Further remarks on the Nemitskii operator in Hölder spaces

Rita Nugari

Abstract. The paper is concerned with the Nemitskii operator in Hölder spaces. Namely conditions are given to ensure acting, continuity, Lipschitz and differentiability properties.

Keywords: Nemitskii operator, Hölder spaces

Classification: 47H15

0. Introduction.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual norm denoted by $|\cdot|$. In what follows Ω will denote an open bounded subset of $\mathbb{R}^n$ unless otherwise stated and $\overline{\Omega}$ its closure.

For $\alpha \in (0,1]$, $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ is the space of all real functions u which are α-Hölder continuous in $\overline{\Omega}$, i.e. are such that: $h_\alpha(u) := \sup\{|u(x)-u(y)|/|x-y|^\alpha,\ x,y \in \overline{\Omega},\ x \neq y\} < \infty$. $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ is a Banach space with the norm: $\|u\|_\alpha = \|u\|_\infty + h_\alpha(u)$ where $\|u\|_\infty = \sup\{|u(x)|;\ x \in \overline{\Omega}\}$.

This paper is concerned with the study in $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ of some properties of the so called Nemitskii operator, i.e. the operator $F(u)(x) = f(x,u(x))$, $x \in \overline{\Omega}$ where $f = f(x,u)$ is a real valued function defined on $\overline{\Omega} \times \mathbb{R}$.

This argument has been deeply studied mainly in eastern Europe (see [1] and [2] for a complete bibliography). Among the others we like to mention P. Drábek [4] who has found necessary and sufficient conditions for $f = f(u)$ to induce a continuous Nemitskii operator mapping $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ into itself.

Theorem 1.1 is simply a translation in words of [2, Theorem 7.3]; Theorem 3.1 extends the analogue in [2] which deals only with the case $f = f(u)$, as Theorems 2.1 and 4.1 do in relation with the ones in [5]. Finally Theorems 1.1, 2.1 and 4.1 extend our previous paper [7] since the actual assumptions are sensibly weaker.

We have now to compare our paper with the very recent one by M. Goebel [6]. First, we prove most of our results for any open bounded $\Omega \subset \mathbb{R}^n$ rather than for $\Omega = (a,b)$ as in [6]. (The extension to the case $f : \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$ is straightforward, see our final remark.)

Also, in [6] only *sufficient* conditions on f are given so that F has the various desired properties in $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$, while we prove also some *necessary* conditions (Theorems 2.2 and 3.1) which in particular — in case $\Omega = (a,b)$ — yield a characterization of the local Lipschitz property of F (Corollary 3.2).

Let us next discuss the conditions given here with those in [6]. To see this in some detail, we state here two basic assumptions — for a given function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ — to be used through the paper:

$$\text{(H)} \qquad \begin{array}{l} g = g(x,u) \text{ is continuous in } \overline{\Omega} \times \mathbb{R} \\ \text{and } \alpha\text{-Hölder continuous in } x, \\ \text{uniformly with respect to } u \text{ in compact intervals of } \mathbb{R}. \end{array}$$

$$\text{(K)} \qquad \begin{array}{l} g = g(x,u) \text{ is } \alpha\text{-Hölder continuous in } x, \\ \text{uniformly with respect to } u \text{ in compact intervals of } \mathbb{R}, \\ \text{and locally Lipschitz continuous in } u, \\ \text{uniformly with respect to } x \in \overline{\Omega}. \end{array}$$

It is quite clear (see also the proof of Theorem 1.1) that (H) is a weaker assumption than (K).

We note that (H) is equivalent to the assumption that g be continuous and satisfy (A) of [6], while (K) is the same as (B) of [6].

As remarked in [6], if f satisfies (A) and is differentiable with respect to u with f'_u continuous, then f satisfies $(B) = (K)$. On the basis of this remark, it is easy to check that the various properties of F (acting, continuity, etc.) are established in our paper under conditions on f that are weaker than those in [6]. In particular, we note that requiring existence and continuity of f'_u in order to prove the acting property of F is an unnecessarily strong assumption (compare Theorem 1.1 with [6, Theorem 1]). Theorem 2.1 and especially Theorem 2.2 below show that existence of f'_u should be required at the level of continuity of F.

We should finally mention that our proofs are sensibly different from those in [6], and in particular the proof of Theorem 4.1 (differentiability) seems to us simpler and more transparent.

1. Acting property.

Theorem 1.1. *In order that the Nemitskii operator F generated by f map $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself and be bounded, it is sufficient that f satisfies the assumption* (K). *If $\Omega = (a,b)$, this condition is also necessary.*

Proof: By Theorem 7.3 in [2] it is sufficient to prove that (K) is equivalent to:

$$\text{(1.1)} \qquad \begin{array}{l} \forall\, R > 0 \; \exists\, M > 0 : \\ |f(x,u) - f(y,v)| \le M\{|x-y|^\alpha + \dfrac{|u-v|}{R}\} \qquad \forall\, |u|, |v| \le R,\ \forall\, x, y \in \overline{\Omega}. \end{array}$$

Indeed if (1.1) holds, then f is α-Hölder in x since if $R > 0$, $|u| \le R$, and $x, y \in \overline{\Omega}$, then $|f(x,u) - f(y,u)| \le M|x-y|^\alpha$. Moreover (1.1) implies that f is locally Lipschitz in u since, given $R > 0$, $\exists\, M > 0 : |f(x,u) - f(x,v)| \le M\frac{|u-v|}{R}$, $\forall\, |u|, |v| \le R$,

$\forall\, x \in \overline{\Omega}$. Assume now that f satisfies (K); Let $R > 0$, and let L be the Lipschitz constant of f in $[-R, R]$ and k its Hölder constant in $\overline{\Omega}$. We get:

$$\begin{aligned}|f(x,u) - f(y,v)| &\le |f(x,u) - f(x,v)| + |f(x,v) - f(y,v)| \\ &\le L|u-v| + k|x-y|^{\alpha} \quad (|u|, |v| \le R,\ x, y \in \overline{\Omega})\end{aligned}$$

and this yields (1.1) with $M = \max(LR, k)$. □

2. Continuity.

Theorem 2.1. *Let f satisfy the assumption* (K) *(so that F acts in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$). If moreover f is differentiable with respect to u and f'_u satisfies the assumption* (H), *then F is continuous.*

Proof: Let $u, v \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$. To estimate $h_\alpha(F(u+v) - F(u))$, we write (for $x, y \in \overline{\Omega}$)

$$\begin{aligned}
w(x,y) \;&\equiv f(x, u(x)+v(x)) - f(x,u(x)) - f(y, u(y)+v(y)) + f(y, u(y)) \\
&= f(x, u(x)+v(x)) - f(x, u(y)+v(y)) + f(x, u(y)+v(y)) - f(x, u(x)) \\
&\quad - f(y, u(y)+v(y)) + f(y, u(x)) - f(y, u(x)) + f(y, u(y)) \\
= {}&(u(x)+v(x)-u(y)-v(y)) \int_0^1 f'_u(x, u(y)+v(y)+ \\
&\qquad \tau(u(x)+v(x)-u(y)-v(y)))\, d\tau \\
&- (u(x)-u(y)-v(y)) \int_0^1 f'_u(x, u(y)+v(y)+\tau(u(x)-u(y)-v(y)))\, d\tau \\
&+ (u(x)-u(y)-v(y)) \int_0^1 f'_u(y, u(y)+v(y)+\tau(u(x)-u(y)-v(y)))\, d\tau \\
&\qquad - (u(x)-u(y)) \int_0^1 f'_u(y, u(y)+\tau(u(x)-u(y)))\, d\tau \\
= {}&(u(x)-u(y)) \int_0^1 \{f'_u(x, u(y)+v(y)+\tau(u(x)+v(x)-u(y)-v(y))) \\
&\qquad - f'_u(x, u(y)+v(y)+\tau(u(x)-u(y)-v(y))) \\
&\qquad + f'_u(y, u(y)+v(y)+\tau(u(x)-u(y)-v(y))) \\
&\qquad - f'_u(y, u(y)+\tau(u(x)-u(y)))\}\, d\tau \\
&+ (v(x)-v(y)) \int_0^1 f'_u(x, u(y)+v(y)+\tau(u(x)+v(x)-u(y)-v(y)))\, d\tau \\
&\quad + v(y) \int_0^1 \{f'_u(x, u(y)+v(y)+\tau(u(x)-u(y)-v(y))) \\
&\qquad - f'_u(y, u(y)+v(y)+\tau(u(x)-u(y)-v(y)))\}\, d\tau.
\end{aligned}$$

Let now $\varepsilon > 0$ be given; set $M = \|u\|_\alpha$, $R = M+1$. Since f'_u is uniformly continuous in $\overline{\Omega} \times [-2R, 2R]$, then:

(a) there exists a constant N such that $N = \max\{|f'_u(x,u)| : x \in \overline{\Omega},\ u \in [-2R, 2R]\}$,

(b) $\forall\, \varepsilon' > 0\ \exists\, \delta'$ such that: $|f(x,u) - f(x,v)| < \varepsilon'$ whenever $x \in \overline{\Omega}$, $u, v \in [-2R, 2R]$ and $|u - v| < \delta'$.

Moreover f'_u is α-Hölder in x, namely there exists a non negative constant L such that: $|f'_u(x,u) - f'_u(y,u)| \le L|x-y|^\alpha$ for any $x, y \in \overline{\Omega}$, and $u \in [-2R, 2R]$. Then, if $\varepsilon' = \varepsilon/2M$ and $\delta = \min\{\delta', 1, \frac{\varepsilon}{N}, \frac{\varepsilon}{L}\}$ one gets, if $\|v\|_\alpha < \delta$:

$$|w(x,y)| \le 4\varepsilon |x-y|^\alpha \qquad (x, y \in \overline{\Omega})$$

whence $h_\alpha(F(u+v) - F(u)) \le 4\varepsilon$.

To conclude, note that $f(x, u(x) + v(x)) - f(x, u(x)) = \int_0^1 f'_u(x, u(x) + \tau v(x)) v(x)\, d\tau$ and hence $\|F(u+v) - F(u)\|_\infty \le N\|v\|_\alpha < \varepsilon$. □

Theorem 2.2. *Let f satisfy the assumption* (K). *If F is continuous, then f is differentiable with respect to u.*

Proof: Since f is α-Hölder continuous in x and locally lipschitzian in u by Theorem 1.1, then f is absolutely continuous in u and hence almost everywhere differentiable with respect to u in $\mathbb{R}$ in the following sense: for every $x \in \Omega$ the set $N_x = \{u : f'_u(x,u) \text{ does not exist}\}$ has zero Lebesgue measure in $\mathbb{R}$. It follows that its complement N_x^c is dense in $\mathbb{R}$. We want to prove that $N_x^c = \mathbb{R}$ for every x.

Let us proceed by contradiction. Assume $N_{x_0} \ne \emptyset$ for some $x_0 \in \Omega$ and let $u_0 \in N_{x_0}$; thus setting

$$l_1 = \liminf_{h \to 0} \frac{f(x_0, u_0 + h) - f(x_0, u_0)}{h}$$
$$l_2 = \limsup_{h \to 0} \frac{f(x_0, u_0 + h) - f(x_0, u_0)}{h}$$

we should have $l_1 < l_2$. Let h_n and χ_n be real sequences converging to zero such that:

$$l_1 = \lim_{n \to \infty} \frac{f(x_0, u_0 + \chi_n) - f(x_0, u_0)}{\chi_n}, \qquad l_2 = \lim_{n \to \infty} \frac{f(x_0, u_0 + h_n) - f(x_0, u_0)}{h_n}$$

and let y_n and x_n be sequences in Ω such that $h_n = |y_n - x_0|^\alpha$ and $\chi_n = |x_n - x_0|^\alpha$ (take e.g. $y_n = x_0 + h_n^{\alpha^{-1}} v$, $|v| = 1$); then x_n and y_n both converge to x_0. By the density of $N_{x_0}^c$ there exists a real sequence θ_m converging to zero such that $f'_u(x_0, u_0 + \theta_m)$ exists for any m and

$$f'_u(x_0, u_0 + \theta_m) = \lim_{\xi \to 0} \frac{f(x_0, u_0 + \xi + \theta_m) - f(x_0, u_0 + \theta_m)}{\xi} \qquad (m \in \mathbb{N}).$$

Hence also:

$$\begin{aligned} f'_u(x_0, u_0 + \theta_m) &= \lim_{n\to\infty} \frac{f(x_0, u_0 + h_n + \theta_m) - f(x_0, u_0 + \theta_m)}{h_n} \\ &= \lim_{n\to\infty} \frac{f(x_0, u_0 + \chi_n + \theta_m) - f(x_0, u_0 + \theta_m)}{\chi_n}. \end{aligned}$$

We will prove that $l_2 = \lim_{m\to\infty} f'_u(x_0, u_0 + \theta_m)$.

Let y_n be defined as above and consider, for any n, m, the following expression:

$$\begin{aligned} &|h_n^{-1}[f(x_0, u_0 + h_n + \theta_m) \\ &\qquad - f(x_0, u_0 + h_n) - f(x_0, u_0 + \theta_m) + f(x_0, u_0)]| \\ &= |h_n^{-1}[f(y_n, u_0 + h_n + \theta_m) - f(x_0, u_0 + \theta_m) \\ &\qquad - f(y_n, u_0 + h_n) + f(x_0, u_0) \\ &\qquad - f(y_n, u_0 + h_n + \theta_m) + f(y_n, u_0 + h_n) \\ &\qquad - f(x_0, u_0 + h_n) + f(x_0, u_0 + h_n + \theta_m)]|. \end{aligned} \tag{2.1}$$

If we define $u(x) = |x - x_0|^\alpha + u_0$, so that $u(y_n) = h_n + u_0$ and $u(x_0) = u_0$, the expression in (2.1) is less than or equal to

$$\|F(u + \theta_m) - F(u)\|_\alpha + \|F(u_0 + h_n) - F(u_0 + h_n + \theta_m)\|_\alpha .$$

Letting $n \to \infty$ and using the continuity of F in $u_0 + \theta_m$ we get for any m:

$$|l_2 - f'_u(x_0, u_0 + \theta_m)| \le \|F(u + \theta_m) - F(u)\|_\alpha + \|F(u_0) - F(u_0 + \theta_m)\|_\alpha .$$

Letting now $m \to \infty$ we get $l_2 = \lim_{m\to\infty} f'_u(x_0, u_0 + \theta_m)$. The same argument shows that $l_1 = \lim_{m\to\infty} f'_u(x_0, u_0 + \theta_m)$, so that $l_1 = l_2$: contradiction. □

Corollary 2.3. *Let $\Omega = (a, b)$ and assume that the Nemitskii operator F induced by f acts in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ is bounded and continuous. Then f is differentiable with respect to u.*

3. Lipschitz property.

Theorem 3.1. *Let f satisfy the assumption* (K). *In order that F be locally lipschitzian, it is sufficient that f be differentiable with respect to u and f'_u satisfy the assumption* (K). *If $\Omega = (a, b)$, this condition is also necessary.*

PROOF: The "if" part can be proved in the same way as [7, Theorem 1.2].

To prove the "only if" part, note that by assumption

$$\begin{aligned} &\forall\, R > 0\ \exists\, k(R) \ge 0 : \\ &\|F(u) - F(v)\|_\alpha \le k(R)\|u - v\|_\alpha \qquad \forall\, \|u\|_\alpha, \|v\|_\alpha \le R. \end{aligned} \tag{3.1}$$

Let $u \in C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ with $\|u\|_\alpha = M$, $R = M+1$ and $\lambda \in (0, 1)$, so that $\|u+\lambda\|_\alpha < R$. Let us consider, for any $x \in [a, b]$, the function: $g(x, \lambda) = \lambda^{-1}[f(x, u(x) + \lambda) - f(x, u(x))]$. As a consequence of (3.1) the function g has the following properties:

(i) $|g(x, \lambda) - g(y, \lambda)| \le k(R)|x - y|^\alpha \ \ (x, y \in [a, b],\ \lambda \in (0, 1))$
(ii) $|g(x, \lambda)| \le k(R) \ \ (x, y \in [a, b],\ \lambda \in (0, 1))$.

Then the set $\{g_\lambda\} := \{g(\cdot,\lambda),\ \lambda \in (0,1)\}$ is a subset of real continuous functions defined on $[a,b]$ which satisfies the assumptions of Ascoli-Arzelà's theorem; hence there exists a sequence λ_n such that:

$\lambda_n \to 0$
$g_{\lambda_n} \to g$ for some g continuous. Observe that, since F is continuous, from Theorem 2.2 we get the differentiability of f with respect to u. Hence for any $x \in [a,b]$ we have $g(x) = f'_u(x,u(x))$.

The rest of the proof consists in showing that the Nemitskii operator G induced by f'_u maps $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ into itself and is bounded, so that we can apply Theorem 1.1 to prove the claim. For $u \in C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ with $\|u\|_\alpha \le R$ we have $|g_{\lambda_n}(x)| \le k(R)$, and thus passing to the limit as $n \to \infty$, we get $|g(x)| \le k(R)$, which implies $\|G(u)\|_\infty \le k(R)$. Likewise, letting $n \to \infty$ in the inequality $|x-y|^{-\alpha}|g_{\lambda_n}(x) - g_{\lambda_n}(y)| \le k(R)$, we get $|x-y|^{-\alpha}|g(x)-g(y)| \le k(R)$, whence $h_\alpha(G(u)) \le k(R)$. We conclude that $\|G(u)\|_\alpha \le 2k(R)$ and finish the proof. □

Corollary 3.2. *Let $\Omega = (a,b)$. Then F maps $C^{0,\alpha}(\overline{\Omega},\mathbb{R})$ into itself and is locally lipschitzian if and only if both f and f'_u satisfy the assumption* (K).

4. Differentiability.

Theorem 4.1. *Let f be twice differentiable with respect to u and assume that both f and f'_u satisfy the assumption* (K), *while f''_u satisfies the assumption* (H). *Then F is continuously differentiable.*

Proof: From the assumptions and Theorem 2.1 the Nemitskii operator G induced by f'_u is continuous. Let us compute:

$$\begin{aligned} w(x,u,v) &= f(x,u(x)+v(x)) - f(x,u(x)) - f'_u(x,u(x))v(x) \\ &= \int_0^1 [f'_u(x,u(x)+\xi v(x)) - f'_u(x,u(x))v(x)]\,d\xi \\ &= \int_0^1 [G(u+\xi v) - G(u)](x)v(x)\,d\xi \end{aligned}$$

whence

$$\|F(u+v) - F(u) - G(u)v\|_\alpha \le \int_0^1 \|G(u+\xi v) - G(u)v\|_\alpha\,d\xi.$$

Moreover,

$$\begin{aligned} &|x-y|^{-\alpha}|w(x,u,v) - w(y,u,v)| \le \\ &\quad \le \int_0^1 |x-y|^{-\alpha}|(G(u+\xi v) - G(u))(x)v(x) - (G(u+\xi v) - G(u))(y)v(y)|\,d\xi \end{aligned}$$

whence

$$h_\alpha[F(u+v) - F(u) - G(u)v] \le \int_0^1 h_\alpha[G(u+\xi v) - G(u)v]\,d\xi.$$

We conclude that

$$\|F(u+v)-F(u)-G(u)v\|_\alpha \le \int_0^1 \|(G(u+\xi v)-G(u))v\|_\alpha \, d\xi$$
$$\le m\|v\|_\alpha \int_0^1 \|G(u+\xi v)-G(u)\|_\alpha \, d\xi.$$

Now let $\varepsilon > 0$. By the continuity of G there exists $\delta > 0$ such that $\|G(u+\xi v) - G(u)\|_\alpha < \varepsilon$ whenever $\|v\|_\alpha < \delta$. Therefore,

$$\|F(u+v)-F(u)-G(u)v\|_\alpha \le \varepsilon\|v\|_\alpha$$

whenever $\|v\|_\alpha < \delta$, showing that F is differentiable at u with derivative $F'(u)[v] = G(u)v$. Finally, to show that the derivative is continuous, let $\mathcal{L}$ denote the Banach space of all linear bounded mappings of $C^{0,\alpha}(\overline{\Omega}, \mathbb{R})$ into itself, equipped with its usual norm $\|T\|_{\mathcal{L}} = \sup\{\|T[v]\|_\alpha : \|v\|_\alpha = 1\}$. Since

$$\|F'(u+w)[v]-F'(u)[v]\|_\alpha = \|G(u+w)v-G(u)v\|_\alpha \le m\|G(u+w)-G(u)\|_\alpha\|v\|_\alpha$$

we have

$$\|F'(u+w)-F'(u)\|_{\mathcal{L}} \le m\|G(u+w)-G(u)\|_\alpha$$

and the conclusion follows again from the continuity of G. □

Remark. If Ω denotes, as before, an open bounded subset of $\mathbb{R}^n$, the conditions stated in Sections 1, 2, 3, 4 are sufficient also in the case $f = f(x,u) = f(x, u_1, \ldots, u_m)$ is a real valued function defined in $\overline{\Omega} \times \mathbb{R}^m$, $(m \ge 1)$. In this case f'_u denotes the gradient of f with respect to the variable $u \in \mathbb{R}^m$, while f''_u will denote the $m \times m$ Hessian matrix $(f''_{u_i u_j})$ $(i,j = 1, \ldots, m)$ of f with respect to the same variable. As a norm in $C^{0,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ we take $\|u\|_{\alpha,m} = \sum_{i=1}^m \|u\|_\alpha$, $(u = (u_1, \ldots, u_m))$.

References

[1] Appell J., *The superposition operator in function spaces – A survey*, Expo. Math. **6** (1975), 209–270.

[2] Appell J., Zabreiko P., *Nonlinear superposition operators*, Cambridge Tracts in Mathematics 95, Cambridge University Press, 1990.

[3] Bondarenko V.A., Zabreiko P., *The superposition operator in Hölder function spaces*, Soviet Math. Dokl. **16** (1975), 739–743.

[4] Drábek P., *Continuity of Němickij's operator in Hölder spaces*, Comment. Math. Univ. Carolinae **16** (1975), 37–57.

[5] Goebel M., *On Fréchet differentiability of Nemitskij operators acting in Hölder spaces*, Glasgow Math. J. **33** (1991).

[6] ———, *Continuity and Fréchet differentiability of Nemitskij operators in Hölder spaces*, Monatshefte für Mathematik **113** (1992), 107–119.

[7] Nugari R., *Continuity and differentiability properties of the Nemitskij operator in Hölder spaces*, Glasgow Math. J. **30** (1988), 59–65.

[8] Valent T., *Boundary value problems in finite elasticity*, Springer Tracts in Natural Philosophy 31, Springer-Verlag, New York, 1988.

Dipartimento di Matematica, Università della Calabria, 87036 Arcavacata di Rende (CS), Italy

(*Received* May 13, 1992, *revised* July 27, 1992)