A note on a theorem of Klee

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Abstract. It is proved that if E, F are separable quasi-Banach spaces, then $E \times F$ contains a dense dual-separating subspace if either E or F has this property.

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Introduction.

In [2] Klee answered (negatively) the following question posed by A. Robertson and W. Robertson: If a topological vector space (tvs) E is dual-separating, i.e. its topological dual E' separates points of E from zero, is the same true of its completion? Klee's Corollary 3.6 of [2] leads to the following: If E is an infinite dimensional separable Banach space and $0 < p < 1$, then the product $L^p \times E$ contains a dense dual-separating subspace. In fact, if τ is the original topology of L^p and ϑ a vector topology on L^p such that $(L^p, \vartheta) \cong E$, then τ and ϑ are orthogonal [2]. Now by Corollary 3.6 of [2] we obtain that the completion of $Z = (L^p, \sup(\tau, \vartheta))$ (Z is dual-separating!) is the product $(L^p, \tau) \times E$. Recall that L^p with τ is without non-trivial continuous linear functionals [1].

In this note we extend this result by showing the following:

Theorem. *Let E, F be two separable quasi-Banach spaces. Then $E \times F$ contains a dense dual-separating subspace if either E or F contains a dense dual-separating subspace.*

A tvs E is quasi-Banach if E is metrizable and complete and E has a bounded neigbourhood of zero; in this case E is locally p-convex for some $0 < p \leq 1$, [5, p. 61].

Proof of Theorem: Our Theorem follows from the following

Lemma. *Let (E, τ) be an infinite dimensional separable quasi-Banach space and (Y, ϑ) an infinite dimensional separable metrizable and complete* tvs. *Let G be a dense dual-separating subspace of (E, τ). Then there exists an injective linear map P from G into Y such that $D = \{(x, P(x)) : x \in G\}$ is a dense dual-separating subspace of the product $(E, \tau) \times (Y, \vartheta)$.*

Proof: Set $\tau_0 = \tau \mid G$. First we find on G a separable normed topology β such that the topology $\inf(\tau_0, \beta)$ is indiscrete. Next we prove that G admits a Hausdorff vector topology $\alpha < \beta$ such that the completion $(G, \alpha)\hat{}$ of (G, α) is isomorphic to (Y, ϑ).

Suppose we have already found such topologies. Then $\inf(\tau_0, \alpha)$ is indiscrete. Hence $\triangle = \{(x,x) : x \in G\}$ is dense in $(G,\tau_0) \times (G,\alpha)$. Since we have $(G, \sup(\tau_0,\alpha)) \cong (\triangle, \tau_0 \times \alpha \mid \triangle)$, then $(G, \sup(\tau_0,\alpha))\hat{} \cong (\triangle, \tau_0 \times \alpha \mid \triangle)\hat{} \cong (E,\tau) \times (G,\alpha)\hat{} \cong (E,\tau) \times (Y,\vartheta)$. Let P be an isomorphism from (G,α) onto a dense subspace of (Y,ϑ). Then $Q : (x,y) \to (x, P(y))$, $x, y \in G$, is an isomorphism from $(G,\tau_0) \times (G,\alpha)$ onto a dense subspace of $(G,\tau_0) \times (Y,\vartheta)$. Hence $Q \mid \triangle$: $(x,x) \to (x, P(x))$ is an isomorphism from $\triangle$ onto a dense subspace $D = \{(x, P(x)) : x \in G\}$ of $(E,\tau) \times (Y,\vartheta)$. This also proves that D is dual-separating. Now we construct β on G. Let $\mu(G,G')$ be the Mackey topology on G associated with τ_0, i.e. the finest locally convex topology on G weaker than τ_0. Let B be the τ_0-unit ball and set $W = \operatorname{conv} B$. Then $\mu(G,G)$ is normed and W is a $\mu(G,G')$-bounded neighbourhood of zero. By [4, Theorem 1], there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of τ_0-dense subspaces of G such that $\dim G_n = c$ and $G = \oplus_{n=1}^{\infty} G_n$. Let p_w be the Minkowski functional of W and set $q_w(x) = \sup_n (n+1)^{-1} p_w(x_n)$, where $x_n \in G_n$, $x = \sum_{n=1}^{\infty} x_n$. Then (G, q_w) is a normed space. Let β be the topology defined by q_w. Set $U_p = \{x \in G : p_w(x) \le 1\}$, $V_q = \{x \in G : q_w(x) \le 1\}$. Clearly $tB \subset U_p$ for some $0 < t < 1$ and $(n+1)U_p \cap G_n \subset V_q$, $n \in \mathbb{N}$. Moreover V_q is τ_0-dense. In fact, let $x \in G$. Then $x \in tnS$ for some $n \in \mathbb{N}$, where S is a balanced τ_0-neighbourhood of zero such that $S + S \subset B$. Since G_n is τ_0-dense, there exists $x_n \in G_n$ such that $x_n - x \in tS \subset S$. Therefore $x_n \in x + tS \subset tnB \subset (n+1)U_p \cap G_n \subset V_q$. Hence we have that $\inf(\tau_0, \beta)$ is indiscrete and β is separable. Now we construct α. It is enough to find such a topology on the completion H of (G,β). Since H is an infinite dimensional separable Banach space, there exists a biorthogonal system $(x_n, f_n)_{n\in\mathbb{N}}$ such that $x_n \in H$, $f_n \in H'$, $(f_n)_{n\in\mathbb{N}}$ is equicontinuous and total on H. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in (Y,ϑ) such that $\sum_{n=1}^{\infty} y_n$ absolutely converges; $\operatorname{lin}\{y_n : n \in \mathbb{N}\}$ is ϑ-dense; $(y_n)_{n\in\mathbb{N}}$ is linearly m-independent, i.e. if $\sum_{n=1}^{\infty} t_n y_n = 0$ for $(t_n)_{n\in\mathbb{N}} \in \ell^{\infty}$, then $t_n = 0$, $n \in \mathbb{N}$, [3, Theorem 1]. Then the linear map $T : H \to Y$, $T(x) = \sum_{n=1}^{\infty} f_n(x) y_n$ is an injective compact map such that $T(H)$ is dense in Y and different from Y. This enables us to find a topology α as required. The proof is complete. $\square$

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