Homology theory in the AST III
Comparison with homology theories of Čech and Vietoris

Jaroslav Guričan*

Abstract. The isomorphism between our homology functor and these of Vietoris and Čech is proved. Introductory result on dimension is proved.

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0. Introduction.

The paper [G2] deals with the construction and basic properties of one homological functor in the Alternative set theory — functor for generalized π-symmetries and also for π-symmetries (and π-equivalencies) via generalized π-symmetries.

It is known (cf. [V]) that we are able to define basic topological notions via indiscernibility relations and by this procedure we can show relationship between our approach to these phenomena and the classical topological structures. This relationship for instance shows that a figure of a set in any indiscernibility relation could be considered to be a compact metric space. There are very deeply developed theories of algebraic topology in the classical mathematics. Therefore we shall try to compare our approach to homology theory with some of the classical ones which could be simply translated into the AST.

1. The comparison with Vietoris's homology theory.

We are going to use the definitions and results of [G1, Section 2]. First of all we shall make some agreements towards some definitions (or, more precisely, towards some assignments) of the [G2]. Our definition [G2, Definition 2.2] defines groups of chains, cycles, boundaries and homology groups of generalized π-symmetries $\mathbf{R}$ (with some additional parameters $\mathbf{F}$, $\mathbf{G}$ which are of no importance now) and we use the assignments $\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$, $\mathbf{Z}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$, $\mathbf{B}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ and finally $\mathbf{H}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$. For related groups defined by π-symmetries $\mathbf{R}$ we use assignments $\mathbf{C}_\nu(\mathbf{r}(\mathbf{R}),\mathbf{F},\mathbf{G}),\dots$. The definition [G2, Definition 4.1] uses the assignment which is a little bit apart the previous ones:

$$\mathbf{C}_\nu(\mathbf{R},\mathbf{u},\mathbf{a}) = \mathbf{C}_\nu(\mathbf{r}(\mathbf{R}))/\mathbf{C}_\nu\left(\mathbf{r}\left(\mathbf{R}\cap\mathbf{a}^2\right)\right)$$

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for the π-symmetry $\mathbf{R}$. This "abbreviation" did not make any confusion, because in [G2, Section 4] we were working only with π-symmetries. But all these definitions actually need to use generalized π-symmetries. Therefore we can use similar assignments to the remaining cases as well, i.e. we can use $\mathbf{C}_\nu(\mathbf{R},\mathbf{F},\mathbf{G})$ instead of $\mathbf{C}_\nu(\mathbf{r}(\mathbf{R}),\mathbf{F},\mathbf{G})$, etc., for π-symmetries.

The actual meaning of these assignments is uniquely "reconstructable".

We shall work directly with generalized π-symmetries now. Let $\mathbf{R}$ be a generalized π-symmetry on $\mathbf{u}$ such that $\mathbf{R}\subseteq\mathcal{P}(\mathbf{u})$ — the set of all subsets of $\mathbf{u}$. Similarly as for π-symmetries we define a **generating sequence** of the generalized π-symmetry $\mathbf{R}$ on $\mathbf{u}$ to be a sequence $\{R_n; n\in\mathbf{FN}\}$ such that for every $n\in\mathbf{FN}$ we have:

$$\mathrm{Sd}(R_n),\quad R_n\subseteq\mathcal{P}(\mathbf{u}),\quad R_{n+1}\subseteq R_n,\quad R_n \text{ is a GS on } \mathbf{u} \text{ and } \mathbf{R}=\bigcap\{R_n; n\in\mathbf{FN}\}.$$

Theorem 1.4 of [G2] asserts that for every generalized π-symmetry $\mathbf{R}$ there exists its generating sequence $\{R_n; n\in\mathbf{FN}\}$. Let $(\mathbf{G},+)$ be an Sd-group in the entire paper. Let us recall that each of the groups $\mathbf{C}_\nu(R_n,\mathbf{G})$, $\mathbf{Z}_\nu(R_n,\mathbf{G})$, $\mathbf{B}_\nu(R_n,\mathbf{G})$, $\mathbf{H}_\nu(R_n,\mathbf{G})$ is an Sd-group in this case.

Let $\mathbf{a}\subseteq\mathbf{u}$. We shall use (in harmony with the above agreement) the assignments $\mathbf{Z}_\nu(R_n,\mathbf{u},\mathbf{a})$ for the groups of cycles of the complex $\mathbf{C}(R_n,\mathbf{u},\mathbf{a})=\left\{\mathbf{C}_\nu(R_n,\mathbf{u},\mathbf{a}),\bar{\partial}_\nu,\nu\in\mathbf{Z}\right\}$ (cf. [G2, Definition 4.1]) and $\mathbf{B}_\nu(R_n,\mathbf{u},\mathbf{a})$ for the groups of boundaries of the same complex. It means that

$$\begin{aligned}
\mathbf{C}_\nu(R_n,\mathbf{u},\mathbf{a}) &= \mathbf{C}_\nu(R_n)/\mathbf{C}_\nu(R_n\cap\mathcal{P}(\mathbf{a}))\\
\mathbf{Z}_\nu(R_n,\mathbf{u},\mathbf{a}) &= \{x\in\mathbf{C}_\nu(R_n,\mathbf{u},\mathbf{a});\ \partial(x)=0\} \qquad\qquad \text{and}\\
\mathbf{B}_\nu(R_n,\mathbf{u},\mathbf{a}) &= \{x\in\mathbf{C}_\nu(R_n,\mathbf{u},\mathbf{a}); (\exists y\in\mathbf{C}_{\nu+1}(R_n,\mathbf{u},\mathbf{a}))\,\partial(y)=x\}.
\end{aligned}$$

We shall abbreviate the group $\mathbf{C}_\nu(R_n\cap\mathcal{P}(\mathbf{a}))$ to $\mathbf{C}_\nu(R_n,\mathbf{a})$. Sometimes we shall use also the assignment $\mathbf{C}_\nu(R_n,\mathbf{u})$ instead of $\mathbf{C}_\nu(R_n)$ to emphasize the role of $\mathbf{u}$ in this group. These groups are also Sd-groups. The same assignments are used for generalized π-symmetries $\mathbf{R}$ (substituting R_n by $\mathbf{R}$).

Theorem 1.1. *For each $\nu\in\mathbf{N}$ we have*

(1) $\mathbf{Z}_\nu(\mathbf{R},\mathbf{u},\mathbf{a})=\bigcap\{\mathbf{Z}_\nu(R_n,\mathbf{u},\mathbf{a});\ n\in\mathbf{FN}\}$
(2) $\mathbf{B}_\nu(\mathbf{R},\mathbf{u},\mathbf{a})=\bigcap\{\mathbf{B}_\nu(R_n,\mathbf{u},\mathbf{a});\ n\in\mathbf{FN}\}$.

Proof: For the first statement, let $x\in\bigcap\{\mathbf{Z}_\nu(R_n,\mathbf{u},\mathbf{a});\ n\in\mathbf{FN}\}$. First we show that $x\in\mathbf{C}_\nu(\mathbf{R},\mathbf{u},\mathbf{a})$. x is the function $x:\mathrm{dom}(x)\to\mathbf{G}\setminus\{0\}$ and $v\in\mathrm{dom}(x)\Leftrightarrow(\forall n\in\mathbf{FN})\,(v\in R_n\ \&\ (v\nsubseteq\mathbf{a}))$. This means that $v\in\mathrm{dom}(x)\Leftrightarrow(v\in\mathbf{R}\ \&\ (v\nsubseteq\mathbf{a}))$, hence $x\in\mathbf{C}_\nu(\mathbf{R},\mathbf{u},\mathbf{a})$.

We can continue the proof now. It holds that $(\forall n\in\mathbf{FN})\,(x\in\mathbf{Z}_\nu(R_n,\mathbf{u},\mathbf{a}))$. So that $(\forall n\in\mathbf{FN})\,(x\in\mathbf{C}_\nu(R_n,\mathbf{u},\mathbf{a})\ \&\ \partial(x)\in\mathbf{C}_{\nu-1}(R_n,\mathbf{a}))$ and finally $x\in\mathbf{C}_\nu(\mathbf{R},\mathbf{u},\mathbf{a})\ \&\ \partial(x)\in\mathbf{C}_{\nu-1}(\mathbf{R},\mathbf{a})$. The opposite inclusion is clear.

For the second statement: the inclusion $\subseteq$ is obvious. Let $c\in\bigcap\{\mathbf{B}_\nu(R_n,\mathbf{u},\mathbf{a});\ n\in\mathbf{FN}\}$. It means that for each $n\in\mathbf{FN}$ there is $d_n\in$

$\mathbf{C}_{\nu+1}(R_n, \mathbf{u}, \mathbf{a})$ such that $\partial(d_n) = c$. By the prolongation of the sequences $\{d_n;\ n \in \mathbf{FN}\}$ and $\{R_n; n \in \mathbf{FN}\}$ we get $d \in \mathbf{C}_{\nu+1}(\mathbf{R}, \mathbf{u}, \mathbf{a})$ such that $\partial(d) = c$. $\square$

Now we can form the following Sd-IS of groups : $(\mathbb{Z}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \bar{i})$ is $\{\mathbf{Z}_\nu(R_n, \mathbf{u}, \mathbf{a}), \overline{i_n^m} \otimes \mathbf{Id}_\mathbf{G}, m \geq n\}$, $(\mathbb{B}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \bar{i})$ is $\{\mathbf{B}_\nu(R_n, \mathbf{u}, \mathbf{a}), \overline{i_n^m} \otimes \mathbf{Id}_\mathbf{G}, m \geq n\}$ (i_n^m are competent inclusions) and finally $(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$ is $\{\mathbf{H}_\nu(R_n, \mathbf{u}, \mathbf{a}), p_n^m, m \geq n\}$ where p_n^m are projections of factor groups induced by $\overline{i_n^m} \otimes \mathbf{Id}_\mathbf{G}$. As one can see, the inverse limit of the last defined Sd-IS, say $\varprojlim(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$, would serve as an analogy of Vietoris's homology groups (considering $\mathbf{R} = \mathbf{r}(\overline{\mathbf{R}})$ and $R_n = \mathbf{r}(\overline{R}_n)$ for an indiscernibility relation $\overline{\mathbf{R}}$ and its generating sequence $\{\overline{R}_n;\ n \in \mathbf{FN}\}$).

We have

Theorem 1.2. *Under the previous assignments and agreements for each* $\nu \in \mathbf{N}$ *we have* $\mathbf{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}) \cong \varprojlim(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$.

PROOF: The assumptions of [G1, Corollary 2.12] are satisfied and therefore $\bigcap\{\mathbf{Z}_\nu(R_n, \mathbf{u}, \mathbf{a});\ n \in \mathbf{FN}\} / \bigcap\{\mathbf{B}_\nu(R_n, \mathbf{u}, \mathbf{a});\ n \in \mathbf{FN}\} \cong \varprojlim(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$. Hence according to the previous theorem we have
$\mathbf{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}) \cong \varprojlim(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$. $\square$

Lemma 1.3. *Let there be given generalized* π*-symmetries* $\mathbf{R}$ *on the set* $\mathbf{u}$ *and* $\mathbf{S}$ *on the set* $\mathbf{v}$ *with generating sequences* $\{R_n; n \in \mathbf{FN}\}$ *and* $\{S_n; n \in \mathbf{FN}\}$*, respectively (we assume that* $\mathbf{R} \subseteq \mathcal{P}(\mathbf{u})$ *and* $\mathbf{S} \subseteq \mathcal{P}(\mathbf{v})$ *unless something other is explicitly stated). Let* $\mathbf{f} : \mathbf{u} \to \mathbf{v}$ *be a map simplicial from* $\mathbf{R}$ *to* $\mathbf{S}$.

Then for each $n \in \mathbf{FN}$ *there exists* $m \in \mathbf{FN}$ *such that for each* $k \geq m$ *the map* $\mathbf{f}$ *is a simplicial map from* R_k *to* S_n.

PROOF: Take the "suitable" Sd* prolongation $\{R_\nu;\ \nu \in \alpha \in \mathbf{N} \setminus \mathbf{FN}\}$ of the $\{R_n; n \in \mathbf{FN}\}$. The class $\mathcal{A} = \{\nu \in \alpha;\ (\forall z \in R_\nu)\,(\mathbf{f}''z \in S_n)\}$ is an Sd*-class. It contains all infinitely large numbers of α inasmuch its minimal element is a finite natural number which satisfies all our requirements. $\square$

Let $\mathbf{f} : (\mathbf{u}, \mathbf{a}) \to (\mathbf{v}, \mathbf{b})$ be a map simplicial from $\mathbf{R}$ to $\mathbf{S}$. By the previous lemma, we can take the generating sequence $\{R_n; n \in \mathbf{FN}\}$ of the generalized π-symmetry $\mathbf{R}$ such that for every $n \in \mathbf{FN}$ the map $\mathbf{f}$ is a map simplicial from R_n to S_n. The map $\mathbf{f}$ then induces the homomorphism $\mathbf{f}_*^n : \mathbf{H}_\nu(R_n, \mathbf{u}, \mathbf{a}) \to \mathbf{H}_\nu(S_n, \mathbf{v}, \mathbf{b})$ for every $n \in \mathbf{FN}$. The map $\bar{\mathbf{f}}$ commutes with injections on the level of chain complexes. Inasmuch the sequence $\{\mathbf{f}_*^n;\ n \in \mathbf{FN}\}$ together with the identity map on $\mathbf{FN}$ defines a homomorphism between the inverse systems $(\mathbf{H}_\nu(R_n, \mathbf{u}, \mathbf{a}), \mathbf{p}_n^m, m \geq n)$ (abbreviated to $(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$) and $(\mathbf{H}_\nu(S_n, \mathbf{v}, \mathbf{b}), \mathbf{p}_n^m, m \geq n)$ (abbreviated to $(\mathbb{H}_\nu(\mathbf{S}, \mathbf{v}, \mathbf{b}), \mathbf{p})$) with inverse limits $\varprojlim(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p})$ and $\varprojlim(\mathbb{H}_\nu(\mathbf{S}, \mathbf{v}, \mathbf{b}), \mathbf{p})$, respectively. For the inverse limit of this map we write $\mathbf{f}_*^\infty$.

Theorem 1.4. *Let*

$$\mathbf{F} : \mathbf{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}) \longrightarrow \varprojlim(\mathbb{H}_\nu(\mathbf{R}, \mathbf{u}, \mathbf{a}), \mathbf{p}) \quad \textit{and}$$
$$\mathbf{G} : \mathbf{H}_\nu(\mathbf{S}, \mathbf{v}, \mathbf{b}) \longrightarrow \varprojlim(\mathbb{H}_\nu(\mathbf{S}, \mathbf{v}, \mathbf{b}), \mathbf{p})$$

be isomorphisms from Theorem 1.2. *Then for all* $\nu \in \mathbf{FN}$ *the diagram*

$$\begin{tikzcd} \mathbf{H}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right) \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{f}_*"'] & \varprojlim(\mathbb{H}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right),\mathbf{p}) \arrow[d, "\mathbf{f}_*^\infty"'] \\ \mathbf{H}_\nu\left(\mathbf{S},\mathbf{v},\mathbf{b}\right) \arrow[r, "\mathbf{G}"] & \varprojlim(\mathbb{H}_\nu\left(\mathbf{S},\mathbf{v},\mathbf{b}\right),\mathbf{p}) \end{tikzcd}$$

commutes.

Proof: The diagram

$$\begin{tikzcd} \mathbf{H}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right) \arrow[r, "i_*"] \arrow[d, "\mathbf{f}_*"'] & \mathbf{H}_\nu\left(R_n,\mathbf{u},\mathbf{a}\right) \arrow[d, "\mathbf{f}_*^n"'] \\ \mathbf{H}_\nu\left(\mathbf{S},\mathbf{v},\mathbf{b}\right) \arrow[r, "i_*"] & \mathbf{H}_\nu\left(S_n,\mathbf{v},\mathbf{b}\right) \end{tikzcd}$$

commutes because $i \circ \mathbf{f} = \mathbf{f} \circ i$. Consider the diagrams (one for each $n \in \mathbf{FN}$):

$$\begin{tikzcd} \mathbf{H}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right) \arrow[r, "\mathbf{F}"] \arrow[d, "\mathbf{f}_*"'] & \varprojlim(\mathbb{H}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right),\mathbf{p}) \arrow[r, "\mathbf{p}_n"] \arrow[d, "\mathbf{f}_*^\infty"'] & \mathbf{H}_\nu\left(R_n,\mathbf{u},\mathbf{a}\right) \arrow[d, "\mathbf{f}_*^n"'] \\ \mathbf{H}_\nu\left(\mathbf{S},\mathbf{v},\mathbf{b}\right) \arrow[r, "\mathbf{G}"] & \varprojlim(\mathbb{H}_\nu\left(\mathbf{S},\mathbf{v},\mathbf{b}\right),\mathbf{p}) \arrow[r, "\mathbf{p}_n"] & \mathbf{H}_\nu\left(S_n,\mathbf{v},\mathbf{b}\right) \end{tikzcd}$$

It is easy to see that for all $n \in \mathbf{FN}$ $\quad \mathbf{p}_n \circ \mathbf{F} = i_*$, $\ \mathbf{p}_n \circ \mathbf{G} = i_*$ and therefore the great rectangle commutes. As we have seen, the right little square commutes as well.

Let $x \in \mathbf{H}_\nu\left(\mathbf{r}\left(\mathbf{R}\right)\right)$. Then for all $n \in \mathbf{FN}$

$$\mathbf{p}_n \circ \mathbf{f}_*^\infty \circ \mathbf{F}(x) = \mathbf{f}_*^n \circ \mathbf{p}_n \circ \mathbf{F}(x) = \mathbf{p}_n \circ \mathbf{G} \circ \mathbf{f}_*(x).$$

So that for all $n \in \mathbf{FN}$ $\qquad \mathbf{f}_*^\infty \circ \mathbf{F}(x) - \mathbf{G} \circ \mathbf{f}_*(x) \in \mathbf{p}_n^{-1}(0)$. To complete the proof it is enough to prove that $\bigcap\left\{\mathbf{p}_n^{-1}{}''(0);\ n \in \mathbf{FN}\right\} = \{0\} \subseteq \varprojlim(\mathbb{H}_\nu\left(\mathbf{R},\mathbf{u},\mathbf{a}\right),\mathbf{p})$. But this fact follows directly from the definition of the inverse limit of an Sd-IS. $\square$

To obtain similar results we can also proceed in a quite different way. Let us suppose that we have Theorem 1.1 only for the case $\mathbf{a} = \emptyset$, it means that we have proved the equalities $\mathbf{Z}_\nu\left(\mathbf{r}\left(\mathbf{R}\right)\right) = \bigcap\left\{\mathbf{Z}_\nu\left(\mathbf{r}\left(R_n\right)\right);\ n \in \mathbf{FN}\right\}$ and $\mathbf{B}_\nu\left(\mathbf{r}\left(\mathbf{R}\right)\right) = \bigcap\left\{\mathbf{B}_\nu\left(\mathbf{r}\left(R_n\right)\right);\ n \in \mathbf{FN}\right\}$. We have Theorem 1.2 for $\mathbf{a} = \emptyset$ in the mentioned case, it means that we have proved the isomorphisms of absolute homology groups.

Now let us consider the homology sequences

$$\cdots \leftarrow \mathbf{H}_{\nu-1}\left(R_n,\mathbf{a}\right) \xleftarrow{\partial_*} \mathbf{H}_\nu\left(R_n,\mathbf{u},\mathbf{a}\right) \xleftarrow{j_*} \mathbf{H}_\nu\left(R_n,\mathbf{u}\right) \xleftarrow{i_*} \mathbf{H}_\nu\left(R_n,\mathbf{a}\right) \leftarrow \cdots$$

(the group $(\mathbf{G},+)$ is implicitly used as group of coefficients).

This sequence is exact and it consists of Sd-groups. The generating sequence $\{R_n; n \in \mathbf{FN}\}$ of $\mathbf{R}$ has the property that $(\forall n \in \mathbf{FN})\,(R_{n+1} \subseteq R_n)$. Therefore the inclusion $i : (\mathbf{u}, \mathbf{a}) \to (\mathbf{u}, \mathbf{a})$ induces homomorphisms in the vertical direction commuting with the given homomorphisms ∂_*, j_* and i_*, so that we have the inverse system of Sd-chains of groups. Hence the inverse limit of this inverse system of groups is the exact "Vietoris's" homology sequence ([G1, Theorem 2.9]). Combining the isomorphisms of "absolute" members and "five lemma" yield to the isomorphisms of "relative" members.

2. The comparison with Čech's homology theory, elements of dimension theory.

Thanks to the results of the previous section we are able to show the connection with Čech theory via the technique due to C.H. Dowker [D]. But we cannot use everything without modifications in the AST and therefore we have to be very careful with its transferring. Especially we are not restricted on the finiteness of the chains in the AST, our chains are set linear combinations. I think that the full meaning of the last two sentences becomes clear after concrete developing of a certain part of the technique.

Let R be an Sd-symmetry on $\mathbf{u}$, such that $R \subseteq \mathbf{u}^2$, let $\mathbf{v} \subseteq \mathbf{u}$ in the entire section. According to the [G2, Corollary 3.4] we need not care about homology groups in infinite dimensions.

Let us recall the

Lemma 2.1. *The class* $\{R''\{x\};\ x \in \mathbf{u}\}$ *is a set.*

Now let us make some necessary definitions and assignments. Let us consider the quadruple $[\mathbf{u}, \mathbf{v}, \alpha^R_{\mathbf{u}}, \alpha^R_{\mathbf{v}}]$, where

$$\alpha^R_{\mathbf{u}} = \{R''\{x\};\ x \in \mathbf{u}\}\,,\ \ \alpha^R_{\mathbf{v}} = \{R''\{x\};\ x \in \mathbf{v}\}\,.$$

The sets $\alpha^R_{\mathbf{u}}$ and $\alpha^R_{\mathbf{v}}$ are set coverings of the sets $\mathbf{u}$ and $\mathbf{v}$, respectively.

Put

$$\mathbf{K}^R_{\mathbf{x}} = \{a;\ a \neq \emptyset\ \ \&\ \ (\exists z \in \mathbf{x})\ a \subseteq R''\{z\}\}\,.$$

Now let us take a couple $[\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}}]$. Obviously this is a couple of generalized symmetries.

Now we define the couple of generalized symmetries $[\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}]$ as follows. $\mathbf{L}^R_{\mathbf{u}}$ and $\mathbf{L}^R_{\mathbf{v}}$ are GS on the sets $\alpha^R_{\mathbf{u}}$ and $\alpha^R_{\mathbf{v}}$, respectively. Let $R''\{x_0\}, \dots, R''\{x_k\}$ be elements of $\alpha^R_{\mathbf{u}}$ $(\alpha^R_{\mathbf{v}})$, i.e. "vertices" of our future generalized symmetries $\mathbf{L}^R_{\mathbf{u}}$ $(\mathbf{L}^R_{\mathbf{v}})$. Then $\{R''\{x_0\}, \dots, R''\{x_k\}\} \in \mathbf{L}^R_{\mathbf{u}}$ $(\in \mathbf{L}^R_{\mathbf{v}})$ iff there is $x \in \mathbf{u}$ $(x \in \mathbf{v})$ such that $x \in R''\{x_0\} \cap \dots \cap R''\{x_k\}$.

So we can say that the set $\{R''\{x_0\}, \dots, R''\{x_k\}\}$ is a **simplex** of $\mathbf{L}^R_{\mathbf{u}}$ $(\mathbf{L}^R_{\mathbf{v}})$ iff the sets $R''\{x_0\}, \dots, R''\{x_k\}$ have nonempty intersection with an element laying in $\mathbf{u}$ $(\mathbf{v})$.

We shall use also the assignments $\mathbf{L}\left(\alpha^R_{\mathbf{u}}\right)$, $\mathbf{L}\left(\{X;\ X \in \alpha^R_{\mathbf{u}}\}\right)$ and similar ones of the above meaning which are self-explanatory.

Now let $\mathbf{X}$ be a GS. By $\mathbf{X}'$ we assign the so called **barycentric subdivision** of $\mathbf{X}$. The "vertices" of $\mathbf{X}'$ are simplexes (i.e. elements) of $\mathbf{X}$. The simplexes (i.e. the elements) of GS $\mathbf{X}'$ are those (now we consider only finite) sets of vertices of $\mathbf{X}'$ which are linearly ordered by the inclusion.

It is clear that the operator of making the barycentric subdivision preserves the inclusion.

Now let $\mathbf{K}_1$ be a generalized Sd-symmetry on $\mathbf{u}$ such that $\bigcup \mathbf{K}_1 = \mathbf{u}$ and let $\mathbf{K}_2$ be a generalized Sd-symmetry on $\mathbf{v}$ such that $\bigcup \mathbf{K}_2 = \mathbf{v}$ ($\mathbf{v} \subseteq \mathbf{u}$).

Let the vertices of the GS $\mathbf{K}_1$ (i.e. elements of the set $\mathbf{u}$) be ordered by an Sd function $\mathbf{F}$ in such a manner that the elements of $\mathbf{v}$ precede the elements of $\mathbf{u} \setminus \mathbf{v}$.

Now we shall define the simplicial map $\varphi : [\mathbf{K}_1', \mathbf{K}_2'] \to [\mathbf{K}_1, \mathbf{K}_2]$ in the following way:
Let $y = [x_0, \dots, x_n]$ be an ordered simplex of $\mathbf{K}_i$ ($i \in \{1, 2\}$) ordered in the ordering $\mathbf{F}$ (i.e. $x_0 <_{\mathbf{F}} \dots <_{\mathbf{F}} x_n$). It means that y is a vertex of $\mathbf{K}_i'$ and each vertex of $\mathbf{K}_i'$ is of this form. Let us put $\varphi(y) = x_0$.

It is easy to verify that φ is a simplicial map. Changing the ordering $\mathbf{F}$ we obtain simplicial map contiguous to the original one. Therefore all such possible maps induce the same homomorphisms of homological groups. (cf. [G2, § 4, Homotopy axiom])

It is well-known that $\varphi_* : \mathbf{H}_k(\mathbf{K}_1', \mathbf{K}_2') \cong \mathbf{H}_k(\mathbf{K}_1, \mathbf{K}_2)$ is an isomorphism in every dimension $k \in \mathbf{FN}$.

The GS $\mathbf{K}_1'$ and $\mathbf{K}_2'$ have the barycentric subdivisions $\mathbf{K}_1''$ and $\mathbf{K}_2''$, respectively. The ordering of vertices of $\mathbf{K}_i'$ by the inclusion (which is set-definable and can be imbedded into some Sd linear ordering) defines simplicial map $\varphi' : [\mathbf{K}_1'', \mathbf{K}_2''] \to [\mathbf{K}_1', \mathbf{K}_1']$ etc. These maps also induce the isomorphisms between convenient homological groups.

Let us consider the second couple of set-definable GS $[\mathbf{L}_1, \mathbf{L}_2]$. Let $\psi : [\mathbf{L}_1, \mathbf{L}_2] \to [\mathbf{K}_1, \mathbf{K}_2]$ be a simplicial map. It induces the following simplicial map $\psi' : [\mathbf{L}_1', \mathbf{L}_2'] \to [\mathbf{K}_1', \mathbf{K}_2']$ of barycentric subdivisions:
If $y = \{x_0, \dots, x_k\}$ is a vertex of $\mathbf{L}_i'$, i.e. simplex of $\mathbf{L}_i$, then $\{\psi(x_0), \dots, \psi(x_k)\}$ is a simplex of $\mathbf{K}_i$ because ψ is a simplicial map. So that it is a vertex of $\mathbf{K}_i'$. We define

$$\psi'(\{x_0, \dots, x_k\}) = \{\psi(x_0), \dots, \psi(x_k)\}.$$

One can see that ψ' preserves the inclusion inasmuch it is a simplicial map.

Let $\bar{\varphi} : [\mathbf{L}_1', \mathbf{L}_2'] \to [\mathbf{L}_1, \mathbf{L}_2]$ be the simplicial map obtained in the same way as φ for $\mathbf{K}$'s, i.e. it is mapping a vertex of $\mathbf{L}_i'$ to its first vertex in some given Sd ordering of the vertices of $\mathbf{L}_i$. Similarly as in the previous case we know that $\bar{\varphi}_* : \mathbf{H}_k(\mathbf{L}_1', \mathbf{L}_2') \cong \mathbf{H}_k(\mathbf{L}_1, \mathbf{L}_2)$ is an isomorphism in every dimension $k \in \mathbf{FN}$.

Lemma 2.2 [D]. *The maps $\varphi \circ \psi'$ and $\psi \circ \bar{\varphi} : [\mathbf{L}_1', \mathbf{L}_2'] \to [\mathbf{K}_1, \mathbf{K}_2]$ are contiguous.*

Proof: Let $y'' = \{y_0', \dots, y_n'\}$ be a simplex of $\mathbf{L}_1'$, let $\hat{y}'$ be its latest (in the ordering by the inclusion) vertex. Then for all $y_i' \in y''$ it holds $y_i' \subseteq \hat{y}'$, hence $\psi'(y_i') \subseteq \psi'(\hat{y}')$. And because $\varphi(\psi'(y_i')) \in \psi'(y_i')$ we have $\varphi(\psi'(y_i') \in \psi'(\hat{y}')$. Also $\bar{\varphi}(y_i') \in y_i' \subseteq \hat{y}'$ inasmuch $\psi(\bar{\varphi}(y_i')) \in \psi'(\hat{y}')$. So that the images of vertices of the simplex y'' in both maps $\varphi \circ \psi'$ and $\psi \circ \bar{\varphi}$ are contained in the simplex $\psi'(\hat{y}')$. Moreover, if y'' is a simplex of $\mathbf{L}'_2$, $\psi'(\hat{y}')$ is a simplex of $\mathbf{K}_2$. $\square$

This proof could be found in [D]. We gave it only to demonstrate the technique and the fact that our translation works. The remaining theorems due to [D] are given without proofs.

Finally we define the map $\psi : [\mathbf{L}^{R'}_{\mathbf{u}}, \mathbf{L}^{R'}_{\mathbf{v}}] \to [\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}}]$ of the barycentric subdivision of $[\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}]$ to $[\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}}]$. This map requires the largest care in the transferring of the Dowker's technique because we have to keep the set-definability of our notions. Every vertex $x' = \{x_0, \ldots, x_k\}$ of the GS $\mathbf{L}^{R'}_{\mathbf{u}}$ is a simplex of $\mathbf{L}^R_{\mathbf{u}}$. We recall that the set $\mathbf{u}$ is ordered by some linear ordering $\mathbf{F}$ such that elements of the given subset $\mathbf{v}$ of $\mathbf{u}$ precede the elements of $\mathbf{u} \setminus \mathbf{v}$ and that $\mathbf{u}$ is the set of all vertices of $\mathbf{K}^R_{\mathbf{u}}$ and $\mathbf{v}$ is the set of vertices of $\mathbf{K}^R_{\mathbf{v}}$. To x' we can attach the minimal element $z \in \mathbf{u}$ with $z \in x_0 \cap \cdots \cap x_k$ (every x_i is $R''\{y\}$ for some $y \in \mathbf{u}$ — i.e. it is a set) in the ordering $\mathbf{F}$. So we put $\psi(x') = z$. It is clear that if x' is a vertex of $\mathbf{L}^{R'}_{\mathbf{v}}$ then $\psi(x') = z \in \mathbf{v}$ owing to the required property of $\mathbf{F}$. Strictly speaking the map we have just defined is the map between the sets of vertices of the mentioned generalized symmetries.

So let $y = \{y_0, \ldots, y_k\}$ be a simplex of $\mathbf{L}^{R'}_{\mathbf{u}}$ and let y_0 be its first element (the ordering is the inclusion now). Then for every $y_i \in y$ $\bar{\varphi}(y_0) \in y_0 \subseteq y_i$, so that $\psi(y_i) \in \bar{\varphi}(y_0)$. Therefore the elements $\psi(y_0), \ldots, \psi(y_k)$ form a simplex of the GS $\mathbf{K}^R_{\mathbf{u}}$. The same result can be obtained also for $\mathbf{L}^{R'}_{\mathbf{v}}$ and $\mathbf{K}^R_{\mathbf{v}}$. So that ψ is a simplicial map because it is set-definable. The definition of ψ depends on the ordering $\mathbf{F}$. But according to the previous consideration, all possible images (in various orderings) of the vertices of y are contained in $\bar{\varphi}(y_0)$. So that all possible maps are congruent. Therefore the homomorphism $\psi_* : \mathbf{H}_k(\mathbf{L}^{R'}_{\mathbf{u}}, \mathbf{L}^{R'}_{\mathbf{v}}) \to \mathbf{H}_k(\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}})$ is uniquely determined.

Let us put

$$\omega = \psi_* \circ \bar{\varphi}_*^{-1} : \mathbf{H}_k(\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}) \to \mathbf{H}_k(\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}})$$

and similarly let

$$\omega_2 = (\psi \restriction \mathbf{L}^{R'}_{\mathbf{v}})_* (\bar{\varphi} \restriction \mathbf{L}^{R'}_{\mathbf{v}})_*^{-1}$$

be the corresponding homomorphism from $\mathbf{H}_k(\mathbf{L}^R_{\mathbf{v}})$ to $\mathbf{H}_k(\mathbf{K}^R_{\mathbf{v}})$. Let ∂_* and $\bar{\partial}_*$ be boundary homomorphisms of homology sequences of couples of GS $[\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}}]$ and $[\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}]$.

Lemma 2.3 [D]**.** *The homomorphism ω commutes with boundary homomorphisms, i.e.*

$$\partial_* \circ \omega = \omega_2 \circ \bar{\partial}_* : \mathbf{H}_k(\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}) \to \mathbf{H}_{k-1}(\mathbf{K}^R_{\mathbf{v}}).$$

Let R and S be Sd-symmetries on the sets $\mathbf{u}$ and $\mathbf{w}$, respectively. Let $\mathbf{f} : [\mathbf{u}, \mathbf{v}] \to [\mathbf{w}, \mathbf{z}]$ be a map continuous from R to S (i.e. $\mathbf{f}''\mathbf{v} \subseteq \mathbf{z} \quad \& \quad x\mathbf{R}y \implies \mathbf{f}(x)\mathbf{S}\mathbf{f}(y)$). The map $\mathbf{f}$ is simplicial from $[\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}}]$ to $[\mathbf{K}^S_{\mathbf{w}}, \mathbf{K}^S_{\mathbf{z}}]$.

The map $f : [\alpha^R_{\mathbf{u}}, \alpha^R_{\mathbf{v}}] \to [\alpha^S_{\mathbf{w}}, \alpha^S_{\mathbf{z}}]$ defined by $f(R''\{x\}) = S''\{\mathbf{f}(x)\}$ is a simplicial map from $[\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}]$ to $[\mathbf{L}^S_{\mathbf{w}}, \mathbf{L}^S_{\mathbf{z}}]$.

Lemma 2.4 [D]. *Let R and S be Sd-symmetries on the sets* $\mathbf{u}$ *and* $\mathbf{w}$, *respectively, let* $\mathbf{f} : [\mathbf{u}, \mathbf{v}] \to [\mathbf{w}, \mathbf{z}]$ *be continuous from R to S. Then* $\mathbf{f}_*$ *and* f_* *commute with* ω, *i.e.*

$$\mathbf{f}_* \circ \omega = \omega \circ f_* : \mathbf{H}_k(\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}) \to \mathbf{H}_k(\mathbf{K}^S_{\mathbf{w}}, \mathbf{K}^S_{\mathbf{z}}).$$

Theorem 2.5 [D]. *Let R be an Sd-symmetry on* $\mathbf{u}$, *let* $\mathbf{v} \subseteq \mathbf{u}$. *Then*

$$\omega : \mathbf{H}_k(\mathbf{L}^R_{\mathbf{u}}, \mathbf{L}^R_{\mathbf{v}}) \to \mathbf{H}_k(\mathbf{K}^R_{\mathbf{u}}, \mathbf{K}^R_{\mathbf{v}})$$

is an isomorphism for every $k \in \mathbf{FN}$.

Let $\mathbf{R}$ be an indiscernibility equivalence on the set $\mathbf{u}$ fixed for the rest of the paper. Let $\{R_n; n \in \mathbf{FN}\}$ be a generating sequence of $\mathbf{R}$ on the set $\mathbf{u}$. Take the Sd-IS $(\mathbf{H}_k(\mathbf{K}^{R_n}_{\mathbf{u}}, \mathbf{K}^{R_n}_{\mathbf{v}}), \mathbf{p}^m_n, m \geq n)$ abbreviated to the $(\mathbb{H}_k(\mathbf{K}), \mathbf{p})$ with the inverse limit $\varprojlim(\mathbb{H}_k(\mathbf{K}), \mathbf{p})$ and the Sd-IS $(\mathbf{H}_k(\mathbf{L}^{R_n}_{\mathbf{u}}, \mathbf{L}^{R_n}_{\mathbf{v}}), \mathbf{p}^m_n, m \geq n)$ abbreviated to the $(\mathbb{H}_k(\mathbf{L}), \mathbf{p})$ with the inverse limit $\varprojlim(\mathbb{H}_k(\mathbf{L}), \mathbf{p})$. All morphisms in the last Sd-IS are projections defined by convenient inclusions.

One can easily see that for any simplex x it holds

$$x \in \mathbf{r}(\mathbf{R}) \iff x \in \bigcap \left\{\mathbf{K}^{R_n}_{\mathbf{u}};\ n \in \mathbf{FN}\right\}.$$

Indeed we have

$$x \in \mathbf{r}(\mathbf{R}) \iff (\exists z)(x \subseteq \mathbf{R}''\{z\} \implies x \subseteq R''_n\{z\} \implies x \in \bigcap \left\{\mathbf{K}^{R_n}_{\mathbf{u}};\ n \in \mathbf{FN}\right\}.$$

For the opposite let $x \in \bigcap \left\{\mathbf{K}^{R_n}_{\mathbf{u}};\ n \in \mathbf{FN}\right\}$. Then $(\forall n \in \mathbf{FN})(\exists z_n \in \mathbf{u})\ x \subseteq R''_n\{z_n\}$. By the prolongation we obtain $z \in \mathbf{u}$ such that $x \subseteq \mathbf{R}''\{z\}$.

Theorem 2.6. *Let* $\mathbf{G}$ *be an Sd group of coefficients. Let us keep the above assignments. Then for every* $k \in \mathbf{FN}$ *we have*

$$\mathbf{H}_k(\mathbf{R}, \mathbf{u}, \mathbf{v}) \cong \varprojlim(\mathbb{H}_k(\mathbf{K}), \mathbf{p}) \cong \varprojlim(\mathbb{H}_k(\mathbf{L}), \mathbf{p})\ .$$

Proof: The first isomorphism follows from Theorem 1.2, the second one from [D, Theorem 5.2a]. □

Lemma 2.7. *Let* $\mathbf{R} \subseteq S$, *S be an Sd-symmetry (on the same set* $\mathbf{u}$ *as the indiscernibility equivalence* $\mathbf{R}$*). Then there is a finite subset* $v \subseteq \mathbf{u}$ *such that* $\{\mathrm{Int}\left(S''\{x\}\right);\ x \in v\}$ *is a finite open covering of* $\mathbf{u}$ *and also* $\{S''\{x\};\ x \in v\}$ *is an Sd-covering of* $\mathbf{u}$.

Proof: As $\{\mathrm{Int}\left(S''\{x\}\right);\ x \in \mathbf{u}\}$ form an open covering of $\mathbf{u}$, there is a finite $v \subseteq \mathbf{u}$ such that $\{\mathrm{Int}\left(S''\{x\}\right);\ x \in v\}$ is an open covering of $\mathbf{u}$ as well. Because for every $x \in \mathbf{u}$ we have that $\mathrm{Int}\left(S''\{x\}\right) \subseteq S''\{x\}$, the set $\{S''\{x\};\ x \in v\}$ is an Sd-covering of $\mathbf{u}$. □

Lemma 2.8. *For every* $n \in \mathbf{FN}$ *let* v_n *be a finite subset of* $\mathbf{u}$ *such that* $\{\mathrm{Int}\left(R''_n\{x\}\right);\ x \in v_n\}$ *is an open covering of* $\mathbf{u}$. *Put* $R'_n = \left\{[x,y]; (\exists z \in v_n)(\{x,y\} \subseteq R''_n\{z\}\right\}$. *Then every* R'_n *is an Sd-class and* $\mathbf{R} = \bigcap\left\{R'_n;\ n \in \mathbf{FN}\right\}$.

PROOF: Set-definability of R'_n is obvious. For every $n \in \mathbf{FN}$ we have $R'_{n+1} \subseteq R_n$ and therefore $\bigcap\left\{R'_n;\ n \in \mathbf{FN}\right\} \subseteq \mathbf{R}$.

For the second inclusion let $[x,y] \in \mathbf{R}$ and let $n \in \mathbf{FN}$. There is $z \in v_n$ such that $x \in \mathrm{Int}\left(R''_n\{z\}\right)$. But $\mathrm{Int}\left(R''_n\{z\}\right)$ is a figure in $\mathbf{R}$ and therefore $y \in \mathrm{Int}\left(R''_n\{z\}\right)$. Hence $[x,y] \in R'_n$. □

Corollary 2.9. *Let* $\mathbf{G}$ *be an Sd-group. Let* $\mathrm{Dim}(\mathbf{R}) \leq d$ *(cf.* [S-W]). *Then*

$$\mathbf{H}_n(\mathbf{R},\mathbf{G}) = \mathbf{0}$$

for every $n \geq d+1$.

PROOF: Proposition 5 of the § 4 of [S-W] says: $\mathrm{Dim}(\mathbf{R}) \leq d$ iff into every finite Sd-covering of $\mathbf{u}$ it can be imbedded a finite Sd-subcovering of the rank $\leq d$. Combining this fact with 2.7 and 2.8 we get such finite Sd-coverings $\mathcal{R}_n$ each of the rank $\leq d$ that $\mathbf{R} = \bigcap\left\{\mathcal{R}'_n;\ n \in \mathbf{FN}\right\}$. Now let $(\mathcal{H}_n, \mathbf{p})$ be an Sd-IS of the groups $\mathbf{H}_n(\mathbf{L}(\mathcal{R}'_n), \mathbf{G})$ with projections induced by convenient inclusions. Let $\varprojlim(\mathcal{H}_n, \mathbf{p})$ be its inverse limit. Then by 2.6 we have

$$\mathbf{H}_n(\mathbf{R},\mathbf{G}) \cong \varprojlim(\mathcal{H}_n, \mathbf{p}) = \mathbf{0}$$

because in the GS $\mathbf{L}(\mathcal{R}'_n)$ there are no simplexes of dimensions $n \geq d+1$. □

Let $\mathbf{R} \subseteq S$, where S is an Sd-symmetry on $\mathbf{u}$, let v be finite subset of $\mathbf{u}$ such that $\left\{S''\{x\};\ x \in v\right\}$ is an Sd-covering of $\mathbf{u}$. Let us consider $\mathbf{L}^S_v$. Let $\{a_0, \dots, a_n\} \subseteq v$ be maximal (under inclusion) subset of v such that $\bigcap\left\{S''\{a_i\};\ 1 \leq i \leq n\right\} \neq \emptyset$. For every such maximal set let us choose an element $x_{a_0 \dots a_n} \in \bigcap\left\{S''\{a_i\};\ 1 \leq i \leq n\right\}$. All these elements of $\mathbf{u}$ form a set, say z. Now let us take $\mathbf{L}^{S\cap z^2}_z = \mathbf{L}(\left\{S''\{x\} \cap z^2;\ x \in z\right\})$. Then $\mathbf{L}^S_v$ and $\mathbf{L}^{S\cap z^2}_z$ are isomorphic as combinatorial structures (shortly speaking combinatorially isomorphic). Clearly, z is a finite set, because $\mathrm{card}(z) \leq \mathrm{card}(\mathcal{P}(v))$.

Theorem 2.10. *Let* $\mathbf{G}$ *be an Sd-group. Then for every* $k \in \mathbf{FN}$ *we have*

$$\mathbf{H}_k(\mathbf{K}^S_v, \mathbf{G}) \cong \mathbf{H}_k(\mathbf{K}^{S\cap z^2}_z, \mathbf{G}).$$

PROOF: In the sequence

$$\mathbf{H}_k(\mathbf{K}^S_v, \mathbf{G}) \cong \mathbf{H}_k(\mathbf{L}^S_v, \mathbf{G}) \cong \mathbf{H}_k(\mathbf{L}^{S\cap z^2}_z, \mathbf{G}) \cong \mathbf{H}_k(\mathbf{K}^{S\cap z^2}_z, \mathbf{G})$$

the first and the third isomorphisms follow from 2.5 and the second one follows from the combinatorial isomorphism of GS $\mathbf{L}^S_v$ and $\mathbf{L}^{S\cap z^2}_z$. □

This theorem enables us to compute homology groups with the help of finite structures.

Now we shall prove the relationship between our and Čech's homology groups.

Let $\alpha = \{X_1, \dots, X_n\}$ and $\beta = \{Y_1, \dots, Y_m\}$ be two coverings. The covering β is said to be finer than the covering α ($\beta \prec \alpha$) iff for every $Y_i \in \beta$ there is $X_j \in \alpha$ such that $Y_i \subseteq X_j$. Let us write $\overset{\circ}{\alpha}$ for $\left\{ \overset{\circ}{X}_1, \dots, \overset{\circ}{X}_n \right\}$ and analogically for β.

Theorem 2.11. *Let Sd-covering $\alpha = \{X_1, \dots, X_n\}$ be an Sd-covering of the set $\mathbf{u}$. Then there is an Sd-covering $\beta = \{Y_1, \dots, Y_m\}$ of $\mathbf{u}$ such that $\beta \prec \overset{\circ}{\alpha}$ and $\mathbf{L}(\overset{\circ}{\alpha})$ and $\mathbf{L}(\overset{\circ}{\beta})$ are combinatorially isomorphic.*

We need a further consideration for this theorem.

Let $\{Y_1, \dots, Y_n\}$ be an open covering of $\mathbf{u}$. Every Y_i is a σ-class because it is an open class. Therefore there are Sd-classes $Z_i^j \subseteq Y_i$ such that $\bigcup \left\{ Z_i^j;\ j \in \mathbf{FN} \right\} = Y_i$ and moreover $Z_i^j \subseteq Z_i^{j+1}$ for every $i, j \in \mathbf{FN}$.

Lemma 2.12. *There is $m \in \mathbf{FN}$ such that $\{Z_1^m, \dots, Z_n^m\}$ is an Sd-covering.*

Proof: Let for every $k \in \mathbf{FN}$ there is $x_k \in \mathbf{u}$ such that $R_k''\{x_k\}$ is contained in no Z_i^k. Then there is $x_\kappa \in \mathbf{u}$ such that $R_\kappa''\{x_\kappa\}$ is contained in no Z_i^κ. Because $R_\kappa''\{x_\kappa\} \subseteq \mathrm{Mon}(x_\kappa)$ and $Y_i \subseteq Z_i^\kappa$ it would mean that $\mathrm{Mon}(x_\kappa)$ is contained in no Y_i — a contradiction. □

Corollary 2.13. *There is $m \in \mathbf{FN}$ such that $\left\{ \overset{\circ}{Z}{}_1^m, \dots, \overset{\circ}{Z}{}_n^m \right\}$ is an open covering of $\mathbf{u}$.*

Lemma 2.14. *There is $m_0 \in \mathbf{FN}$ such that $\left\{ \overset{\circ}{Z}{}_1^{m_0}, \dots, \overset{\circ}{Z}{}_n^{m_0} \right\}$ is an open covering of $\mathbf{u}$ and $\mathbf{L}\left(\left\{ \overset{\circ}{Z}{}_1^{m_0}, \dots, \overset{\circ}{Z}{}_n^{m_0} \right\} \right)$ and $\mathbf{L}(\{Y_1, \dots, Y_n\})$ are combinatorially isomorphic.*

Proof: For every maximal subset $\{a_1, \dots, a_k\} \subseteq n+1$ such that $Y_{a_1} \cap \dots \cap Y_{a_k} \neq \emptyset$ we choose an element $x_{a_1 \dots a_k} \in Y_{a_1} \cap \dots \cap Y_{a_k}$. Because there is only the finite number of such maximal subsets of $n+1$, there is $l \in \mathbf{FN}$ such that for every chosen element $x_{a_1 \dots a_k}$ it holds $x_{a_1 \dots a_k} \in \overset{\circ}{Z}{}_{a_1}^l \cap \dots \overset{\circ}{Z}{}_{a_k}^l$. For such l we have the implication

$$Y_{a_1} \cap \dots \cap Y_{a_k} \neq \emptyset \implies \overset{\circ}{Z}{}_{a_1}^l \cap \dots \cap \overset{\circ}{Z}{}_{a_k}^l \neq \emptyset.$$

The converse implication is always trivial because for every $i, m \in \mathbf{FN}$, $\overset{\circ}{Z}{}_i^m \subseteq Y_i$. Hence taking m_0 greater than l and m obtained in 2.13 we get a required number. □

Proof of 2.11: Now it is enough to notice that for the given α the covering $\overset{\circ}{\alpha}$ is an open covering of $\mathbf{u}$. □

Corollary 2.15. *For every* $m \in \mathbf{FN}$ *satisfying* 2.14 *GS* $\mathbf{L}\left(\{Z_1^m, \dots, Z_n^m\}\right)$ *and* $\mathbf{L}\left(\left\{\overset{\circ}{Z}{}_1^m, \dots, \overset{\circ}{Z}{}_n^m\right\}\right)$ *are combinatorially isomorphic.*

It follows from 2.11 and 2.15 that for 2.8 we can choose such Sd-coverings $\alpha_n = \{R''_n\{x\};\ x \in v_n\}$ of $\mathbf{u}$ that GS $\mathbf{L}(\alpha_n)$ and $\mathbf{L}(\overset{\circ}{\alpha}_n)$ are combinatorially isomorphic. Let us take Sd-IS $(\mathbb{H}, \mathbf{p})$ and $(\overset{\circ}{\mathbb{H}}, \mathbf{p})$ with the groups $\mathbf{H}_k(\mathbf{L}(\alpha_n), \mathbf{G})$ and $\mathbf{H}_k(\mathbf{L}(\overset{\circ}{\alpha}_n), \mathbf{G})$ ($\mathbf{G}$ is as usually an Sd-group) with homomorphisms induced by convenient inclusions. Let $\varprojlim(\mathbb{H}, \mathbf{p})$ and $\varprojlim(\overset{\circ}{\mathbb{H}}, \mathbf{p})$ respectively be their inverse limits. We have

Theorem 2.16. *For every* $k \in \mathbf{FN}$

$$\mathbf{H}_k(\mathbf{R}, \mathbf{G}) \cong \varprojlim(\mathbb{H}, \mathbf{p}) \cong \varprojlim(\overset{\circ}{\mathbb{H}}, \mathbf{p})\,.$$

The similar theorem holds also for the relative case. From the next results of [D] it follows that our homological sequences of couples [figure, subfigure] in an indiscernibility relation are isomorphic with Čech's homology sequences generated by Sd-coverings of [set, subset] and also with Čech's homology sequences generated by open coverings of [set, subset]. This could be considered as the solution of the problem of [C] for compact metric spaces.

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Department of Algebra and Number Theory, Faculty of Mathematics and Physics, Mlynská dolina, 842 15 Bratislava, Slovak Republic

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