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Monotonic valuations of $\pi\sigma$-triads and evaluations of ideals

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Abstract. We develop problems of monotonic valuations of triads. A theorem on monotonic valuations of triads of the type $\pi\sigma$ is presented. We study, using the notion of the monotonic valuation, representations of ideals by monotone and subadditive mappings. We prove, for example, that there exists, for each ideal J of the type π on a set A, a monotone and subadditive set-mapping h on $P(A)$ with values in non-negative rational numbers such that $J = h^{-1}{}''\{r \in Q;\, r \geq 0 \ \&\ r \doteq 0\}$. Some analogical results are proved for ideals of the types σ, $\sigma\pi$ and $\pi\sigma$, too. A problem of an additive representation is also discussed.

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We develop problems of monotonic valuations of triads of the type $\pi\sigma$ by proving a theorem on such valuations of so called limit $\pi\sigma$-triads. This theorem completes a list of theorems on monotonic valuations of triads of the type $\sigma\pi$ and $\pi\sigma$. (See [M3].) Moreover, we study, using the general theorems on valuations, representations of ideals on sets by monotone and subadditive mappings. We prove, for example, that there exists, for each ideal J of the type π on a set A, a monotone and subadditive set-mapping h on $P(A)$ with values in non-negative rational numbers Q^+ such that $J = h^{-1}{}''\{r \in Q^+;\ r \doteq 0\}$. Some analogical results are proved for ideals of the type σ, $\sigma\pi$ and $\pi\sigma$, too. We discuss also the existence of such additive representations; it means that the mapping in question is additive on A.

We work in the alternative set theory; we shall use the usual notations of this theory. Recall, that small Latin letters range over sets and i, j, k, l, m, n range over finite natural numbers. By a collection we mean a collection of classes which satisfies a given formula of the language FL_V. The collection of all set-theoretically definable classes is denoted by Sd_V; it is a codable system. We say that a class X is of the type π- (σ- resp.) if there exists a set-theoretically definable relation R such that $X = \bigcap_n R''\{n\}$ ($X = \bigcup_n R''\{n\}$ resp.). Let E be a π-equivalence on a set-definable class A. We say that a class $X \subseteq A$ is E-closed in A if we have the following: $(\forall a \in A - X)(\exists U \in Sd_V)(U \cap X = \emptyset \ \&\ E''\{a\} \subseteq U)$.

Monotonic valuations of $\pi\sigma$-triads

Monotonic valuations of triads.

We recall briefly some notions about triads (see [M2]). We say that a structure $\langle A, F, E\rangle$ is an *e-structure* if we have the following:

(1) $\langle A, F\rangle$ is a semigroup (i.e. F is an associative operation on A),

(2) $E \circ E$ is the identity on A,
(3) we have either $F(E(x), E(y)) = E(F(x, y))$ for each x, $y \in A$ or $F(E(x), E(y)) = E(F(y, x))$ for each x, $y \in A$.

Let $\langle A, F, E\rangle$ be an e-structure. We define a *canonical relation* $\lhd_A$ *of an e-structure* $\langle A, F, E\rangle$ on A by:

$$x \lhd_A y \Leftrightarrow (\exists z \in A)(F(x, z) = y).$$

Let $\langle A, F, E\rangle$ be an e-structure. We can see that

(1) E is a one-one mapping on A and, thus, E is an automorphism or an anti-automorphism of the semigroup $\langle A, F\rangle$.
(2) $\lhd_A$ is a transitive relation on A.

Assume that $\mathbb{A}, \hat{\mathbb{A}}$ are two e-structures. A mapping $H : A \to \hat{A}$ is said to be a *valuation of* $\mathbb{A}$ *in* $\hat{\mathbb{A}}$ if we have:

(a) $H(F(x, y)) \lhd_{\hat{A}} \hat{F}(H(x), H(y))$ holds for each $x, y \in A$,
(b) $H(F(x)) = \hat{F}(H(x))$ holds for each $x \in A$.

Let $\mathbb{A}$ be an e-structure. Then the triple $\langle \mathbb{A}, \mathbb{A}\restriction U, \mathbb{A}\restriction B\rangle$, where $B \subseteq U \subseteq A$ and $\mathbb{A}\restriction B$, $\mathbb{A}\restriction U$ are substructures of $\mathbb{A}$, is said to be a *triad over the e-structure* $\mathbb{A}$. We denote it as

$$\mathbb{A}(U, B).$$

A mapping H is called a *valuation of the triad* $\mathbb{A}(U, B)$ *in a triad* $\langle \hat{A}, \hat{F}, \hat{E}\rangle(\hat{U}, \hat{B})$, if H is a valuation of $\langle A, F, E\rangle$ in $\langle \hat{A}, \hat{F}, \hat{E}\rangle$ and we have, moreover,

(c) $H^{-1''}\hat{U} = U$, $H^{-1''}\hat{B} = B$.

Let us recall the notion of monotonic valuations (see [M3]).

A valuation H of an e-structure $\mathbb{A}$ in $\hat{\mathbb{A}}$ is called a *monotonic valuation* if we have

$$(x, y \in A \ \& \ x \lhd_A y) \Rightarrow (H(x) \lhd_{\hat{A}} H(y)).$$

By a *monotonic valuation of a triad* $\mathcal{T}$ in a triad $\hat{\mathcal{T}}$ we mean such a valuation of $\mathcal{T}$ in $\hat{\mathcal{T}}$ which is a monotonic valuation of the relevant e-structures.

Our intention is to present a theorem on monotonic valuations of a closed (w.r.t. $\lhd$) triad $\langle A, F, E\rangle(U, B)$ in some canonical one under assumption that $\langle A, F, E\rangle$ and B belong to Sd_V and U is a $\pi\sigma$-class. We refer to this situation as to a problem of monotonic valuations of $\pi\sigma$-triads. We shall solve a more general problem assuming that the classes $\langle A, F, E\rangle$ and B belong to a so called saturated standard universe (see below). Note that a similar problem of monotonic valuations of π- , σ- and $\sigma\pi$- triads is solved in [M3].

Now, let us define *canonical τ-triads* $\mathcal{T}_\tau$, where τ is the symbol σ, π, $\sigma\pi$ or $\pi\sigma$. Put, at first,

$$[0]^+ = \{r \in Q^+;\ r \doteq 0\}.$$

We define

$$\mathcal{T}_\sigma = \langle N, +, Id\rangle(FN, \{0\}), \quad \mathcal{T}_\pi = \langle Q^+, +, Id\rangle([0]^+, \{0\}).$$

The triad $\mathcal{T}_{\sigma\pi}$ is defined as follows: Let ζ be fixed, $\zeta \in N - FN$. We define, on ${}^{\zeta}Q^{+}$, the mapping $+$ by the relation: $(f+g)(\alpha) = f(\alpha)+g(\alpha)$. Then $\langle {}^{\zeta}Q^{+}, +, Id\rangle$ is an e-structure and the canonical relation of this structure is the ordering $\leq$ defined by $f \leq g \Leftrightarrow (\forall \alpha \in \zeta)(f(\alpha) \leq g(\alpha))$. Now, $\langle {}^{\zeta}N, +, Id\rangle$ is an e-structure, too. Put

$$0_{\zeta} = \zeta \times \{0\}, \quad U(\sigma\pi) = \{f \in {}^{\zeta}N;\ (\exists n)(\forall m) f(m) < n\}.$$

We can see that $U(\sigma\pi)$ is a $\sigma\pi(Sd_V)$-class and $\langle {}^{\zeta}N, +, Id\rangle(U(\sigma\pi), \{0_{\zeta}\})$ is a triad. We define

$$\mathcal{T}_{\sigma\pi} = \langle {}^{\zeta}N, +, Id\rangle(U(\sigma\pi), \{0_{\zeta}\}).$$

Finally, let us define a *canonical $\pi\sigma$-triad* $\mathcal{T}_{\pi\sigma}$. Let $\zeta \in N - FN$ be fixed as above. We put

$$U(\pi\sigma) = \{f \in {}^{\zeta}Q^{+};\ (\forall \gamma \in \zeta - FN) f(\gamma) \doteq 0\}.$$

Then we define

$$\mathcal{T}_{\pi\sigma} = \langle {}^{\zeta}Q^{+}, +, Id\rangle(U(\pi\sigma), \{0_{\zeta}\}).$$

Before we formulate a theorem on monotonic valuations, let us give some useful notions. We say that an *e-structure* $\langle A, F, E\rangle$ *is commutative*, whenever F is a commutative operation on A.

Assume $U \subseteq A$ and let $\mathbb{A} = \langle A, F, E\rangle$ be an e-structure. The class U is said to be *closed in* $\mathbb{A}$ if U is closed under the canonical relation, i.e. if we have

$$(\forall x \in A)(\forall y \in U)(x \lhd_A y \Rightarrow x \in U).$$

We say that a triad $\mathbb{A}(U, B)$ is *closed* if U and B are closed in $\mathbb{A}$.

A structure $\langle A, F, E, G\rangle$ is called a *u-expansion of an e-structure* $\langle A, F, E\rangle$, if G is a binary function and if we have:

(1) $x \lhd_A y \Rightarrow G(y, x) = x$,
(2) $G(x, y) \lhd_A x$.

We say that an *e-structure* $\mathbb{A}$ *has a u-expansion*, if there exists a u-expansion $\langle \mathbb{A}, G\rangle$ of the e-structure $\mathbb{A}$.

Example. The structure $\langle P(a), \cup, Id, \cap\rangle$ is a u-expansion of the e-structure $\langle P(a), \cup, Id\rangle$.

It is not difficult to prove that every triad $\mathcal{T}_{\tau}$,where τ is $\sigma, \pi, \sigma\pi$ or $\pi\sigma$, is commutative and closed. The commutativity is clear. Let $U \in Sd_V$ be a relation with $dom(U) = \zeta^2$ such that

$$U(\alpha, \beta) = \{f \in {}^{\zeta}Q^{+};\ (\forall \gamma \in [\beta, \zeta)) f(\gamma) \leq 2^{-\alpha}\}.$$

We have, for $\alpha+1, \beta+1 \in \zeta$, $U(\alpha, \beta+1) \supseteq U(\alpha, \beta) \supseteq U(\alpha+1, \beta)$. Put $U_m = \bigcup_n U(m, n)$. Then we have $U(\pi\sigma) = \bigcap_m U_m$, $U_{m+1} \subseteq U_m$, $U_{m+1} + U_{m+1} \subseteq U_m$, $(g \in U(\alpha, \beta)\ \ \&\ \ f \leq g\ \ \&\ \ f \in {}^{\zeta}Q^{+}) \Rightarrow f \in U(\alpha, \beta)$. We deduce from this that the triad $\mathcal{T}_{\pi\sigma}$ is a closed $\pi\sigma(Sd_V)$-triad.

We can show quite analogously that all remaining canonical triads are closed, too.

Monotonic valuation of $\pi\sigma$-triads.

We study our problem of valuations of triads with respect to a so called standard universe of classes. It means that the relevant e-structures belong to this universe and we are looking for valuations from this universe, too. Note that the system Sd_V is such a special collection.

We say that a collection of classes is a *universe of classes* if it is closed under the definitions by normal formulas of the language FL_V with class-parameters from this collection. Thus, having a universe $\mathfrak{U}$ of classes and a normal formula $\varphi(x, X_1, X_2, \ldots, X_k)$ of the language FL_V such that the classes $X_1, X_2, \ldots, X_k$ belong to $\mathfrak{U}$, we see that the class $\{x; ; \varphi(x, X_1, X_2, \ldots, X_k)\}$ belongs to $\mathfrak{U}$, too. Note that every universe of classes contains all sets. More generally, every set-theoretically definable class belongs to each universe of classes. By a *standard universe of classes* we call each universe of classes which contains only such non-empty subclasses of the class of natural numbers which have the first element. We can see that the following proposition holds (see [M1]).

Proposition. *Every standard universe of classes contains only revealed classes and does not contain any proper semiset. It satisfies all axioms of Gödel-Bernays's theory of finite sets.*

A standard universe $\mathfrak{U}$ of classes is said to be a *saturated standard universe of classes* if we have the following: Let $\{X_n\}_{n\in FN}$ be a sequence of classes of this universe. Then there exists a relation R from $\mathfrak{U}$ such that

$$(\forall n) R''\{n\} = X_n.$$

Example. The system Sd_V is a standard universe of classes which is not a standard saturated universe of classes. Its revealment Sd_V^* is a standard saturated universe of classes.

By a π- (σ- resp.) *string* we mean a relation R such that $dom(R) = FN$ and, for each $n \in FN$, $R''\{n+1\} \subseteq R''\{n\}$ ($R''\{n\} \subseteq R''\{n+1\}$ resp.) holds. Assume that $\mathfrak{S}$ is a standard universe of classes. A class $\bigcap_n X_n$, where $\{X_n\}_{n\in FN} \subseteq \mathfrak{S}$, is called a $\pi(\mathfrak{S})$-*class* and a class of the form $\bigcup_n X_n$, where $\{X_n\}_{n\in FN} \subseteq \mathfrak{S}$, is called a $\sigma(\mathfrak{S})$-*class*.

Let $\mathbb{A} = \langle A, F, E\rangle$ be an e-structure, $\mathbb{A} \in \mathfrak{S}$. We say that a class U is a *limit $\pi\sigma(\mathfrak{S})$-universe in* $\mathbb{A}$ if there exists a non-increasing sequence $\{U_n\}_{n\in FN}$ of $\sigma(\mathfrak{S})$-classes such that $U = \bigcap_m U_m$ and $F''U^2_{m+1} \subseteq U_m$, $E''U_{m+1} \subseteq U_m$ hold. It is a *limit closed $\pi\sigma(\mathfrak{S})$-universe in* $\mathbb{A}$ if we have, moreover, $\lhd\, ''U_{m+1} \subseteq U_m$. A triad $\mathbb{A}(U, B)$ is said to be a *limit $\pi\sigma(\mathfrak{S})$-triad* (a *limit closed $\pi\sigma(\mathfrak{S})$-triad* resp.) if $\mathbb{A}, B \in \mathfrak{S}$ and U is a limit $\pi\sigma(\mathfrak{S})$-universe in $\mathbb{A}$ (a limit closed $\pi\sigma(\mathfrak{S})$-universe in $\mathbb{A}$ resp.). Thus every limit closed $\pi\sigma(\mathfrak{S})$-triad is a closed triad.

It is not difficult to prove that $\mathcal{T}_{\pi\sigma}$ is a commutative limit closed $\pi\sigma(Sd_V)$-triad.

Theorem. *Let $\mathfrak{S}$ be a saturated standard universe of classes. Let $\mathbb{A}(U, B)$ be a closed limit $\pi\sigma(\mathfrak{S})$-triad such that $\mathbb{A}$ is commutative and has a u-expansion in $\mathfrak{S}$.*

Then there exists a monotonic valuation of the triad $\mathbb{A}(U,B)$ in $\mathcal{T}_{\pi\sigma}$ which belongs to $\mathfrak{S}$.

PROOF: Writing $[F,E,\ \lhd\ ](X,Y)$ we mean that $F''X^2 \subseteq Y$, $E''X \subseteq Y$, $\lhd\,''X \subseteq Y$ hold. We define a mapping $F_3 : A^3 \to A$ by $F_3(x,y,z) = F(F(x,y),z)$. By a matrix we mean a relation M such that $dom(M) = \xi^2$ for some $\xi \in N - FN$. Let us use the following notation: $\mathcal{H}_{\pi\sigma}(M) = \bigcap_m \bigcup_n M(m,n)$. We deduce from the proposition in 2.1.2, [M3], that there exists a matrix $T \in \mathfrak{S}$ such that (1) $B \subseteq T(\alpha,\beta) \subseteq A$ holds for each $\alpha,\beta \in dom(T)$, (2) $\mathcal{H}_{\pi\sigma}(T) = U$ and (3) for each $m \in FN$, $[F,E,\ \lhd\ ](\bigcup_n T(m+1,n), \bigcup_n T(m,n))$. We can construct as in [M3, 2.2.3] a matrix $R \in \mathfrak{S}$ such that (1), (2) hold for R instead of T and $[F,E,\ \lhd\ ](R(\alpha+1,\beta), R(\alpha,\beta))$ is satisfied for each $\alpha+1,\beta \in \xi$, where $\xi = dom(R)$. Let $2\theta \leq \xi$, $\theta \notin FN$. Let $M \in \mathfrak{S}$ be such a matrix that we have $dom(M) = \theta^2$, $M(0,\beta) = B$, $M(\theta-1,\beta) = A$ for each $\beta \in \theta$ and $M(\alpha,\beta) = P(2\alpha,\beta)$ for each $\alpha,\beta \leq \theta$ where $P(\gamma,\beta) = \trianglelefteq''(R(\gamma,\beta) \cap E''R(\gamma,\beta))$. We can see similarly as in [M3, 1.1.0] that $F''M^2(\alpha+1,\beta) \subseteq M(\alpha,\beta)$ holds for each $\alpha+1,\beta \in \theta$, $E''M(\alpha,\beta) \subseteq M(\alpha,\beta)$ holds for each $\alpha,\beta \subseteq \theta$ and $\trianglelefteq''\ M^2(\alpha+1,\beta) \subseteq M(\alpha,\beta)$. We have, moreover, $R(2\alpha,\beta+1) \subseteq M(\alpha,\beta+1) \subseteq R(2\alpha,\beta)$. We deduce from this that $\mathcal{H}_{\pi\sigma}(M) = \mathcal{H}_{\pi\sigma}(R) = U$. We have, for $\gamma+1,\beta \in \theta$, $F''P^2(\gamma+1,\beta) \subseteq P(\gamma,\beta)$ and, consequently, for $\gamma+2,\beta \in \theta$ holds the following: $F_3''P^3(\gamma+2,\beta) \subseteq F''(F''P^2(\gamma+2,\beta))^2 \subseteq F''P^2(\gamma+1,\beta) \subseteq P(\gamma,\beta)$. Thus $F_3''M^3(\alpha+1,\beta) = F_3''P^3(2(\alpha+1),\beta) \subseteq P(2\alpha,\beta) = M(\alpha,\beta)$ holds for each $\alpha+1,\beta \in \theta$.

Put, for each $\beta \in \theta$,

$$S(\beta) = \{\langle \alpha, x\rangle;\ x \in M(\alpha,\beta)\}.$$

Then each $S(\beta)$ has the following properties:

(a) $S(\beta) \in \mathfrak{S}$,
(b) $\alpha+1 \in \theta \Rightarrow [F,F_3](S(\beta)(\alpha+1), S(\beta)(\alpha))$ ($[F,F_3](\dots)$ has a similar meaning as in the previous proof),
(c) $\alpha \in \theta \Rightarrow E''S(\beta)(\alpha) \subseteq S(\beta)(\alpha)$,
(d) $\alpha \in \theta \Rightarrow \trianglelefteq''\ (E''S(\beta)(\alpha)) \subseteq S(\beta)(\alpha)$.

It is not difficult to see that we can assume that $dom(S) = \zeta^2$ with some $\zeta \in N - FN$ and that, for each $\beta \in \zeta$, $S(\beta)(0) = A$, $S(\beta)(\zeta-1) = B$ hold. Such $S(\beta)$ is called a monotonic $\pi^{\mathfrak{S}}$-string in $\mathbb{A}$ over B. We can see, similarly as in the proof of the theorem on monotonic valuations of $\sigma^{\mathfrak{M}}$- and $\pi^{\mathfrak{M}}$-triads in [M3, p. 383-384], that there exists a normal formula $\Psi(x,y,X,Y)$ of the language FL_V such that the following holds:

Let $\mathbb{A}(B,B) \in \mathfrak{S}$ be a triad and let $D \in \mathfrak{S}$ be a monotonic $\pi^{\mathfrak{S}}$-string in $\mathbb{A}$ over B. Then $H = \{\langle x,y\rangle;\ \Psi(x,y,\mathbb{A},D)\}$ is a monotonic valuation of $\mathbb{A}(B,B)$ in $\langle Q^+,+,Id\rangle(\{0\},\{0\})$ and $D(\alpha+1) \subseteq \{x \in A;\ H(x) \leq 2^{-\alpha}\} \subseteq D(\alpha)$ holds for each $\alpha \in dom(D)$.

Let

$$W = \{\langle \beta, \langle x,y\rangle\rangle;\ \Psi(x,y,\mathbb{A},S(\beta))\ \ \&\ \ \beta \in \zeta\}.$$

Put $W_\beta = W''\{\beta\}$. Then W_β is a monotonic valuation of $\mathbb{A}(B,B)$ in $\langle Q^+, +, Id\rangle(\{0\},\{0\})$ and $W_\beta \in \mathfrak{S}$. We have

$$x \in U \Leftrightarrow (\forall \beta \in \zeta - FN)(x \in \bigcap_m M(m,\beta)) \Leftrightarrow (\forall \beta \in \zeta - FN) W_\beta \doteq 0.$$

Let $H : A \to {}^{\zeta}Q^+$ be defined by

$$H(x) = \{\langle \alpha, W_\alpha(x)\rangle;\ \alpha \in \zeta\}.$$

We have, for each $x, y \in A$, $H(F(x,y))(\alpha) = W_\alpha(F(x,y)) \leq W_\alpha(x) + W_\alpha(y)$. Thus $H(F(x,y)) \leq H(x) + H(y)$ holds. We can see similarly that $H(E(x)) = H(x)$ and $x \lhd_A y \Rightarrow H(x) \leq H(y)$ hold, too. Thus H is a monotonic valuation of $\mathbb{A}$ in $\langle {}^{\zeta}Q^+, +, Id\rangle$. It is easy to see that $H(x) = \zeta \times \{0\}$ iff $x \in B$. Finally, we have $x \in U \Leftrightarrow (\forall \beta \in \zeta - FN)(W_\beta \doteq 0) \Leftrightarrow (\forall \beta \in \zeta - FN) H(x)(\beta) \doteq 0 \Leftrightarrow H(x) \in U_{\pi\sigma}$, which completes the proof. □

Remark. Let $\mathbb{A} = \langle A, F, E\rangle$ be an e-structure. We say that a class $U \subseteq A$ is a *limit $\pi\sigma^{\mathfrak{S}}$-universe in* $\mathbb{A}$ if there exists a matrix $M \in \mathfrak{S}$ such that

(1) $\mathcal{H}_{\pi\sigma}(M) = U$,
(2) $(\forall m \in FN)([F,E](\bigcup_n M(m+1,n), \bigcup_n M(m,n)))$.

U is a *limit closed $\pi\sigma^{\mathfrak{S}}$-universe in* $\mathbb{A}$ if we have, moreover,

$$\lhd '' \bigcup_n M(m+1) \subseteq \bigcup_n M(m,n).$$

A triad $\mathbb{A}(U,B)$ is said to be a *limit $\pi\sigma^{\mathfrak{S}}$-triad* (a *limit closed $\pi\sigma^{\mathfrak{S}}$-triad* resp.) if $\mathbb{A}, B \in \mathfrak{S}$ and U is a limit $\pi\sigma^{\mathfrak{S}}$-universe in $\mathbb{A}$ (a limit closed $\pi\sigma^{\mathfrak{S}}$-universe in $\mathbb{A}$ resp.). Thus every limit closed $\pi\sigma^{\mathfrak{S}}$-triad is a closed triad. The triad $\mathcal{T}_{\pi\sigma}$ is a limit closed $\pi\sigma^{\mathfrak{S}}$-triad.

We can see that the last proof guarantees that if we assume, in the last theorem, that $\mathfrak{S}$ is only a standard universe and that the triad in question is a limit closed $\pi\sigma^{\mathfrak{S}}$-triad, we obtain a true proposition.

Proposition. *There exists a $\pi\sigma(Sd_V)$-triad $\mathbb{A}(U,B)$ (i.e. $\mathbb{A} \in Sd_V$, $B \in Sd_V$ and U is a $\pi\sigma$-class) which is not a limit $\pi\sigma(Sd_V)$-triad.*

Proof: Let $(\mathcal{E})$ be an equivalence on N defined by

$$\langle \alpha, \beta\rangle \in (\mathcal{E}) \Leftrightarrow (\exists n)(\alpha, \beta < n) \vee (\forall n)(\alpha, \beta > n).$$

Let $\langle N^2 \cup \{\emptyset\}, F, E\rangle$ be the e-structure defined by the following relations:

$$\begin{aligned} F(\langle x,y\rangle, \langle \widetilde{y}, z\rangle) &= \langle x, z\rangle \Leftrightarrow y = \widetilde{y} \\ &= \emptyset \Leftrightarrow y \neq \widetilde{y}, \\ F(u,\emptyset) = F(\emptyset, u) &= \emptyset \Leftrightarrow u \in A. \end{aligned}$$

The function $E : N^2 \cup \{\emptyset\} \to N^2 \cup \{\emptyset\}$ is defined by $E(\langle x,y\rangle) = \langle y,x\rangle$ for each $\langle x,y\rangle$ and $E(\emptyset) = \emptyset$.

Then $\langle N^2 \cup \{\emptyset\}, F, E\rangle((\mathcal{E}) \cup \{\emptyset\}, Id\restriction N^2 \cup \{\emptyset\})$ is a $\pi\sigma(Sd_V)$-triad which is not a limit $\pi\sigma(Sd_V)$-one.

Indeed, assume, contrariwise, that it is. Then there exists its valuation $H \in Sd_V^*$ in $\mathcal{T}_{\pi\sigma}$ (where Sd_V^* is a revealment of Sd_V). This follows from 2.1.2 and 3.0.4 in [M3]. Put $D = H\restriction N^2$. We have $(\mathcal{E}) = \bigcap\{D^{-1}{''}U(m,\beta);\ m \in FN\ \ \&\ \ \beta \in \zeta - FN\}$, where $U(\alpha,\beta)$ are as above. Put, for $\alpha,\beta < \zeta$, $W(\alpha,\beta) = D^{-1}{''}U(\alpha,\beta)$. We can see that $W(m+1,\beta) \circ W(m+1,\beta) \subseteq W(m,\beta)$. Let $\langle a,b\rangle \in N^2 - (\mathcal{E})$, $a \in FN$, $b \in N - FN$. Then there exist $m \in FN$ and $\beta \in \zeta - FN$ such that $\langle a,b\rangle \notin W(m,\beta)$. Thus $W(m+1,\beta){''}\{a\} \cap W(m+1,\beta){''}\{b\} = \emptyset$. Put $A = W(m+1,\beta){''}\{a\}$ and $B = W(m+1,\beta){''}\{b\}$. We have $FN = (\mathcal{E}){''}\{a\} \subseteq A$, $N - FN = (\mathcal{E}){''}\{b\} \subseteq B$ and, moreover, $A \cap B = \emptyset$, $A \in Sd_V^*$, $B \in Sd_V^*$. The class A is a fully revealed class and $A \cap (N - FN) = \emptyset$, which is impossible. $\square$

Evaluations of ideals

Evaluations of ideals of the type $\sigma\pi$ and $\pi\sigma$.

Throughout this section, let A be a non-empty set and let $\zeta \in N - FN$ be fixed.

We say that J is an *ideal on A* if we have: $J \subseteq P(A)$, $A \notin J$, $u \in J\ \&\ v \in J \Rightarrow u \cup v \in J$ and $v \subseteq u \in J \Rightarrow v \in J$.

Let $H : P(A) \to {}^{\zeta}Q^+$ be a mapping. We say that H is *monotone on $P(A)$* if $u \subseteq v \Rightarrow H(u) \leq H(v)$ holds for each $u, v \subseteq A$. The mapping H is said to be *subadditive on $P(A)$* if we have, for each $u, v \subseteq A$, $H(u \cup v) \leq H(u) + H(v)$. We say that H is an *evaluation on $P(A)$ in ${}^{\zeta}Q^+$* if it is a monotone and subadditive mapping on $P(A)$ and $H^{-1}{''}\{0_\zeta\} = \{\emptyset\}$. (Recall that $0_\zeta = \zeta \times \{0\}$.)

The presented definitions can be naturally applied to a mapping $H : P(A) \to K$, where K is N or Q^+. (We identify K with the subclass $\{\zeta \times \{x\};\ x \in K\}$ of ${}^{\zeta}Q^+$.)

Theorem. *Let J be an ideal on a non-empty set A.*

(1) *Let J be a σ-class. Then there exists a set-evaluation h on $P(A)$ in N such that $h^{-1}{''}FN = J$.*
(2) *Let J be a π-class. Then there exists a set-evaluation h on $P(A)$ in Q^+ such that $h^{-1}{''}[0]^+ = J$.*
(3) *Let ζ be fixed. Let J be a $\sigma\pi$-class. Then there exists a set-evaluation h on $P(A)$ in ${}^{\zeta}Q^+$ such that $h^{-1}{''}U(\sigma\pi) = J$.*

Proof: Let J be an ideal on A. We see that $\langle P(A), \cup, Id\rangle(J, \{\emptyset\})$ is a closed triad and $\langle P(A), \cup, Id\rangle$ is commutative. Moreover, $\langle P(A), \cup, Id, \cap\rangle$ is a u-expansion of the e-structure in question. We can find, for τ equal to π, σ or $\sigma\pi$, a monotonic set-valuation h of the triad $\langle P(A), \cup, Id\rangle(J, \{\emptyset\})$ in the canonical τ-triad $\mathcal{T}_\tau$.

The existence of the mapping h is guaranteed by the following proposition on monotonic set-valuations of π-, σ- and $\sigma\pi$-triads, which follows easily from the theorems on monotonic valuations in [M3].

Let $\mathbb{A}(U, B)$ be a triad such that $\mathbb{A}$, B are sets and let $\mathbb{A}$ be commutative and have a u-expansion which is a set. If U is a τ-class, where τ is π, σ or $\sigma\pi$, then $\mathbb{A}(U, B)$ has a monotonic set-valuation in $\mathcal{T}_\tau$.

Now, let h be a monotonic valuation of the triad $\langle P(A), \cup, Id\rangle(J, \{\emptyset\})$ in $\mathcal{T}_\tau$, where τ is σ, π or $\sigma\pi$. We see that h is an evaluation on $P(A)$ in K_τ, where $K_\sigma = N$, $K_\pi = Q^+$ and $K_{\sigma\pi} = {}^\zeta Q^+$. The required equalities from the items (1), (2) and (3) clearly hold. $\square$

Now, we shall formulate a theorem on evaluations of $\pi\sigma$-ideals.

Theorem. *Let J be an ideal on a nonempty set A and let J be a $\pi\sigma$-class. Then there exists a set-evaluation h on $P(A)$ in ${}^\zeta Q^+$ such that $h^{-1\prime\prime}U(\pi\sigma) = J$ iff there exists a non-increasing sequence $\{J_n\}_{n\in FN}$ of σ-classes such that we have $\bigcap_m J_m = J$, $\{u \cup v;\ u, v \in J_{m+1}\} \subseteq J_m$ and $v \subseteq u \in J_{m+1} \Rightarrow v \in J_m$ hold for each m.*

Proof: We deduce quite analogously as in the previous proof, by using the theorem on monotonic valuations of $\pi\sigma$-triads, that there exists a monotone valuation $H \in Sd^*_V$ of $\langle P(A), \cup, Id\rangle(J, \{\emptyset\})$ in $\mathcal{T}_{\pi\sigma}$. The mapping H is necessarily a set and $h = H$ has the required properties.

Let us prove the implication from the left to the right. Let U_m be, for each $m \in FN$, as above. Assume that $h : P(A) \to {}^\zeta Q^+$ is such that $h^{-1\prime\prime}\{0_\zeta\} = \{\emptyset\}$ and $h^{-1\prime\prime}U(\pi\sigma) = J$. Put $J_m = h^{-1\prime\prime}U_m$. Then the classes J_m have the required properties. $\square$

Additive evaluations of π-ideals.

In this section, let A be a set which has at least two elements.

A mapping $h : P(A) \to Q^+$ is said to be *additive on* $P(A)$ if we have for each u, $v \subseteq A$:

$$u \cap v = \emptyset \Rightarrow h(u \cup v) = h(u) + h(v).$$

Then $h(\emptyset) = 0$ and, for each $u \subseteq A$, the equality $h(u) = \sum_{x\in u} h(\{x\})$ holds. Thus, h is monotone.

We shall describe a class of π-ideals on A of such a kind that, having such an ideal J, there is no additive set-mapping $h : P(A) \to Q^+$ such that $J = h^{-1\prime\prime}[0]^+$. At first, we denote by $|u|$ the set-cardinality of the set u. It means that there exists a one-one set-mapping between u and a natural number α.

A partition p of A is said to be *relatively bounded*, whenever $(\forall t \in p)(|t|/|p| \in BQ)$ holds.

By a *set-selector* on a partition p on A we mean a set $u \subseteq A$ such that $(\forall t \in p)(|t \cap u| = 1)$.

Proposition. *Let A be a non-empty set. Put $J = w^{-1\prime\prime}[0]^+$ and let w be an additive mapping on $P(A)$. Let p be a relatively bounded partition on A such that $J \cap p = \emptyset$. Then there exists a set-selector on p which does not belong to J.*

Proof: Put, for each $x \in A$, $\widetilde{w}(x) = w(\{x\})$. There exists a set $u = \{a_t;\ t \in p\}$ such that $a_t \in t$ holds for each $t \in p$ and $\widetilde{w}(a_t) = \max(\widetilde{w}^{\prime\prime}t)$. We have, for each $t \in p$: $0 \not\doteq \widetilde{w}(t) \leq |t| \cdot \widetilde{w}(a_t)$. Thus, there exists a number $k \in FN$ such that $1/k \leq |t| \cdot \widetilde{w}(a_t)$ holds for each $t \in p$.

Put $\theta = \max\{|t|;\ t \in p\}$. We deduce from the assumption that the partition p is relatively bounded that there exists a number $m \in FN$ such that $\theta/|p| \leq m$. Thus we have the following: $w(u) = \sum_{t\in p} \widetilde{w}(a_t) \geq 1/k \cdot \sum_{t\in p} 1/|t| \geq 1/k \cdot |p|/\theta \geq 1/(k\cdot m)$. We can conclude that $u \notin J$. □

We say that a partition p of a set A is *relatively non-zero* if we have $(\forall t \in p)(|t|/|p| \not\doteq 0)$.

Proposition. *Let A be an infinite set and let p be an infinite relatively bounded and relatively non-zero set-partition on A. Put*

$$J = \{u \subseteq A;\ (\forall t \in p)(|t \cap u|/|p| \doteq 0)\}.$$

Then we have:

(1) *The class J is an ideal on A of the type π and $[A]^1 \subseteq J$.*
(2) *Every set-selector on p belongs to J.*
(3) *There is no additive set-mapping h on $P(A)$ such that $J = h^{-1\prime\prime}[0]^+$.*

Proof: The items (1) and (2) are easy. The item (3) follows from (2) and from the previous proposition. □

It is clear that, on each infinite set A, there exists an infinite relatively bounded and relatively non-zero set-partition p.

Now, let us briefly pay attention to some classes of evaluations. Let A be a non-empty set. We put

$$Ev(A) = \{h;\ h \text{ is a set-evaluation on } P(A) \text{ in } Q^+\}$$

and let $\sim$ be a relation on $Ev(A)$ defined as follows:

$$f \sim g \Leftrightarrow f^{-1\prime\prime}[0]^+ = g^{-1\prime\prime}[0]^+.$$

Proposition. *Let A be an infinite set. Then $\sim$ is a non-compact equivalence on $Ev(A)$ of the type $\pi\sigma$. The system $\mathcal{W} = \{W_{\kappa,m};\ \kappa \in N - FN\ \&\ m \in FN\}$, where for each α, β,*

$$\begin{aligned} &W_{\alpha,\beta} = \{\langle f, g\rangle \in (Ev(A))^2;\\ &f(u) < 2^{-\alpha} \Rightarrow g(u) < 2^{-\beta}\ \&\ g(u) < 2^{-\alpha} \Rightarrow f(u) < 2^{-\beta}\},\end{aligned}$$

is a uniformity basis on $Ev(A)$ over $\sim$ (that is, $\sim = \bigcap \mathcal{W}$ and $\mathcal{W}$ has the usual properties).

Proof: It is clear that $\sim$ is an equivalence on $Ev(A)$. Let us prove that $\sim$ is not compact. Let $\delta \notin FN$ be such that there exists a set-partition $\{A_\alpha;\ \alpha < \delta\}$ of A and $\delta \leq |A_\alpha| \leq 2\delta$ holds. Let $w_\alpha : A \to Q^+$ be a function such that we have, for each $x \in A$, $w_\alpha(x) = \delta^{-1}$ and w_α is equal to zero on $A - A_\alpha$. Let $h_\alpha : P(A) \to Q^+$ be a function defined by $f_\alpha(u) = \sum_{x\in u} w_\alpha(x)$ whenever

$\emptyset \neq u \subseteq A$ and let $f_\alpha(\emptyset) = 0$. Then $\{f_\alpha;\ \alpha < \delta\}$ is an infinite $\sim$-net which guarantees that $\sim$ is not compact. We prove that $\sim$ is a $\pi\sigma$-class. The explicit definition of $\sim$ has the form $(\forall u \subseteq A)(\forall k)(\exists m)\varphi$ where φ is a set-formula of the language FL_V. We deduce from this that the formula in question is equivalent to a formula $(\forall k)(\exists n)\psi$ where ψ is a set-formula of the language FL_V.

Let us prove that $\mathcal{W}$ has the required properties. We can see that

$$f \sim g \Leftrightarrow (\forall \kappa \in N - FN)(\forall m)(\forall u \subseteq A)(f(u) < 2^{-\kappa} \Rightarrow \\ g(u) < 2^{-m} \ \&\ g(u) < 2^{-\kappa} \Rightarrow f(u) < 2^{-m})$$

and, consequently, $\sim\, = \bigcap \mathcal{W}$ holds. We see also that the system $\mathcal{W} \subseteq Sd_V$ is a system of reflexive and symmetric relations on $Ev(A)$ such that $(\forall W_1, W_2 \in \mathcal{W})(\exists W \in \mathcal{W})(W \subseteq W_1 \cap W_2)$, $(\forall W \in \mathcal{W})(\exists W_0 \in \mathcal{W})(W_0 \circ W_0 \subseteq W)$. □

Let us define, finally,

$$Ev_0(A) = \{h \in Ev(A);\ h''[A]^1 \subseteq [0]^+\}$$
$$Ad_0(A) = \{h \in Ev_0(A);\ h \text{ is an additive set-mapping on } P(A)\}.$$

Proposition. *Let A be an infinite set. Then*

$$Ad_0(A) \subsetneq \sim'' Ad_0(A) \subsetneq Ev_0(A)$$

and the classes $Ad_0(A)$ and $Ev_0(A)$ are π-classes.

Proof: Let us prove the first inclusion. Assume that $h \in Ad_0(A)$ and let $r \doteq 0, r > 0$. Put, for each $u \subseteq A$, $u \neq \emptyset$, $h_r(u) = h(u) + r$ and $h_r(\emptyset) = 0$. Then $h_r \sim h$ and $h_r \notin Ad_0(A)$.

Let us prove the second inclusion. Let J be an ideal from the last but one proposition and let h be an evaluation on $P(A)$ *in* Q^+ such that $h^{-1}{}''[0]^+ = J$. Then $h \in Ev_0(A)$. Let $w \in Ad_0(A)$ and suppose that $w \sim h$. We have $J = w^{-1}{}''[0]^+$, which is a contradiction. □

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