_{n+1}\| \le s_{n+1}-t_{n+1}.$$

(III_{n+1}):

$$\begin{gathered}\|Y_{n+1}-X_0\| \le \|Y_{n+1}-X_{n+1}\|+\|X_{n+1}-Y_n\|\\ +\|Y_n-X_n\|+\|X_n-X_0\|\\ \le (s_{n+1}-t_{n+1})+(t_{n+1}-s_n)+(s_n-t_n)+(t_n-t_0)=s_{n+1}.\end{gathered}$$

(IV_{n+1}):

$$P'(X_{n+1})-P'(X_0)=\int_0^1 P''[X_{n+1}+t(X_n-X_0)]\,dt\,(X_{n+1}-X_0),$$

so

$$\begin{gathered}\|P'(X_{n+1})-P'(X_0)\| \le M\|X_{n+1}-X_0\|\\ \le K(t_{n+1}-t_0)=Kt_{n+1}<Kr_1\le \frac{1}{\beta}\le \frac{1}{\|P'(X_0)^{-1}\|}\end{gathered}$$

and by Banach Theorem [5, p. 164], $P'(X_{n+1})^{-1}$ exists and

$$\begin{aligned}\|P'(X_{n+1})^{-1}\| &\le \frac{\|P(X_0)^{-1}\|}{1-\|P'(X_0)^{-1}\|}\|P'(X_{n+1})-P'(X_0)\|\\ &= \frac{1}{\frac{1}{\beta}-K\|X_{n+1}-X_0\|}\le \frac{1}{\frac{1}{\beta}-K(t_{n+1}-t_0)}\\ &= \frac{1}{\frac{1}{\beta}-Kt_{n+1}}=-g'(t_{n+1})^{-1}.\end{aligned}$$

(V_{n+1}):

$$\begin{aligned}&\big\|\big[I+\frac{1}{2}P'(X_{n+1})^{-1}P''(X_{n+1})(Y_{n+1}-X_{n+1})\big]^{-1}\big\|\\ &\le \frac{1}{1-\frac{1}{2}\|P'(X_{n+1})^{-1}\|\|P''(X_{n+1})\|\|Y_{n+1}-X_{n+1}\|}\\ &= \big[1+\frac{1}{2}g'(t_{n+1})^{-1}g''(t_{n+1})(s_{n+1}-t_{n+1})\big]^{-1}.\end{aligned}$$

(VI_{n+1}): Note that

$$\begin{aligned}
X_{n+2} - Y_{n+1} &= -\frac{1}{2} P'(X_{n+1})^{-1} \\
&\quad \cdot \big[I - \frac{1}{2} H(X_{n+1}, Y_{n+1})\big]^{-1} P''(X_{n+1})(Y_{n+1} - X_{n+1})^2, \\
\|X_{n+2} - Y_{n+1}\| &\leq \frac{1}{2} \|P'(X_{n+1})^{-1}\| \|\big[I - \frac{1}{2} H(X_{n+1}, Y_{n+1})\big]^{-1}\| \\
&\quad \cdot \|P''(X_{n+1})\| \|Y_{n+1} - X_{n+1}\|^2 \\
&\leq -\frac{1}{2} g'(t_{n+1}) \big[1 - \frac{1}{2} h_g(t_{n+1}, s_{n+1})\big]^{-1} \cdot g''(t_{n+1})(s_{n+1} - t_{n+1})^2 \\
&= t_{n+2} - s_{n+1}.
\end{aligned}$$

Now we are ready to prove (4.12). Notice that

$$g(t_n) = \frac{K}{2}(r_1 - t_n)(r_2 - t_n),$$

and

$$g'(t_n) = -\frac{K}{2}\big[(r_1 - t_n) + (r_2 - t_n)\big].$$

Then by (3.2), we have

$$\begin{aligned}
(r_1 - t_{n+1}) &= r_1 - t_n - \frac{(r_1 - t_n)(r_2 - t_n)\big[(r_1 - t_n) + (r_2 - t_n)\big]}{(r_1 - t_n)^2 + (r_1 - t_n)(r_2 - t_n) + (r_2 - t_n)^2} \\
&= \frac{(r_1 - t_n)^3}{(r_1 - t_n)^2 + (r_1 - t_n)(r_2 - t_n) + (r_2 - t_n)^2},
\end{aligned} \tag{4.13}$$

and similarly, we get

$$r_2 - t_{n+1} = \frac{(r_2 - t_n)^3}{(r_2 - t_n)^2 + (r_2 - t_n)(r_1 - t_n) + (r_1 - t_n)^2}. \tag{4.14}$$

So we obtain

$$\frac{r_1 - t_n}{r_2 - t_n} = \big[\frac{r_1 - t_{n-1}}{r_2 - t_{n-1}}\big]^3 = \cdots = \big[\frac{r_1 - t_0}{r_2 - t_0}\big]^{3^n} = \theta^{3^n}. \tag{4.15}$$

Then we solve this equation for $r_1 - t_n$ by using the fact that $r_2 - t_n = r_1 - t_n + (1 - \theta^2)\eta/\theta$. It is easy to see that

$$r_1 - t_n = \frac{(1 - \theta^2)\eta}{1 - \theta^{3^n}} \theta^{3^n - 1}.$$

□

5. *S*-order of convergence and sufficient asymptotic error bound.

In this section, we try present the sufficient conditions for finding order of convergence for Halley method. First let us recall the original definition of S-order of convergence [1], [2].

Definition 5.1. Let $g(t)$ be a testing function with order 2. Assume that $g(t) = \frac{K}{2}t^2 - \frac{1}{\beta}t + \frac{\eta}{\beta}$, K, β, η and $h = K\beta\eta < \frac{1}{2}$ are any positive real numbers. A sequence of iterations (one step or multistep iterations without memory) defined in certain metric spaces is said to converge with the order p to a point X^* if there is the maximum p such that

one step case:

$$E(g(t_{n+1}), t_n, s_n) = g(t_{n+1}) - C(t_n, t_{n+1})(t_{n+1} - t_n)^p = 0.$$

Multistep case:

$$E(g(t_{n+1}), t_n, s_n) = g(t_{n+1}) - C(t_n, s_n)(s_n - t_n)^p = 0$$

for some $C > 0$, where

$$E(P(X_{n+1}), X_n, Y_n, X_{n+1}) = P(X_{n+1}) - R(X_n, Y_n, X_{n+1});$$

R is called Ostrowski-Kantorovich representation of $P(X_{n+1})$.

Definition 5.2. The sufficient asymptotic error bounds $C(t^*)$ is defined as follows in the sense of S-order of convergence:

One step case:

$$C(t^*) = \lim_{n\to\infty} \frac{g(t_{n+1})}{(t_{n+1} - t_n)^p}.$$

Multistep case:

$$C(t^*) = \lim_{n\to\infty} \frac{g(t_{n+1})}{(s_n - t_n)^p},$$

where $t^* = r_1 = \frac{1-\sqrt{1-2h}}{h}\eta$. Now we are ready to find the order of convergence for Halley method without assuming the existence of the solution X^*. In (2.4) we replace $P(X)$ by $g(t)$. Then we have

$$
\begin{aligned}
g(t_{n+1}) = & \int_0^1 g''\big[s_n + t(t_{n+1} - s_n)\big](1-t)\,dt\ (t_{n+1} - s_n)^2 \\
& - \frac{1}{2}\int_0^1 g''\big[t_n + t(s_n - t_n)\big]\,dt\ (s_n - t_n) g'(t_n)^{-1} \\
& \cdot \big[1 - \frac{1}{2} h_g(t_n, s_n)\big]^{-1} g''(t_n)(s_n - t_n)^2 \\
& + \frac{1}{2}\big[1 - \frac{1}{2} h_g(t_n, s_n)\big]^{-1} g'(t_n)^{-1} g''(t_n)(s_n - t_n) \\
& \cdot \int_0^1 g''\big[t_n + t(s_n - t_n)\big](1-t)\,dt\ (s_n - t_n)^2 \\
& + \big[1 - \frac{1}{2} h_g(t_n, s_n)\big]^{-1} \int_0^1 \big[g''\big[t_n + t(s_n - t_n)\big](1-t) \\
& - \frac{1}{2} g''(t_n)\big]\,dt\ (s_n - t_n)^2 \\
= & \frac{K}{2}(t_{n+1} - s_n)^2 - \frac{\frac{K^2}{4}(s_n - t_n)^3}{\big[1 + \frac{K(s_n - t_n)}{2g'(t_n)}\big]} g'(t_n)^{-1} \\
= & \frac{K}{2}\left[\frac{-\frac{K}{2}(s_n - t_n)^2}{g'(t_n)\big[1 + \frac{K(s_n - t_n)}{2g'(t_n)}\big]}\right]^2 - \frac{\frac{K^2}{4}(s_n - t_n)^3}{g'(t_n)\big[1 + \frac{K(s_n - t_n)}{2g'(t_n)}\big]} \\
= & \left[\frac{\frac{K^3}{8}(s_n - t_n)}{g'(t_n)^2\big[1 + \frac{K(s_n - t_n)}{2g'(t_n)}\big]^2} - \frac{\frac{K^2}{4}}{g'(t_n)\big[1 + \frac{K(s_n - t_n)}{2g'(t_n)}\big]}\right](s_n - t_n)^3 \\
= & C_H(t_n, s_n)(s_n - t_n)^3.
\end{aligned} \tag{5.3}
$$

Hence p_S (sufficient order of convergence) $= 3 = p_N$ (necessary order of convergence). And the sufficient error bound is given by

$$
C_H(t^*) = \lim_{n\to\infty} \frac{g(t_{n+1})}{(s_n - t_n)^3} = \frac{K^2\beta}{4\sqrt{1 - 2h}}\,.
$$

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