On analyticity in cosmic spaces

Oleg Okunev

Abstract. We prove that a cosmic space (= a Tychonoff space with a countable network) is analytic if it is an image of a K-analytic space under a measurable mapping. We also obtain characterizations of analyticity and σ-compactness in cosmic spaces in terms of metrizable continuous images. As an application, we show that if X is a separable metrizable space and Y is its dense subspace then the space of restricted continuous functions $C_p(X \mid Y)$ is analytic iff it is a $K_{\sigma\delta}$-space iff X is σ-compact.

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All spaces below are assumed to be Tychonoff (= completely regular Hausdorff). We denote by $\mathbf{P}$ the space $\mathbf{N}^{\mathbf{N}}$ where $\mathbf{N}$ is the space of naturals (endowed with the discrete topology). *Analytic* spaces are images of $\mathbf{P}$ under continuous mappings. A space is called K*-analytic* if it is a continuous image of a $K_{\sigma\delta}$-space (see [9]), that is, a space representable as the intersection of a sequence of σ-compact subspaces of some embracing space. A space X is called *cosmic* if it has a countable network, that is, a countable family of subsets F such that any open set in X is a union of elements of F. Note that in a regular space the family of closures of elements of a network is again a network. A space X is analytic iff it is cosmic and K-analytic (see [9]).

The family of *Baire sets* in a space X is defined as the minimal family of sets that contains all zero-sets of continuous real functions and is closed with respect to countable unions and complements. Of course, in perfectly normal spaces (as all cosmic spaces are), a set is Baire iff it is Borel. A mapping $f : X \to Y$ is called *measurable* if the preimage of any Baire set in Y is a Baire set in X. One readily checks that if Y is cosmic (in fact, the hereditary Lindelöf property is used) then a mapping $f : X \to Y$ is measurable as soon as $f^{-1}(U)$ are Baire sets for all elements U of an open subbase for Y.

Clearly, a continuous image of an analytic space is analytic. In 1938 Kuratowski proved the following beautiful theorem (see [6]).

Theorem 0.1. *Let $f : X \to Y$ be a measurable mapping of a separable metric space X onto a separable metric space Y. If X is analytic, then so is Y.*

In [5] Frolík generalized this theorem to

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Theorem 0.2. *Let $f : X \to Y$ be a measurable mapping of a K-analytic space X onto a separable metrizable space Y. Then Y is analytic.*

Surprisingly enough, no generalization of this theorem to nonmetrizable spaces Y has been hitherto known (or, at least, no such version was found by the author). On the other hand, analytic spaces need not be metrizable, but are always cosmic, so the class of cosmic spaces looks the most natural domain for considering analyticity. In this paper we generalize Theorem 0.2 to cosmic spaces and infer some corollaries. Some results of this paper were announced in [8].

1. The following lemma is crucial for our proofs. The construction is well-known, only the observation that the ensuing topologies are measurably isomorphic is new.

Lemma 1.1. *Let X be a cosmic space. Then there exist separable metrizable spaces X_u and X_l and continuous bijections $i_u : X_u \to X$ and $i_l : X \to X_l$ such that the inverse mappings i_u^{-1} and i_l^{-1} are measurable.*

PROOF: Let $\mathcal{F} = \{F_n : n \in \mathbf{N}\}$ be a countable network for X. Without loss of generality, $\mathcal{F}$ is closed with respect to finite intersections and all elements of $\mathcal{F}$ are closed in X.

Define a new topology, $\mathcal{T}_u$ on the same set X by declaring that the elements of $\mathcal{F}$ are open. Clearly, $\mathcal{F}$ is an open base for this topology, and $\mathcal{T}_u$ is stronger than the original topology of X. Furthermore, $\mathcal{F}$ is a countable base for $\mathcal{T}_u$ consisting of clopen sets, whence $\mathcal{T}_u$ is a completely regular second countable topology. Let X_u be the set X equipped with the topology $\mathcal{T}_u$; the identity mapping $i_u : X_u \to X$ is a continuous bijection. To check the measurability of i_u^{-1}, note that any open set in X_u is the union of a subfamily of F, hence an F_σ-set in X.

To construct X_l, choose for every $n \in \mathbf{N}$ a continuous function $f_n : X \to \mathbf{R}$ so that $F_n = f_n^{-1}(0)$ and define the topology $\mathcal{T}_l$ on X as the weakest making all functions f_n, $n \in \mathbf{N}$, continuous. Clearly, $\mathcal{T}_l$ is completely regular and second countable. Let X_l be the set X equipped with the topology $\mathcal{T}_l$; the identity mapping of X determines a continuous bijection $i_l : X \to X_l$. Note that all sets F_n, $n \in \mathbf{N}$, are closed in X_l. Every open set in X is a union of a subfamily of $\mathcal{F}$, hence an F_σ-set in X_l, whence the mapping i_l^{-1} is measurable. □

Theorem 1.2. *Let $f : X \to Y$ be a measurable mapping of a K-analytic space X onto a cosmic space Y. Then Y is analytic.*

PROOF: Define Y_u and $i_u : Y_u \to Y$ as in Lemma 1.1. The mapping $f \circ i_u^{-1} : X \to Y_u$ is measurable, and we are now in the conditions of Theorem 0.2, whence Y_u is analytic. Now Y is analytic because it is a continuous image of Y_u. □

Theorem 1.3. *Let X be a cosmic space. If every metrizable continuous image of X is analytic then X is analytic.*

PROOF: Let X_l and i_l be as in Lemma 1.1. X_l is a metrizable continuous image of X, so X_l is analytic. The mapping $i_l^{-1} : X_l \to X$ is measurable and onto, so X is analytic by Theorem 1.2. □

Theorem 1.3 is specific for cosmic spaces. Indeed, $\mathbf{N}^{\omega_1}$ is not Lindelöf, not to say K-analytic; nevertheless, all metrizable continuous images of $\mathbf{N}^{\omega_1}$ are analytic. To see that note that every metrizable continuous image of $\mathbf{N}^{\omega_1}$ is second countable because $\mathbf{N}^{\omega_1}$ satisfies c.c.c. Furthermore, every continuous mapping of $\mathbf{N}^{\omega_1}$ to a second countable space admits a continuous factorization through a countable face of $\mathbf{N}^{\omega_1}$, homeomorphic to $\mathbf{P}$, hence has an analytic image. Furthermore, neither the Lindelöf Σ-property not the hereditary Lindelöf property can replace the cosmicity in Theorem 1.3. To see the first, consider a Lindelöf Σ-space with a single nonisolated point which is not K-analytic [10]. Every disjoint covering of this space by G_δ-sets is countable, hence every metrizable image of this space is countable. The example for the second is the Sorgenfrey line (see [7]). Possibly, the answer to the following question is positive.

Question 1.4. Let X be a perfectly normal Lindelöf Σ-space all metrizable continuous images of which are analytic. Must X be K-analytic?

Theorem 1.5. *Let X be a cosmic space. If every metrizable continuous image of X is σ-compact, then X is σ-compact.*

PROOF: By Theorem 1.3, X is analytic. Now assume for contradiction that X is not σ-compact. Then by a theorem in [3] (a generalization of the Hurewicz theorem to cosmic spaces), X must contain a closed subset F homeomorphic to $\mathbf{P}$. The space $\mathbf{P}$ is homeomorphic to a subspace of the real line $\mathbf{R}$ (namely, to the space of irrationals), so we can extend the homeomorphism $F \to \mathbf{P}$ to a continuous function $f_1 : X \to \mathbf{R}$. Choose a continuous function $f_2 : X \to \mathbf{R}$ so that $F = f_2^{-1}(0)$ and define $f : X \to \mathbf{R}^2$ by $f(x) = (f_1(x), f_2(x))$. The set $\{(a,b) \in f(X) : b = 0\}$ is closed in $f(X)$ and is homeomorphic to $\mathbf{P}$ whence $f(X)$ is not σ-compact, a contradiction. $\square$

2. Denote by $C_p(X)$ the space of all continuous real functions on the space X endowed with the topology of pointwise convergence, that is, the topology of subspace of $\mathbf{R}^X$ endowed with the product topology. Recall that the sets of the form $\{f : f(x) \in U\}$ where $x \in X$ and U is an open subset of $\mathbf{R}$, constitute an open subbase for this topology. Christensen proved in [4] that if X is a separable metrizable space then $C_p(X)$ is analytic iff X is σ-compact. We extend this result to the following modification of $C_p(X)$.

Let A be a subset of X. Denote $C_p(X \mid A) = \{f \in C_p(A) : f$ admits a continuous extension over $X\}$. In different words, $C_p(X \mid A)$ is the subspace of $C_p(A)$ consisting of the functions on A which are restrictions to A of continuous functions on X. Note that if A is a dense subspace of X then the restriction mapping $r_A : C_p(X) \to C_p(X \mid A)$ is one-to-one.

In our proof below we will also use spaces $C_p(X \mid A, Y)$ defined as the spaces of all functions of A to Y that admit a continuous extension over X, with the topology of pointwise convergence on A.

Theorem 2.1. *Let X be a separable metrizable space and A a dense subspace of X. Then the following conditions are equivalent.*

(i) *X is σ-compact,*

(ii) $C_p(X \mid A)$ *is a* $K_{\sigma\delta}$*-space,*
(iii) $C_p(X \mid A)$ *is analytic.*

Proof: (i) $\Rightarrow$ (ii). Let $X = \bigcup\{X_l : l \in \mathbf{N}\}$ where each X_l, $l \in \mathbf{N}$, is compact. Fix a metric d on X compatible with the topology of X and put

$$M_{kln} = \{f \in [0,1]^A : |f(x) - f(y)| \le 1/k \text{ whenever } d(x,z) < 1/n \text{ and } d(y,z) < 1/n \text{ for some } z \in K_l\}, \quad k, l, n \in \mathbf{N}.$$

Claim 1. $C_p(X \mid A, [0,1]) = \bigcap_{k\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} M_{kln}$.

Suppose $f \in \bigcap_{k\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} M_{kln}$. We need to check that f admits a continuous extension to (or is already continuous at) every point $z \in X$. Find an $l \in \mathbf{N}$ with $z \in X_l$. It suffices to verify that the oscillation of f on A near z is arbitrarily small, which is clear from $f \in \bigcap_{k\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} M_{kln}$. Thus, $\bigcap_{k\in\mathbf{N}} \bigcap_{l\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} M_{kln} \subset C_p(X \mid A, [0,1])$. To check the inverse inclusion, let $f \in C_p(X \mid A, [0,1])$ and $\hat{f}$ be the continuous extension of f over X. Fix k and l. Since $\hat{f}$ is continuous and K_l is compact, there exists $n \in \mathbf{N}$ such that $|\hat{f}(x) - \hat{f}(y)| \le 1/k$ as soon as $d(x,z) < 1/n$ and $d(y,z) < 1/n$ for some $z \in K_l$. Then $f \in M_{kln}$.

Claim 2. The sets M_{kln} are compact.

Indeed, the complement $[0,1]^A \setminus M_{kln} = \{f \in [0,1]^A : |f(x) - f(y)| > 1/k$ for some $x \in A$ and $y \in A$ such that $d(x,z) < 1/n$ and $d(y,z) < 1/n$ for some $z \in K_l\}$. is an open set because the evaluation mapping $f \mapsto f(x)$, $x \in A$, are continuous on $[0,1]^A$. Thus, M_{kln} is closed in the compact space $[0,1]^A$, hence compact.

It follows from the claims 1 and 2 that $C_p(X \mid A, [0,1])$ is a $K_{\sigma\delta}$-space. Let $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty, +\infty\}$ be the two-point compactification of $\mathbf{R}$ homeomorphic to $[0,1]$. Clearly, $C_p(X \mid A, \overline{\mathbf{R}})$ is homeomorphic to $C_p(X \mid A, [0,1])$, hence is a $K_{\sigma\delta}$-space.

Put $S = \bigcap_{l\in\mathbf{N}} \bigcup_{n\in\mathbf{N}} \{f \in \overline{\mathbf{R}}^A : |f(x)| \le n$ for all $x \in A \cap K_l\}$. Clearly, S is an $F_{\sigma\delta}$-set in $\overline{\mathbf{R}}^A$, hence a $K_{\sigma\delta}$-space. Furthermore, $C_p(X \mid A) \subset S \subset \overline{\mathbf{R}}^A$, whence $C_p(X \mid A) = S \cap C_p(X \mid A, \overline{\mathbf{R}})$; $C_p(X \mid A)$ is an intersection of two $K_{\sigma\delta}$-spaces, hence is a $K_{\sigma\delta}$-space.

(ii) $\Rightarrow$ (iii). The space A is second countable, hence $C_p(A)$, and consequently $C_p(X \mid A)$ are cosmic [1]. Every $K_{\sigma\delta}$-space is K-analytic, and every K-analytic cosmic space is analytic [9], so we are done.

(iii) $\Rightarrow$ (i). Let $e : C_p(X \mid A) \to C_p(X)$ be the "extension mapping" inverse to the restriction mapping $r_A : C_p(X) \to C_p(X \mid A)$. The mapping e is measurable. Indeed, $C_p(X)$ is cosmic [1], hence hereditary Lindelöf, so it suffices to check that $e^{-1}(V)$ are Baire sets for all elements of some subbase for $C_p(X)$. Let $V = \{f \in C_p(X) : f(x) \in U\}$ where $x \in X$ and U is open in $\mathbf{R}$; as we already noted, the sets of this form constitute a subbase for $C_p(X)$. Choose a sequence $\{x_n : n \in \mathbf{N}\}$ of points of A converging to x; we have

$$V = \bigcup_{n\in\mathbf{N}} \bigcap_{k>n} \{f \in C_p(X) : f(x_k) \in U\},$$

and

$$e^{-1}(V) = r_A(V) = \bigcup_{n\in\mathbf{N}} \bigcap_{k>n} \{f \in C_p(X \mid A) : f(x_k) \in U\},$$

a $G_{\delta\sigma}$-set in $C_p(X \mid A)$. Thus, e is measurable, and $C_p(X)$ is analytic by Theorem 1.2. By theorem of Christensen [4], X is σ-compact. □

Corollary 2.2. *If A is a dense subspace of $\mathbf{P}$ then $C_p(\mathbf{P} \mid A)$ is a coanalytic nonanalytic subspace of $\mathbf{R}^A$.*

Proof: Only coanalyticity now requires a proof. Put

$$\omega(f, U) = \sup\{|f(x) - f(y)| : x, y \in U\}$$

for $f \in \mathbf{R}^A$ and $U \subset A$. Clearly

> $C_p(\mathbf{P} \mid A) = \{f \in \mathbf{R}^A$: for any $x \in \mathbf{P}$ and $n \in \mathbf{N}$ there is a neighbourhood U of x with $\omega(f, U \cap A) \leq 1/n\}$.

For $x \in \mathbf{P} = \mathbf{N}^{\mathbf{N}}$ denote by $x \mid n$ the sequence of the first n coordinates of x. Clearly, the sets of the form $V(x \mid k) = \{y \in \mathbf{P} : y \mid k = x \mid k\}$, $k \in \mathbf{N}$, form a base for $\mathbf{P}$ at x. Hence $C_p(\mathbf{P} \mid A) = \{f \in \mathbf{R}^A$: for any $x \in \mathbf{P}$ and $n \in \mathbf{N}$ there exists $k \in \mathbf{N}$ with $\omega(f, V(x \mid k) \cap A) \leq 1/n\} = \bigcap_{x\in\mathbf{P}} \bigcap_{n\in\mathbf{N}} \bigcup_{k\in\mathbf{N}} F(x \mid k, n)$ where $F(x \mid k, n) = \{f \in \mathbf{R}^A : \omega(f, V(x \mid k) \cap A) \leq 1/n\}$. A straightforward check shows that the sets $F(x \mid k, n)$ are closed, hence $C_p(\mathbf{P} \mid A)$ is coanalytic. □

Corollary 2.2 answers a question of T. Dobrowolski. W. Marciszewski communicated at the Seventh Prague Topological Symposium, August 1991, that this corollary had been also obtained by A. Krawczyk, who used quite a different approach. It is interesting to note that one can derive the nonanalyticity of $C_p(\mathbf{P} \mid A)$ also from Theorem 1.3 if he uses the facts that every mapping of $C_p(X)$ to a second countable space admits a factorization through $C_p(X \mid A)$ for some countable and, without loss of generality, dense A [2] and that all $C_p(\mathbf{P} \mid A)$ with dense countable A are homeomorphic (because the pairs $(\mathbf{P}, A)$ are).

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Tver State University, Pedagogical Department, Box 0236, 170002 Tver, Russia

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