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Special finite-difference approximations of flow equations in terms of stream function, vorticity and velocity components for viscous incompressible liquid in curvilinear orthogonal coordinates

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Abstract. The Navier-Stokes equations written in general orthogonal curvilinear coordinates are reformulated with the use of the stream function, vorticity and velocity components. The resulting system id discretized on general irregular meshes and special monotone finite-difference schemes are derived.

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There are effective universal numerical methods (finite-difference and finite-element methods) for the solution of boundary value problems of hydrodynamics based on nonlinear Navier-Stokes equations for small Reynold numbers. However, the presence of large parameters at first order derivatives or small parameters at second order derivatives in the system of differential equations cause additional difficulties for the application of general methods which become uneffective (small speed of convergence, low precision). Thus a topical task is to work out special methods of solution — the so-called regular-convergence computational methods for the regarded problems [1]–[3]. Such methods can be applied to the system of flow equations for viscous incompressible fluid in curvilinear orthogonal coordinates.

The subject of examination are the finite-difference approximations of flow equations describing two-dimensional incompressible flow in curvilinear orthogonal coordinates in the case when it is possible to introduce the stream function of the liquid. This gives the possibility to develop special monotone difference schemes.

The flow of the liquid is governed by the Navier-Stokes equations which can be written in the Crocco-Lamb form [5]:

$$
(1) \qquad \begin{cases} \partial \vec{v}/\partial t - \vec{v} \times \vec{\omega} = -\operatorname{grad}\Pi - \nu \operatorname{rot}\vec{\omega} + \vec{F} \\ \operatorname{div}\vec{v} = 0, \end{cases}
$$

where $\vec{v}, \vec{\omega}, \vec{F}$, denote the vectors of velocity, vorticity and external force ($\vec{\omega} = \operatorname{rot}\vec{v}$), $\Pi = \varrho^{-1}p + (v_1^2 + v_2^2 + v_3^2)/2$ is the full pressure, ϱ, p, ν are the density, pressure and kinematic viscosity. Further, by v_1, v_2, v_3 we denote the components of the vector $\vec{v}$ in curvilinear orthogonal coordinates q_1, q_2, q_3; similarly, $\omega_1, \omega_2, \omega_3$ and F_1, F_2, F_3 are the corresponding components of the vectors $\vec{\omega}$ and $\vec{F}$, respectively. By t we denote time.

Operators grad, div, rot have the following representation in the coordinates q_1, q_2, q_3 [6]:

$$\operatorname{grad}\Pi = \sum_{k=1}^{3} H_k^{-1}\partial\Pi/\partial q_k \cdot \vec{\iota}_k,$$

$$\operatorname{div}\vec{v} = (H_1H_2H_3)^{-1}\Big[\frac{\partial}{\partial q_1}(H_2H_3v_1) + \frac{\partial}{\partial q_2}(H_3H_1v_2) + \frac{\partial}{\partial q_3}(H_1H_2v_3)\Big], \tag{2}$$

$$\operatorname{rot}\vec{v} = (H_1H_2H_3)^{-1}\begin{pmatrix} H_1\vec{\iota}_1 & H_2\vec{\iota}_2 & H_3\vec{\iota}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ H_1v_1 & H_2v_2 & H_3v_3 \end{pmatrix},$$

where $\vec{\iota}_k$ are unit vectors in the directions q_k, $k = 1,2,3$, and H_1, H_2, H_3 are Lame's coefficients. The boundary value problem for the system (1) in a bounded domain includes non-slip conditions on solid walls ($\vec{v} = 0$) and vanishing normal components of the viscous stress tensor τ at free surfaces.

The components of the tensor τ have the following form [6]:

$$\tau_{mk} = \eta[H_k^{-1}\partial v_m/\partial q_k + H_m^{-1}\partial v_k/\partial q_m - H_m^{-1}H_k^{-1}(v_m\partial H_m/\partial q_k + \\ + v_k\partial H_k/\partial q_m) + 2\delta_{mk}\sum_{l=1}^{3} v_l H_l^{-1}\partial\ln H_m/\partial q_l], \quad m,k = 1,2,3, \tag{3}$$

where δ_{mk} is the Kronecker symbol and η is the dynamic viscosity ($\eta\varrho^{-1} = \nu$).

In order to transform system (1) into curvilinear orthogonal coordinates, we use the following relations:

$$(\operatorname{grad}\Pi)_m = H_m^{-1}\partial\Pi/\partial q_m,$$
$$(\operatorname{rot}\vec{v})_m = H_{m+1}^{-1}H_{m+2}^{-1}(\partial(H_{m+2}v_{m+2})/\partial q_{m+1} - \partial(H_{m+1}v_{m+1})/\partial q_{m+2}),$$
$$(\vec{v}\times\vec{\omega})_m = v_{m+1}\omega_{m+2} - v_{m+2}\omega_{m+1},$$

$v_{m+3} = v_m$, $\omega_{m+3} = \omega_m$, $H_{m+3} = H_m$, $1 \le m \le 3$. Thus, system (1) becomes

$$\begin{cases} \partial v_m \partial t - v_{m+1}\omega_{m+2} + v_{m+2}\omega_{m+1} = -H_m^{-1}\partial\Pi/\partial q_m - \\ -\nu H_{m+1}^{-1}H_{m+2}^{-1}[\partial(H_{m+2}\omega_{m+2})/\partial q_{m+1} - \partial(H_{m+1}\omega_{m+1})/\partial q_{m+2}] + F_m, \\ \frac{\partial}{\partial q_1}(H_2H_3v_1) + \frac{\partial}{\partial q_2}(H_3H_1v_2) + \frac{\partial}{\partial q_3}(H_1H_2v_3) = 0, \end{cases} \tag{4}$$

where

$$\omega_m = H_{m+1}^{-1}H_{m+2}^{-1}(\partial(H_{m+2}v_{m+2})/\partial q_{m+1} - \partial(H_{m+1}v_{m+1})/\partial q_{m+2}).$$

If we apply the operator rot to system (1), then we eliminate the pressure Π and rewrite the flow equations in the form

$$\partial\vec{\omega}/\partial t - \mathrm{rot}(\vec{v}\times\vec{\omega}) = -\nu\,\mathrm{rot}\,\mathrm{rot}\,\vec{\omega} + \mathrm{rot}\,\vec{F}, \tag{5}$$

or in components,

$$\begin{aligned}&\partial\omega_m/\partial t - H_{m+1}^{-1}H_{m+2}^{-1}\Big[\frac{\partial}{\partial q_{m+1}}(H_{m+2}(v_m\omega_{m+1} - v_{m+1}\omega_m)) -\\ &-\frac{\partial}{\partial q_{m+2}}(H_{m+1}(v_{m+2}\omega_m - v_m\omega_{m+2}))\Big] = f_m - \nu H_{m+1}^{-1}H_{m+2}^{-1}\cdot\\ &\cdot\Big[\frac{\partial}{\partial q_{m+1}}\Big(\frac{H_{m+2}}{H_m H_{m+1}}\Big(\frac{\partial}{\partial q_m}(H_{m+1}\omega_{m+1}) - \frac{\partial}{\partial q_{m+1}}(H_m\omega_m)\Big)\Big) -\\ &-\frac{\partial}{\partial q_{m+2}}\Big(\frac{H_{m+1}}{H_m H_{m+2}}\Big(\frac{\partial}{\partial q_{m+2}}(H_m\omega_m) - \frac{\partial}{\partial q_m}(H_{m+2}\omega_{m+2})\Big)\Big)\Big],\end{aligned} \tag{6}$$

where $\vec{f} = \mathrm{rot}\,\vec{F}$.

In what follows, we consider the class of axially symmetric flows. This means we assume the existence of an index $k \in \{1,2,3\}$ such that the derivatives with respect to q_k, which appear in system (4), (6), vanish. Then, taking into account that

$$\omega_{k+1} = H_{k+2}^{-1}H_k^{-1}\partial(H_k v_k)/\partial q_{k+2}\,, \qquad \omega_{k+2} = -H_{k+1}^{-1}H_k^{-1}\partial(H_k v_k)/\partial q_{k+1}\,,$$

system (4) reduces to the equation

$$\begin{aligned}&\partial v_k/\partial t + v_{k+1}H_k^{-1}H_{k+1}^{-1}\partial(H_k v_k)/\partial q_{k+1} + v_{k+2}H_k^{-1}H_{k+2}^{-1}\partial(H_k v_k)/\partial q_{k+2} =\\ &= \frac{\nu}{H_{k+1}H_{k+2}}\Big[\frac{\partial}{\partial q_{k+1}}\Big(\frac{H_{k+2}}{H_k H_{k+1}}\frac{\partial}{\partial q_{k+1}}(H_k v_k)\Big) +\\ &+\frac{\partial}{\partial q_{k+2}}\Big(\frac{H_{k+1}}{H_k H_{k+2}}\frac{\partial}{\partial q_{k+2}}(H_k v_k)\Big)\Big] + F_k\,.\end{aligned} \tag{7}$$

The continuity equation has now the form

$$\frac{\partial}{\partial q_{k+1}}(H_{k+2}H_k v_{k+1}) + \frac{\partial}{\partial q_{k+2}}(H_k H_{k+1} v_{k+2}) = 0. \tag{8}$$

Hence, the stream function ψ can be defined by

$$v_{k+1} = \frac{1}{H_k H_{k+1}}\frac{\partial\psi}{\partial q_{k+2}}\,, \qquad v_{k+2} = -\frac{1}{H_k H_{k+1}}\frac{\partial\psi}{\partial q_{k+1}}\,. \tag{9}$$

If such a function exists, then (8) is satisfied automatically.

From the definition of the vorticity function ω_k we obtain the Poisson equation for the function ψ:

$$\frac{\partial}{\partial q_{k+1}}\Big(\frac{H_{k+2}}{H_k H_{k+1}}\frac{\partial\psi}{\partial q_{k+1}}\Big) + \frac{\partial}{\partial q_{k+2}}\Big(\frac{H_{k+1}}{H_k H_{k+2}}\frac{\partial\psi}{\partial q_{k+2}}\Big) = -H_{k+1}H_{k+2}\omega_k\,. \tag{10}$$

Equations (6) for the vorticity function ω_k become

$$\begin{aligned}
&\partial\omega_k/\partial t-(H_{k+1}H_{k+2})^{-1}\Big\{\frac{\partial}{\partial q_{k+1}}\Big[\frac{v_k}{H_k}\frac{\partial}{\partial q_{k+2}}(H_kv_k)-H_{k+2}v_{k+1}\omega_k\Big]-\\
&\qquad-\frac{\partial}{\partial q_{k+2}}\Big[H_{k+1}v_{k+2}\omega_k+\frac{v_k}{H_k}\frac{\partial}{\partial q_{k+1}}(H_kv_k)\Big]\Big\}=\\
&\qquad=f_k+\frac{\nu}{H_{k+1}H_{k+2}}\Big[\frac{\partial}{\partial q_{k+1}}\Big(\frac{H_{k+2}}{H_kH_{k+1}}\frac{\partial}{\partial q_{k+1}}(H_k\omega_k)\Big)+\\
&\qquad+\frac{\partial}{\partial q_{k+2}}\Big(\frac{H_{k+1}}{H_kH_{k+2}}\frac{\partial}{\partial q_{k+2}}(H_k\omega_k)\Big)\Big].
\end{aligned}\tag{11}$$

We see that equations (7) and (11) contain source terms of the form $a\omega_k$, bv_k where the functions a and b can change their signs. This causes difficulties in the derivation of monotone difference schemes, since the maximum principle ([4]) is not valid for such equations.

Using the transformation

$$u_k=\omega_kH_k^{-1},\qquad w_k=H_kv_k\tag{12}$$

and taking into account the relations (8) and

$$M_k\equiv\frac{\partial}{\partial q_{k+1}}\Big(\frac{H_{k+2}}{H_{k+1}}\frac{\partial H_k}{\partial q_{k+1}}\Big)+\frac{\partial}{\partial q_{k+2}}\Big(\frac{H_{k+1}}{H_{k+2}}\frac{\partial H_k}{\partial q_{k+2}}\Big)=0,\tag{13}$$

we transform (7) and (11) to the equations

$$\begin{aligned}
&H_{k+1}H_{k+2}\frac{\partial w_k}{\partial t}=\\
&=\nu H_k\Big[\frac{\partial}{\partial q_{k+1}}\Big(\frac{H_{k+2}}{H_kH_{k+1}}\frac{\partial w_k}{\partial q_{k+1}}\Big)+\frac{\partial}{\partial q_{k+2}}\Big(\frac{H_{k+1}}{H_kH_{k+2}}\frac{\partial w_k}{\partial q_{k+2}}\Big)\Big]-\\
&-H_{k+2}v_{k+1}\partial w_k/\partial q_{k+1}-H_{k+1}v_{k+2}\partial w_k/\partial q_{k+2}+H_{k+1}H_{k+2}H_kF_k
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
&H_{k+1}H_{k+2}\frac{\partial u_k}{\partial t}=\\
&=\nu H_k^{-3}\Big[\frac{\partial}{\partial q_{k+1}}\Big(H_k^3\frac{H_{k+2}}{H_{k+1}}\frac{\partial u_k}{\partial q_{k+1}}\Big)+\frac{\partial}{\partial q_{k+2}}\Big(H_k^3\frac{H_{k+1}}{H_{k+2}}\frac{\partial u_k}{\partial q_{k+2}}\Big)\Big]-\\
&-H_{k+2}v_{k+1}\partial u_k/\partial q_{k+1}-H_{k+1}v_{k+2}\partial u_k/\partial q_{k+2}+H_{k+1}H_{k+2}H_k^{-1}f_k-\\
&-H_k^{-4}\big(\partial H_k/\partial q_{k+1}\cdot\partial w_k^2/\partial q_{k+2}-\partial H_k/\partial q_{k+2}\cdot\partial w_k^2/\partial q_{k+1}\big).
\end{aligned}\tag{15}$$

We used the fact that (13) implies the identity

$$\begin{aligned}
&H_k^{-1}\Big[\frac{\partial}{\partial q_{k+1}}\Big(\frac{H_{k+2}}{H_kH_{k+1}}\frac{\partial(H_k^2u_k)}{\partial q_{k+1}}\Big)+\frac{\partial}{\partial q_{k+2}}\Big(\frac{H_{k+1}}{H_kH_{k+2}}\frac{\partial(H_k^2u_k)}{\partial q_{k+2}}\Big)\Big]=\\
&\qquad=H_k^{-3}\Big[\frac{\partial}{\partial q_{k+1}}\Big(H_k^3\frac{H_{k+2}}{H_{k+1}}\frac{\partial u_k}{\partial q_{k+1}}\Big)+\frac{\partial}{\partial q_{k+2}}\Big(H_k^3\frac{H_{k+1}}{H_{k+2}}\frac{\partial u_k}{\partial q_{k+2}}\Big)\Big].
\end{aligned}$$

Let us note that in any orthogonal coordinate system ([7]) there exists always at least one direction q_k for which $M_k = 0$ $(k = 3)$ and the equation for the normalized vorticity function u_k can be represented in form (15). The following special cases can be considered: (1) $M_k = 0$, if $H_k = \text{const}$, (2) $M_3 = 0$, if $H_1 = H_2$ and $\partial^2 H_3/\partial q_1^2 + \partial^2 H_3/\partial q_2^2 = 0$, (3) $M_1 \neq 0$, $M_2 \neq 0$, if H_1, H_2 depend only on q_1, q_2. For example, for rectangular and cylindrical coordinate systems $M_1 = M_2 = M_3 = 0$, and for spherical coordinates $M_1 = M_3 = 0$, $M_2 \neq 0$.

Equations (14), (15) do not contain source terms and are, therefore, suitable for obtaining difference schemes. We begin with the analysis of finite-difference schemes for the following system of ordinary differential equations:

$$\left\{ \begin{array}{l} (b_1 u')' + a_1 u' + c w' = f_1 \\ (b_2 w')' + a_2 w' = f_2, \end{array} \right. \tag{16}$$

where the functions $a_1, b_1, a_2, b_2, c, f_1, f_2$ depend on x,

$$b_1 > 0,\ b_2 > 0,\ u' \equiv \frac{du}{dx},\ u'' \equiv \frac{d^2u}{dx^2},\ \dots\ .$$

System (16) can be rewritten in the matrix form

$$(B\vec{u}')' + A\vec{u}' = \vec{f}, \tag{17}$$

where

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad A = \begin{pmatrix} a_1 & c \\ 0 & a_2 \end{pmatrix}$$

are 2×2 matrices, $\vec{u} = (u, w)$ and $\vec{f} = (f_1, f_2)$. Let us consider an irregular mesh formed by mesh points x_i and introduce the matrix function

$$w(x) = \exp(\int_{x_i}^{x} \alpha(t)\, dt), \qquad x \in (x_{i-1}, x_{i+1}),$$

where

$$\alpha = AB^{-1}.$$

Then equation (17) can be expressed in the selfadjoint form

$$w^{-1}(wB\vec{u}')' = \vec{f} \ \text{ or } \ (wB\vec{u}')' = w\vec{f} \ \text{ in } \ (x_{i-1}, x_{i+1}).$$

Integrating over $(x_{i-1/2}, x_{i+1/2})$ and using the rectangle formula, we obtain the vector equation of balance ([4])

$$\vec{\mathcal{J}}_{i+1/2} - \vec{\mathcal{J}}_{i-1/2} = \int_{x_{i-1/2}}^{x_{i+1/2}} w\vec{f}\, dx \approx \hbar_i \vec{f}_i, \tag{18}$$

where $\vec{\mathcal{F}}(x) = wB\vec{u}'$, $\vec{\mathcal{F}}_{i\pm1/2} = \vec{\mathcal{F}}(x_{i\pm1/2})$, $x_{i\pm1/2} = (x_i + x_{i\pm1/2})/2$, $\hbar_i = (h_i + h_{i+1})/2$, $h_{i+1} = x_{i+1} - x_i$, $h_i = x_i - x_{i-1}$. Since $\vec{u}' = B^{-1}w^{-1}\vec{\mathcal{F}}$, we have

$$\begin{aligned}
\vec{u}_{i+1} - \vec{u}_i &= \int_{x_i}^{x_{i+1}} B^{-1}w^{-1}\vec{\mathcal{F}}\,dx \approx B^{-1}_{i+1/2}\int_{x_i}^{x_{i+1}} w^{-1}\,dx \cdot \vec{\mathcal{F}}_{i+1/2} = \\
&= -B^{-1}_{i+1/2}\alpha^{-1}_{i+1/2}(\exp(-\alpha_{i+1/2}h_{i+1}) - E)\cdot\vec{\mathcal{F}}_{i+1/2}, \\
\vec{u}_i - \vec{u}_{i-1} &= \int_{x_{i-1}}^{x_i} B^{-1}w^{-1}\vec{\mathcal{F}}\,dx \approx B^{-1}_{i-1/2}\int_{x_{i-1}}^{x_i} w^{-1}\,dx \cdot \vec{\mathcal{F}}_{i-1/2} = \\
&= -B^{-1}_{i-1/2}\alpha^{-1}_{i-1/2}(E - \exp(\alpha_{i-1/2}h_i))\cdot\vec{\mathcal{F}}_{i-1/2}
\end{aligned}$$

and from (18) we get the vector finite-difference equations

$$\Lambda\vec{u}_i \equiv \tilde{B}_i(\vec{u}_{i+1} - \vec{u}_i) - \tilde{A}_i(\vec{u}_i - \vec{u}_{i-1}) = \vec{f}_i, \tag{19}$$

where

$$\begin{aligned}
\tilde{B}_i &= \hbar_i^{-1}h_{i+1}^{-1}s(-\alpha_{i+1/2}h_{i+1})B_{i+1/2}, \\
\tilde{A}_i &= \hbar_i^{-1}h_i^{-1}s(\alpha_{i-1/2}h_i)B_{i-1/2}, \\
s(z) &= z(\exp(z) - E)^{-1} = (\exp(z) - E)^{-1}z.
\end{aligned}$$

Here E is the identity matrix. $\alpha_{i\pm1/2}$, $B_{i\pm1/2}$ are the average values of the entries of the matrices α, B in intervals (x_{i-1}, x_i), (x_i, x_{i+1}) and $u_i = u(x_i)$.

Since the matrix function $s(z)$ associated with the matrix $z = \alpha h$ has nonnegative eigenvalues, we have $\tilde{B}_i > 0$, $\tilde{A}_i > 0$ and the corresponding difference scheme is monotone. Calculating the matrix function $s(z)$ on the spectrum of the matrix z ([10]), we get

$$\begin{aligned}
\tilde{B}_i &= \frac{1}{\hbar_i h_{i+1}}\begin{pmatrix} s(-\lambda_1^+)b_1^+ & c^+\frac{s(-\lambda_2^+)-s(-\lambda_1^+)}{\lambda_2^+-\lambda_1^+}h_{i+1} \\ 0 & s(-\lambda_2^+)b_2^+ \end{pmatrix}, \\
\tilde{A}_i &= \frac{1}{\hbar_i h_i}\begin{pmatrix} s(\lambda_1^-)b_1^- & c^-\frac{s(\lambda_2^-)-s(\lambda_1^-)}{\lambda_2^--\lambda_1^-}h_i \\ 0 & s(\lambda_2^-)b_2^-, \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&\lambda_k^+ = a_k^+h_{i+1}/b_k^+,\ \lambda_k^- = a_k^-h_i/b_k^-,\ b_k^\pm = (b_k)_{i\pm1/2}, \\
&a_k^\pm = (a_k)_{i\pm1/2},\ k = 1,2;\ c^\pm = c_{i\pm1/2},\ s(\lambda) = \lambda/(\exp(\lambda) - 1).
\end{aligned}$$

If we consider $\vec{f} = 0$ in (16), then the corresponding system (19) can also be obtained directly from (16) integrating this over segments (x_{i-1}, x_i), (x_i, x_{i+1}), assuming that the coefficients b_1, a_1, b_2, a_2, c have constant values b_j^-, a_j^-, c^- and b_j^+, a_j^+, c^+ $(j = 1, 2)$ on the intervals (x_{i-1}, x_i) and (x_i, x_{i+1}), respectively, and using the following boundary conditions

$$u(x_{i\pm1}) = u_{i\pm1},\ w(x_{i\pm1}) = w_{i\pm1},\ u_i = u(x_i^-) = u(x_i^+),$$
$$w(x_i^-) = w(x_i^+) = w_i,\ b_1^+u'(x_i^+) = b_1^-u'(x_i^-),$$
$$b_2^+w'(x_i^+) = b_2^-w'(x_i^-).$$

In the case of constant coefficients and regular meshes we have

$$\begin{aligned}\Lambda\vec{u}_i \equiv h^{-2}[s(-z)B(\vec{u}_{i+1} - \vec{u}_i \pm (\vec{u}_{i+1} + \vec{u}_{i-1})/2)-\\ -s(z)B(\vec{u}_i - \vec{u}_{i-1} \pm (\vec{u}_{i+1} + \vec{u}_{i-1})/2) =\\ = \gamma(z)B\vec{u}_{x\bar{x}_i} + A\vec{u}_{\dot{x}_i},\end{aligned} \tag{20}$$

where

$$z = AB^{-1}h = s(-z) - s(z), \quad \gamma(z) = (s(z) + s(-z))/2 = (z/2)cth(z/2).$$

Here γ is the so-called perturbation of the coefficient matrix and $u_{\dot{x}}$, $u_{x\bar{x}}$ are the central differences of the first and the second order ([4]).

By computing $\gamma(z)$ on the spectrum of the matrix z, we get ([10])

$$\gamma(z) = \begin{pmatrix} \gamma_1 & cb_2^{-1}h \vartriangle \gamma \\ 0 & \gamma_2 \end{pmatrix},$$

where $\gamma_1 = \gamma(\lambda_1)$, $\gamma_2 = \gamma(\lambda_2)$, $\vartriangle \gamma = (\gamma_2 - \gamma_1)/(\lambda_2 - \lambda_1)$ and $\lambda_1 = a_1h/b_1$, $\lambda_2 = a_2h/b_2$ are the eigenvalues of the matrix z.

Thus, the difference equations for the system of equations (16) on regular mesh have the form ([9])

$$\begin{cases} b_1\gamma_1u_{x\bar{x}} + a_1u_{\dot{x}} + cw_{\dot{x}} + ch \vartriangle \gamma w_{x\bar{x}} = f_1 \\ b_2\gamma_2w_{x\bar{x}} + a_2w_{\dot{x}} = f_2. \end{cases} \tag{21}$$

If $\lambda_1 \to \lambda_2$, then $\vartriangle \gamma \to \gamma'(\lambda)$. The advantage of the difference equations (21) with the perturbation coefficient $\vartriangle \gamma$ for large values of parameter c is shown in [9].

For the approximation of equations (14), (15) irregular mesh with points $(q_i, \tilde{q}_j)$ is used where $q \equiv q_{k+1}$, $\tilde{q} \equiv q_{k+2}$. Let $H \equiv H_{k+1}$, $\tilde{H} \equiv H_{k+2}$, $G \equiv H_k$, $v \equiv v_{k+1}$, $\tilde{v} \equiv v_{k+2}$, $u \equiv u_k$, $w \equiv w_k$, $f \equiv f_k$, $F \equiv F_k$. Then, using the results derived for the model equation (19), we obtain the following difference equations:

$$\begin{aligned}B_{i,j}^{(1)}(u_{i+1,j} - u_{i,j}) - A_{i,j}^{(1)}(u_{i,j} - u_{i-1,j}) + \tilde{B}_{i,j}^{(1)}(u_{i,j+1} - u_{i,j})-\\ -\tilde{A}_{i,j}^{(1)}(u_{i,j} - u_{i,j-1}) + D_{i,j}(w_{i+1,j} - w_{i,j}) - E_{i,j}(w_{i,j} - w_{i-1,j})+\\ +\tilde{D}_{i,j}(w_{i,j+1} - w_{i,j}) - \tilde{E}_{i,j}(w_{i,j} - w_{i,j-1}) = F_{i,j}^{(1)}\hbar_i\vartheta_j,\end{aligned} \tag{22}$$

$$
\begin{aligned}
(23) \qquad & B_{i,j}^{(2)}(w_{i+1,j} - w_{i,j}) - A_{i,j}^{(2)}(w_{i,j} - w_{i-1,j}) + \tilde{B}_{i,j}^{(2)}(w_{i,j+1} - w_{i,j}) - \\
& \quad - \tilde{A}_{i,j}^{(2)}(w_{i,j} - w_{i,j-1}) = F_{i,j}^{(2)} \hbar_i \bar{g}_j,
\end{aligned}
$$

where

$$
\begin{aligned}
B_{i,j}^{(1)} &= \nu \bar{g}_j \cdot h_{i+1}^{-1} (G_{i+1/2,j}/G_{i,j})^{\kappa} (\tilde{H}/H)_{i+1/2,j} s(\alpha_{i+1/2,j} \cdot h_{i+1}), \\
A_{i,j}^{(1)} &= \nu \bar{g}_j \cdot h_i^{-1} (G_{i-1/2,j}/G_{i,j})^{\kappa} (\tilde{H}/H)_{i-1/2,j} s(-\alpha_{i-1/2,j} \cdot h_i), \\
\tilde{B}_{i,j}^{(1)} &= \nu \hbar_i g_{j+1}^{-1} (G_{i,j+1/2}/G_{i,j})^{\kappa} (H/\tilde{H})_{i,j+1/2} s(\tilde{\alpha}_{i,j+1/2} \cdot g_{j+1}), \\
\tilde{A}_{i,j}^{(1)} &= \nu \hbar_i g_j^{-1} (G_{i,j-1/2}/G_{i,j})^{\kappa} (H/\tilde{H})_{i,j-1/2} s(-\tilde{\alpha}_{i,j-1/2} \cdot g_j), \\
D_{i,j} &= -2\bar{g}_j w_{i+1/2,j} G_{i,j}^{-3} (\partial \ln G/\partial \tilde{q})_{i+1/2,j} s'(\alpha_{i+1/2,j} \cdot h_{i+1}), \\
E_{i,j} &= 2\bar{g}_j w_{i-1/2,j} G_{i,j}^{-3} (\partial \ln G/\partial \tilde{q})_{i-1/2,j} s'(-\alpha_{i-1/2,j} \cdot h_i), \\
\tilde{D}_{i,j} &= 2\hbar_i w_{i,j+1/2} G_{i,j}^{-3} (\partial \ln G/\partial q)_{i,j+1/2} s'(\tilde{\alpha}_{i,j+1/2} \cdot g_{j+1}), \\
\tilde{E}_{i,j} &= -2\hbar_i w_{i,j-1/2} G_{i,j}^{-3} (\partial \ln G/\partial q)_{i,j-1/2} s'(-\tilde{\alpha}_{i,j-1/2} \cdot g_j), \\
F^{(1)} &\equiv H\tilde{H}(\partial u/\partial t - fG^{-1}), \\
F^{(2)} &\equiv \tilde{H}H(\partial w/\partial t - FG), \\
u_{i,j} &= u(q_i, \tilde{q}_j) \\
w_{i,j} &= w(q_i, \tilde{q}_j), \\
s'(\lambda) &= \frac{ds}{d\lambda},\ -s'(-\lambda) - s'(\lambda) = 1,\ -s'(-\lambda) + s'(\lambda) = 2\gamma'(\lambda), \\
\gamma(\lambda) &= (\lambda/2)cth(\lambda/2),\ \alpha = vH/\nu,\ \tilde{\alpha} = \tilde{v}\tilde{H}/\nu,\ h_{i+1} = q_{i+1} - q_i, \\
g_{j+1} &= \tilde{q}_{j+1} - \tilde{q}_j,\ \hbar_i = (h_i + h_{i+1})/2,\ \bar{g}_j = (g_j + g_{j+1})/2,\ \kappa = 3.
\end{aligned}
$$

Coefficients $B_{i,j}^{(2)}$, $A_{i,j}^{(2)}$, $\tilde{B}_{i,j}^{(2)}$, $\tilde{A}_{i,j}^{(2)}$ can be calculated analogously to $B_{i,j}^{(1)}$, $A_{i,j}^{(1)}$, $\tilde{B}_{i,j}^{(1)}$, $\tilde{A}_{i,j}^{(1)}$ with $\kappa = -1$.

Since the coefficients $D_{i,j}$, $E_{i,j}$, $\tilde{D}_{i,j}$, $\tilde{E}_{i,j}$ contain the unknown function w, an iterative method must be used for the solution of equations (22). The parameters with indices $i \pm 1/2$, $j \pm 1/2$ denote corresponding average values of mesh functions in the intervals (q_{i-1}, q_i), (q_i, q_{i+1}) and $(\tilde{q}_{j-1}, \tilde{q}_j)$, $(\tilde{q}_j, \tilde{q}_{j+1})$, respectively.

If $w_k = 0$ (or axially symmetric flow), the difference equations have form (22) with $D_{i,j} = E_{i,j} = \tilde{D}_{i,j} = \tilde{E}_{i,j} = 0$. In this case the model equation

$$
(bu')' + au' = f \tag{24}
$$

can be used with functions a, b, f which depend on argument x and $b > 0$.

The corresponding difference equations have the form

$$
\Lambda u_i \equiv \tilde{B}_i(u_{i+1} - u_i) - \tilde{A}_i(u_i - u_{i-1}) = f_i,
$$

where

$$
(25) \qquad \begin{aligned}
&B_i = \hbar_i^{-1} h_{i+1}^{-1} b_{i+1/2} s(-\alpha_{i+1/2} h_{i+1}) > 0,\\
&A_i = \hbar_i^{-1} h_i^{-1} b_{i-1/2} s(\alpha_{i-1/2} h_i) > 0,\\
&\alpha = ab^{-1}.
\end{aligned}
$$

In the case of constant coefficients and regular meshes we have

$$
\begin{gathered}
\Lambda u_i \equiv h^{-2}[s(-z)b(u_{i+1} - u_i \pm (u_{i+1} + u_{i-1})/2) -\\
-s(z)b(u_i - u_{i-1} \pm (u_{i+1} + u_{i-1})/2)] =\\
= bh^{-2}[(s(-z) + s(z))(u_{i+1} - 2u_i + u_{i-1})/2 + (s(-z) - s(z))(u_{i+1} - u_{i-1})/2] =\\
= \gamma(z) b u_{x\bar{x}_i} + a u_{\dot{x}_i},
\end{gathered}
$$

where $z = ah/b$, $s(-z) - s(z) = z$, $\gamma(z) = (s(z) + s(-z))/2 = (z/2)cth(z/2)$ (see [11]).

The Ilhyn difference scheme can also be used in the case of variable coefficients a, b, supposing, e.g., that $a = a(x_i)$, $b = b(x_i)$, $f = f(x_i)$, $z = z(x_i) = a(x_i)h/b(x_i)$, in the interval (x_{i-1}, x_{i+1}).

Since equation (10) for the stream function has a similar form as (14) (where we put $\partial w_k/\partial t = 0$, $v_{k+1} = v_{k+2} = 0$, $\nu = 1$, $F_k = \omega_k = u_k H_k$), the discretization of (10) again yields a finite-difference system of form (23) where we now set $s(0) = 1$.

If we consider the heat transfer equation

$$
(26) \qquad \partial T/\partial t + \operatorname{div}(\vec{v}\, T) = \operatorname{div}(\chi \operatorname{grad} T) + Q,
$$

then equation (14) can be written as

$$
(27) \qquad \begin{aligned}
&H_{k+1}H_{k+2}(\partial T/\partial t - Q) + h_{k+2} v_{k+1} \frac{\partial T}{\partial q_{k+1}} + H_{k+1} v_{k+2} \frac{\partial T}{\partial q_{k+2}} =\\
&= \frac{\chi}{H_k}\left[\frac{\partial T}{\partial q_{k+1}}\left(H_k \frac{H_{k+2}}{H_{k+1}} \frac{\partial T}{\partial q_{k+1}}\right) + \frac{\partial T}{\partial q_{k+2}}\left(H_k \frac{H_{k+1}}{H_{k+2}} \frac{\partial T}{\partial q_{k+2}}\right)\right],
\end{aligned}
$$

where χ, Q and T denote the coefficient of heat capacity, the density of heat sources and the temperature, respectively. This means that the finite-difference equations have again the form (22), where we put $\kappa = 1$, $F^{(1)} = H\tilde{H}(\partial T/\partial t - Q)$, $D_{i,j} = E_{i,j} = \tilde{D}_{i,j} = \tilde{E}_{i,j} = 0$ and substitute $T_{i,j}$ and χ for $u_{i,j}$ and ν, respectively.

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