Cotorsion-free algebras as endomorphism algebras in L — the discrete and topological cases

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Abstract. The discrete algebras A over a commutative ring R which can be realized as the full endomorphism algebra of a torsion-free R-module have been investigated by Dugas and Göbel under the additional set-theoretic axiom of constructibility, $V = L$. Many interesting results have been obtained for cotorsion-free algebras but the proofs involve rather elaborate calculations in linear algebra. Here these results are rederived in a more natural topological setting and substantial generalizations to topological algebras (which could not be handled in the previous linear algebra approach) are obtained. The results obtained are independent of the usual Zermelo-Fraenkel set theory ZFC.

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1. Introduction.

The question of which rings can occur as endomorphism rings of torsion-free Abelian groups and modules has attracted considerable interest ever since the result of A.L.S. Corner in 1963 in [1]: Every countable, reduced, torsion-free ring is the endomorphism ring of a countable, reduced, torsion-free Abelian group. This result has been substantially generalized in the last eight years using techniques from model theory and set theory; see e.g. [3], [4], [5] and the references therein. One of the first major breakthroughs came in the work of Dugas and Göbel [4] in 1982; having introduced the notion of a cotorsion-free module they succeeded in showing that every cotorsion-free ring is an endomorphism ring. Their approach was based on the set-theoretic axiom $V = L$ and derived from earlier work of Eklof and Mekler [8] on the construction of indecomposable groups. Both of the works [4] and [8] are essentially based on linear algebra and require rather elaborate calculations. Subsequent work (see [3], [6], [11], [12] and [13]) has indicated that a topological setting using completions seems more natural. This is not too surprising when one considers that cotorsion-free is a topological notion. The purpose of the first part of the present work is to derive, in $V = L$, results similar to the main result of [4] in a much simpler and concise form. Specifically the complicated calculations in linear algebra are discarded and the results are obtained in a fashion which shows

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their natural relation with similar work on p-groups, mixed groups and torsion-free modules over complete discrete valuation rings.

Recall that, for a cardinal κ, an R-module G is said to be κ-free if every submodule of cardinality less than κ is contained in a free submodule of G. An R-module G is said to be strongly κ-free if G is κ-free and any submodule of cardinality less than κ is contained in a submodule U of the same cardinality, where G/U is κ-free. We will show the following

Theorem 1 $(V = L)$**.** *If κ is a regular, not weakly compact cardinal $> |A|$, where A is an R-algebra which as R-module is cotorsion-free, then there exists a family of 2^κ strongly κ-free A-modules H_κ^α of size κ ($\alpha < 2^\kappa$) such that*

(i) $\mathrm{End}_R(H_\kappa^\alpha) = A$

(ii) *if $(\alpha, \kappa) \neq (\beta, \lambda)$ then every R-homomorphism : $H_\kappa^\alpha \to H_\lambda^\beta$ is trivial.*

Moreover if A is free as an R-module, then each H_κ^α is a strongly κ-free R-module.

Here, as throughout, $\mathrm{End}(G) = \mathrm{End}_R(G)$ denotes the algebra of R-endomorphisms of the R-module G.

The consequences of such a discrete realization theorem for direct decompositions are by now widely known. It suffices to say that one can derive the usual pathological decompositions from a result such as the above by suitable choice of the algebra A; see [3] for details of such pathologies. Moreover the existence of rigid families of maximal size and rigid proper classes in both the discrete and topological cases is easily deduced (cf. [11]). It follows, of course (see [7]), that such results are independent of the usual ZFC axioms of set theory.

As a consequence of our more elementary proof of Theorem 1, it is also possible and quite easy to extend this to obtain a topological realization theorem. This was impossible in [4] because of the difficulties arising from the "linear algebra approach". Recall some standard definitions from the corresponding topological theorems in ZFC; see [3], [7].

The finite topology fin on $\mathrm{End}(H)$ is the linear topology having the annihilators

$$U_E = \{\sigma \in \mathrm{End}(H) \mid E\sigma = 0\}$$

of all finite subsets E of H as a basis of neighbourhoods of 0. Then $(\mathrm{End}(H), \mathrm{fin})$ is a complete topological endomorphism algebra, $\mathrm{End}(H)/U_E$ is cotorsion-free if H is cotorsion-free and $\bigcap U_E = 0$. We now derive the converse.

Theorem 2 $(V = L)$**.** *Let (A, τ) be a topological R-algebra A with complete Hausdorff topology τ which admits a basis $\mathcal{N}$ of neighbourhoods of 0 such that each $N \in \mathcal{N}$ is a right ideal of A with A/N cotorsion-free. Let κ be any regular, not weakly compact cardinal with $\kappa > \rho = \sum_{N \in \mathcal{N}} |A/N| \cdot |\mathcal{N}|$. Then there exists a strongly κ-$\oplus_{\mathcal{N}}$ A/N-module H such that*

$$(\mathrm{End}(H), \mathrm{fin}) \cong (A, \tau)$$

is a topological isomorphism.

Remarks. "Strongly κ-$\oplus_{\mathcal{N}}$ A/N" is an obvious generalization of strongly κ-free: each submodule U of H of cardinality $< \kappa$ is contained in a direct sum of $< \kappa$ copies of A/N with quotient isomorphic to some submodule of $\oplus$ $(\oplus_{\mathcal{N}}$ $A/N)$. If $\kappa = \aleph_1$, then $V = L$ can be replaced by $2^{\aleph_0} < 2^{\aleph_1}$; see [4] for the obvious changes and set-theoretic details involving weak diamonds. Theorem 1 follows from Theorem 2 if $0 \in \mathcal{N}$ and τ becomes discrete; e.g. put $\mathcal{N} = \{0\}$.

Finally we apply our topological realization Theorem 2 to derive the existence of some very decomposable almost free abelian groups. A proof of the existence of such groups was expected some time ago but is apparently new. We use the rings A of size κ described by Corner in [2]: Observe that A/N is free for each $N \in \mathcal{N}$ (see [2]). Hence $\oplus$ $\oplus_{\mathcal{N}}$ A/N is free and the module H obtained from Theorem 2 is strongly κ-free. The topological isomorphism and the basic properties of A ensure that H is κ-decomposable, i.e. every non-zero summand of H is a direct sum of κ non-zero summands. It is remarkable that H is "almost free" on the one hand but on the other is κ-decomposable which is a strong measure of being not free.

In conclusion we would like to make the following observation. Comparison of the principal results in e.g. [4] and [5] or in [3] and the present paper, make it tempting to conjecture that there should be a general theorem — possibly stated in terms of model theory or categories — which transports existence results (such as realization theorems) in ZFC to stronger existence theorems (having more restrictions towards freeness) in $V = L$ and vice versa.

2. Algebraic preliminaries.

We shall assume throughout that R is a fixed non-zero commutative ring with 1, with a given countable multiplicatively closed subset S of non-zero divisors such that $1 \in S$. R shall always be S-reduced i.e. $\bigcap_{s\in S} sR = 0$. Recall that S-topology on an R-module M has the submodules $\{sM : s \in S\}$ as a basis of neighbourhoods of zero. Such a topology is, of course, Hausdorff precisely if $\bigcap_{s\in S} sM = 0$. This is equivalent, in algebraic terminology, to saying that M is S-reduced. We shall denote the completion of an S-reduced R-module M (in S-topology) by $\hat{M}$. Similarly one may define the notions of S-pure, S-divisible, S-torsion-free etc. (cf. [3], [4] where such notions have been used extensively). Since the set S is fixed throughout no ambiguity will arise if we drop the prefix S from the above terms. If the elements of S are labelled as $\{s_1, s_2, \dots\}$, then we form the elements q_n $(n \in \omega)$ of S by setting $q_n = \prod_{i=1}^{n} s_i$; observe that q_m/q_n is well defined if $m \geq n$.

Recall that if M is any torsion-free R-module then M is said to be cotorsion-free provided $\mathrm{Hom}(\hat{R}, M) = 0$, where $\hat{R}$ denotes the completion of R. If $\hat{A}$ is the completion of a torsion-free reduced R-algebra A, then every element of $\hat{A}$ is a limit of a Cauchy sequence of elements of A and so we may represent an element $a \in \hat{A}$ by $a = \sum_{n<\omega} a_n t_n$ where $a_n \in A$, $t_n \in S$ and for each k, $q_k \mid t_n$ for almost all n. We shall assume throughout that A is cotorsion-free.

Let $F = \oplus_{i\in I} e_i A$ be a free A-module and $x \in F$. The support $[x]$ of x (with respect to the given decomposition of F) is defined by

$$[x] = \{i \in I : a_i \neq 0 \ \text{ where } \ x = \sum e_i a_i\}.$$

Clearly $[x]$ is a finite subset of I. Moreover if $y \in \hat{F}$ then it is well known that y may be represented uniquely as $y = \sum e_i a_i$ where $a_i \in \hat{A}$ and $\{a_i\}$ is a null sequence. Thus the support of y may be defined in a similar manner and in this case $[y]$ is a countable subset of I. More generally if X is a subset of $\hat{F}$, we may define $[X] = \bigcup_{x \in X} [x]$. If $\phi \in \mathrm{End}_R(F)$ and G is a submodule of F, then we define the ϕ-closure of G as follows:
Let $I_0 = [G]$, $I_{n+1} = I_n \cup [\{e_j a\phi : j \in I_n, a \in A\}]$ and set $I_\omega = \bigcup_{n<\omega} I_n$. Then the ϕ-closure of G is defined by $G^{c\phi} = \oplus_{i \in I_\omega} e_i A$. Clearly $G \leq G^{c\phi}$ and the latter is a canonical summand of F which is invariant under ϕ. Moreover if G has infinite rank $\geq |A|$ then $G^{c\phi}$ has rank equal to $\mathrm{rk}(G)$.

Our algebraic terminology and notation is standard following Fuchs [10] with the exceptions that maps are written on the right and $[$ denotes a direct summand. Terminology and notation relating to set theory may be found in the standard work of Jech [13].

Constructions of the type we are interested in separate nicely in $V = L$ into two distinct phases: algebraic step-lemmas and combinatorial set-theoretic arguments. We now derive the necessary algebraic step-lemmas.

3. Step-Lemmas for discrete realizations.

Let F be a free A-module with a strictly increasing chain of summands $\{F_n\}$, say $F_{n+1} = F_n \oplus D_n$, so that $F = F_0 \oplus D$ where $D = \oplus_{n<\omega} D_n$. An element $y \in \hat{F}$ is said to be a *branch* (relative to the chain of summands) if there exist basis elements e_i in $F \setminus F_0$ with $y = \sum e_i q_i$ such that the set $\{n \mid e_i \in D_n \text{ for some } i\}$ is infinite. An element z in $\hat{F}$ is said to be *branch-like* if $z = y + x$, where $x = \bar{x}\pi$ with $\bar{x} \in F$, $\pi \in \hat{R}$ and y is a branch satisfying $[y] \cap [x] = \emptyset$. There is a clear similarity between the branch elements introduced above and the concept of branch used in ZFC constructions such as [3]; similar constructions in $V = L$ have been used by the present authors in [12].

Lemma 1. *Let F be a free A-module with strictly increasing chain of summands $\{F_n\}$ and $y \in \hat{F}$ a branch. Then the submodule $F' = \langle F, yA \rangle_*$ of $\hat{F}$ is a free A-module and $F_n \,[\, F'$ for all n.*

Proof: Let $F = F_0 \oplus \ \oplus_{n<\omega} D_n$ and suppose $y = \sum e_i q_i$ is a branch. For each n let $I_n = \{i \mid e_i \in D_n\}$; so $D_n = \oplus_{i \in I_n} e_i A \oplus C_n$. Thus $F = \oplus e_n A \oplus \ \oplus C_n \oplus F_0 = \oplus e_n A \oplus F^*$ say. For convenience we shall write $B = \oplus_{n<\omega}\ e_n A$. Now define elements $y^n \in \hat{B}$ by $y^n = \sum_{j \geq n} e_j q_j / q_n$ and note that $y^0 = y$. We claim that the pure submodule F' of F is equal to $X = F^* \oplus \ \oplus_{n<\omega}\ y^n A$.

It is a simple (and standard) exercise to see that the sum in X is direct. Moreover since $y^n - s_{n+1} y^{n+1} = e_n \in F$, it is immediate that $F \leq X$. The purification of $\langle F, yA \rangle$ ensures that X is contained in F' and so it suffices to establish the reverse inclusion. If $g \in F'$ then we have $q_N g = f + ya$ for some $a \in A$, $f \in F$ and $N < \omega$. However

$$q_N y^N = y - (e_0 + \cdots + e_{N-1} q_{N-1}) \ \text{ and so } \ q_N(g - y^N a) \in F.$$

It follows immediately from the purity of F in $\hat{F}$ that $g = y^N a + f_0$ for some $f_0 \in F$. Since $F \leq X$ we conclude that $F' \leq X$.

It remains only to show that $F_n \,[\, F'$ for all n and it clearly suffices to show that for any N, $\oplus_{j<N}\ e_j A$ is a direct summand of $\oplus_{j<\omega}\ y^j A$. We establish this directly by showing that

$$\oplus_{j<\omega}\ y^j A = \oplus_{j<N}\ e_j A \oplus\ \oplus_{j\geq N}\ y^j A.$$

Observe firstly that the sum on the RHS is direct: this follows immediately by examining supports. Moreover as $e_j = y^j - s_{j+1} y^{j+1}$ we can deduce that RHS $\subseteq$ LHS. Thus it only remains to show that for $i < N$, $y \in$ RHS. But $y^{N-1} - s_N y^N = e_{N-1}$ and so $y^{N_1} \in$ RHS. Repeating this argument we get $y^{N-2} - s_{N-1} y^{N-1} = e_{N-2}$ and so y^{n-2} is also in RHS. Continuing this process completes the proof. $\square$

Step-Lemma A. *Let F be a free A-module with a strictly ascending chain of summands $\{F_n\}$. If $\phi : F \to G$ is a non-zero homomorphism into a cotorsion-free A-module G, then there exists a free A-module F' containing F such that*

(i) *F'/F is a torsion-free, divisible rank one A-module*
(ii) *ϕ does not extend to a homomorphism : $F' \to G$*
(iii) *$F_n \,[\, F'$ for all $n < \omega$.*

Proof: If there is a branch $y \in \hat{F}$ with $y\phi \notin G$, choose $F' = \langle F, yA\rangle_* \leq \hat{F}$. The result then follows from Lemma 1. If no such branch exists then $y\phi \in G$ for all branches $y \in \hat{F}$. Since ϕ is non-zero, there exists $x \in F$ with $x\phi \neq 0$. However G cotorsion-free implies that, for some $\pi \in \hat{R}$, $(x\pi)\phi$ is not in G. (We are here identifying $x\pi$ in a natural way as an element of $\hat{F}$.) Choose a branch element $b = \sum_{n<\omega} e_n q_n$ such that $[b] \cap [x] = \emptyset$; this is clearly possible since $[x]$ is a finite set. Define for each $k < \omega$, $b^k = \sum_{n\geq k} e_n q_n / q_k$ and note that $b^0 = b$. The element π is in $\hat{R}$ and there will be no loss of generality in assuming π has the form $\sum_{n<\omega} r_n q_n$ with $r_n \in R$; set, for each $k < \omega$, $\pi^k = \sum_{n\geq k} r_n q_n / q_k$ and define $x_k = x\pi^k \in \hat{F}$. Note that $x_0 = x\pi$. Consider now the elements of $\hat{F}$ given by $z_k = x_k + b^k$. A simple calculation shows that

$$z_n - s_{n+1} z_{n+1} = e_n + r_n x.$$

Let $F = \oplus_{n<\omega}\ e_n A \oplus F^*$; observe that $q_n x_n - q_{n+1} x_{n+1} \in F^*$.

Set $F' = \langle F, z_0 A\rangle_* \leq \hat{F}$; we claim that F' has the required properties. Properties (i) and (ii) are immediate as $z_0\phi = x_0\phi + b\phi \notin G$ and so we only need to verify that F' is free and $F_n \,[\, F'$ for all n.

We show freeness directly by proving that $F' = X$, where $X = F^* \oplus\ \oplus_{n<\omega}\ z_n A$. It is a straightforward exercise to show that the sum on the RHS above is direct. Moreover since $z_n - s_{n+1} z_{n+1} = e_n + r_n x$ and $r_n x \in F^*$, it follows that $F \leq X$. Next observe that for each n, $z_n \in F'$. This follows by direct calculation: $q_n z_n = q_n x_n + q_n b^n$ and $q_n b^n = b + f_n$ where $f_n \in F$. Hence we have $q_n z_n = b + f_n + q_n x_n = b + f_n + (q_n x_n - x_0) + x_0$. But we also have $q_n x_n - x_0 = rx$ for some $r \in R$ and so we conclude that $q_n z_n = (b + x_0) + \bar{f}$ where $\bar{f} \in F$. Thus $z_n \in \langle F, z_0 A\rangle_*$ as claimed. The final step in showing $X = F'$ is to establish that $F' \leq X$. So suppose

$g \in F'$, then $q_n g = f + z_0 a$ for some $n < \omega$, $f \in F$, $a \in A$. But $z_0 = q_n z_n - \bar{f}$ and so $q_n g = q_n z_n - \bar{f} + f$. Thus we have $q_n(g - z_n) = f - \bar{f} \in F$. Now purity of F in $\hat{F}$ gives $g - z_n \in F$ and so, since $F \leq X$, $g \in X$ as required.

Our proof of the Step-Lemma will be completed by showing $F_n \,[\, F'$ for all n. For this it clearly suffices to show that $\oplus_{i<n}\, e_i A \,[\, X$ for each n. Again we show this directly by establishing the decomposition

$$X = F^* \oplus \ \oplus_{i<n}\, e_i A \oplus \oplus_{i \geq n}\, z_i A.$$

Observe firstly, by a simple support argument, that this sum is direct and that the RHS is certainly in X. It then suffices to show that $z_i \in$ RHS for $i < n$. However

$$\begin{aligned} z_{n-1} - s_n z_n &= (x_{n-1} - s_n x_n) + (b^{n-1} - s_n b^n) = x r_{n-1} + e_{n-1} \\ &= e_{n-1} + f \quad \text{where} \quad f = x r_{n-1} \in F^*. \end{aligned}$$

Thus $z_{n-1} = e_{n-1} + s_n z_n + f \in$ RHS. Repeating this argument we obtain $z_{n-2} - s_{n-1} z_{n-1} = e_{n-2} + f_1$, etc. This completes the proof. □

An examination of the argument in the last portion of the above Step-Lemma shows that we have the following corollary (cf. [13, Lemma 2]).

Corollary. *If F is a free A-module with a strictly ascending chain of summands $\{F_n\}$ and the pair (z, y) is branch-like, then $F' = \langle F, zA\rangle_* \leq \hat{F}$ is a free A-module and $F_n \,[\, F'$ for all $n < \omega$.*

Lemma 2. *Let F be a free A-module and $\phi \in \mathrm{End}(F) \setminus A$, then there exists a canonical summand P of F such that*
(i) $\mathrm{rk}(P) \leq |A|$, (ii) $\phi_P = \phi \restriction P \in \mathrm{End}(P)$, (iii) $\phi_P \notin A$.

Proof: Suppose Lemma 2 does not hold and let B be any free summand of F of rank $\leq |A|$. Set $B_0 = B^{c\phi}$, the ϕ-closure of B. Clearly B_0 is a canonical summand of F and satisfies (i) and (ii). Thus $\phi \restriction B_0 \in A$ by hypothesis; say $\phi \restriction B_0 = a$. Since $\phi \in \mathrm{End}(F) \setminus A$, there is an $x \in F$ such that $x(\phi - a)$ is non-zero. Let $B_1 = \langle B_0, xA\rangle^{c\phi}$, a canonical summand of F which clearly satisfies (i) and (ii). By assumption $\phi \restriction B_1 \in A$ and since $B_0 \subseteq B_1$, we conclude that $\phi \restriction B_1 = a$. But then $x(\phi - a) = 0$ — a contradiction. This establishes the lemma. □

For further reference we point out (without proof) the following rather simple result on purity.

Lemma 3. *If F is a free A-module of infinite rank then A is pure in* $\mathrm{End}(F)$.

If F has rank $\leq |A|$, we can choose $P = F$. Hence we may assume that F has infinite rank.

Step-Lemma B. *Let F be a free A-module of rank $> |A|$, with a strictly ascending chain of summands $\{F_n\}$ and suppose $\phi \in \mathrm{End}(F) \setminus A$. Then there exists a free A-module F' containing F such that*

(i) *F'/F is a torsion-free, divisible rank one A-module*
(ii) *ϕ does not extend to an endomorphism of F'*
(iii) *$F_n \,[\, F'$ for all $n < \omega$.*

Proof: By assumption and Lemma 2 we may write $F = P \oplus B$ where P is a ϕ-canonical summand and B has the same rank as F. If there is a branch element y with respect to $\{F_n\}$ in $\hat{B}$ with $y\phi \notin \langle B, yA\rangle_* \le \hat{F}$, then we choose $F' = P \oplus \langle B, yA\rangle_* \le \hat{F}$ and the result follows from Lemma 1. So suppose that no such branch y exists. Then for every branch $y \in \hat{B}$ there exists a pair $(n, a) \in \omega \times A$ such that $y(s_n\phi - a) \in B$. However it follows from Lemma 3 that $s_n\phi - a \neq 0$ and so, since A is cotorsion-free and P is a free A-module, there is an x_{na} in $\hat{P}$ of the form $x\pi (x \in P, \pi \in \hat{R})$ such that such that $x_{na}(s_n\phi - a) \in \hat{P} \setminus P$. Now consider the branch-like pair (z, y) where $z = y + x_{na}$. We claim $z\phi \notin \langle F, zA\rangle_* \le \hat{F}$. For if not, there exists a pair $(m, c) \in \omega \times A$ such that $z(s_m\phi - c) \in F$. By absorbing appropriate multiples there is no loss in assuming $n = m$. So $(y + x_{na})(s_n\phi - c) \in P \oplus B$. However $y(s_n\phi - a) \in B$ and so on subtracting we get

$$y(a - c) + x_{na}(s_n\phi - c) \in P \oplus B.$$

But $x_{na} \in \hat{P}$ and P is ϕ-invariant, hence $x_{na}(s_n\phi - c) \in \hat{P}$ and $y(a - c) \in \hat{B}$. However $y(a - c) \in \hat{B} \setminus B$ unless $a = c$, since y is a branch. Hence $a = c$ follows and $x_{na}(s_n\phi - c) = x_{na}(s_n\phi - a) \in \hat{P} \cap P \oplus B = P$ — a contradiction. Since the pair (z, y) is branch-like, it follows from the Corollary to Step-Lemma A that $F' = \langle F, zA\rangle_* \le \hat{F}$ has the desired properties. This completes the proof. $\square$

4. Step-Lemmas for topological realizations.

In order to adopt part of [3], we make some further adjustments to our notation. Let κ be a regular cardinal greater than $\rho = \sum_{n\in\mathcal{N}} |A/N| \cdot |\mathcal{N}|$, with A, N as in Theorem 2. Let $(\alpha, N) \in \kappa \times \mathcal{N}$ be a generator of an A-algebra $(\alpha, N)A$ with $Ann(\alpha, N) = N$. Define

$$F[N] = \oplus_{\alpha\in\kappa}(\alpha, N)A \quad \text{and} \quad F = \oplus_{N\in\mathcal{N}} F[N].$$

Observe that $A \subseteq \operatorname{End} F$ can be identified by scalar multiplication. Also A/N is cotorsion-free and $S \subseteq R$ (as in § 2) gives rise to S-topology on A/N and on F. As usual we denote the S-completion of F by $\hat{F}$. Supports now refer to κ only: if $f \in \hat{F}$, then $f = \sum(\alpha_n, N_n)a_n$ with $a_n \in (A/N_n)\hat{}$ and we denote the support of f by $[f] = \{\alpha_n \in \kappa \mid a_n \in (A/N_n)\hat{} \setminus \{0\}\}$. A direct summand D of F will be called a (topological) canonical summand if there exists $I \subseteq \kappa$ with $|I| \le \rho$ and $D = \oplus_{\mathcal{N}} \oplus_{\alpha\in I} (\alpha, N)A$.

We are interested in the case where F is the union of a strictly increasing chain $\{F_n\}$ $(n \in \omega)$ of summands, say $F = \bigcup_{n\in\omega} F_n$, $D = \oplus_{n\in\omega} D_n$, $F_{n+1} = F_n \oplus D_n$ and $D^n = \oplus_{m\ge n} D_m$. An element $y \in \hat{F}$ of the form $y = \sum_{i\in\omega} e_i q_i$ with $e_i \in D^n$ for large enough i, will be called a *branch*. Moreover if $N \in \mathcal{N}$ and $e_i \in F[N]$, then y is called an N-branch of $\hat{F}$. An element $z \in \hat{F}$ is said to be N-branch-like, provided $z = y + x$ with y an N-branch, $x \in (F[N])\hat{}$ and $[x] \cap [y] = \emptyset$.

The crucial algebraic step for Theorem 2 is the following

Step-Lemma B*. *Let F, F_n $(n \in \omega)$ be as above and suppose $\phi \in \operatorname{End} F \setminus A$. Then there exist $N \in \mathcal{N}$ and an N-branch-like element $z \in \hat{F}$ such that*

(i) $F' = \langle F, zA \rangle_* \leq \hat{F}$ *is isomorphic to* F

(ii) ϕ *does not extend to an endomorphism of* F'

(iii) $F_n \,[\, F'$ *for all* $n \in \omega$.

Proof: We provide only an outline of the proof; it is a combination of ideas from Step-Lemma B and [3], Lemmas 4.5 and 4.6. The essential steps are as follows: suppose $\phi \in \operatorname{End} F \setminus A$. An examination of the proof of Lemma 4.5 in [3] shows that, in the notation above, there exist $N \in \mathcal{N}$ and a canonical submodule $P \subseteq F$ such that the following property (*) holds:

$$\hat{P}[N](s\phi - a) \nsubseteq F \quad \text{for all} \quad (s, a) \in S \times A \quad \text{with} \quad s = 1 \quad \text{or} \quad a \notin sA + N.$$

Using the Corollary to Step-Lemma A and the arguments from the proof of Lemma 4.6 in [3] with (*) in place of (4.5), we derive the existence of an N-branch-like element z having the appropriate properties to ensure (i), (ii) and (iii) hold. □

5. The combinatorics.

The construction of the modules with the properties stated in Theorems 1 and 2 follows now in a very standard fashion. It is only at this point that the hypothesis $V = L$ is used and then indirectly in the form of Jensen's Diamond Principle. Since the construction is by now routine we omit the details and refer the reader to such works as [4], [6], [8], [11] and [12] where the combinatorics and the inductive construction are worked out in detail.

References

[1] Corner A.L.S., *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) **13** (1963), 687–710.

[2] ———, *On the existence of very decomposable Abelian groups*, in Abelian Group Theory, Proceedings Honolulu 1982/83, LNM 1006, Springer-Verlag, Berlin,1983.

[3] Corner A.L.S., Göbel R., *Prescribing endomorphism algebras, a unified treatment*, Proc. London Math. Soc. (3) **50** (1985), 447–479.

[4] Dugas M., Göbel R., *Every cotorsion-free ring is an endomorphism ring*, Proc. London Math. Soc.(3) **45** (1982), 319–336.

[5] ———, *Every cotorsion-free algebra is an endomorphism algebra*, Math. Z. **181** (1982), 451–470.

[6] ———, *Almost Σ-cyclic Abelian p-groups in L*, in Abelian Groups and Modules (Udine 1984), CISM Courses and Lectures No. 287, Springer-Verlag, Wien-New York, 1984.

[7] ———, *Torsion-free Abelian groups with prescribed finitely topologized endomorphism rings*, Proc. Amer. Math. Soc. **90** (1984), 519–527.

[8] Eklof P., Mekler A., *On constructing indecomposable groups in L*, J. Algebra **49** (1977), 96–103.

[9] ———, *Almost Free Modules: Set-Theoretic Methods*, North Holland, 1990.

[10] Fuchs L., *Infinite Abelian Groups*, Vol. I (1970), vol. II (1973), Academic Press, New York.

[11] Göbel R., Goldsmith B., *Essentially indecomposable modules which are almost free*, Quart. J. Math. (Oxford) (2) **39** (1988), 213–222.

[12] ———, *Mixed modules in L*, Rocky Mountain J. Math. **19** (1989), 1043–58.

[13] ———, *On almost-free modules over complete discrete valuation rings*, Rend. Sem. Mat. Univ. Padova **86** (1991), 75–87.
[14] Jech T., *Set Theory*, Academic Press, New York, 1978.

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