Bifurcation for some semilinear elliptic equations when the linearization has no eigenvalues

Wolfgang Rother

Abstract. We prove existence and bifurcation results for a semilinear eigenvalue problem in $\mathbb{R}^N$ $(N \geq 2)$, where the linearization $- \triangle$ has no eigenvalues. In particular, we show that under rather weak assumptions on the coefficients $\lambda = 0$ is a bifurcation point for this problem in H^1, H^2 and L^p $(2 \leq p \leq \infty)$.

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1. Introduction and presentation of the results.

In the present paper, we consider the nonlinear eigenvalue problem

$$- \triangle\, u - q(x)|u|^{\sigma_1} u + r(x)|u|^{\sigma_2} u = \lambda u \ \text{ in } \ \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$ and σ_1 and σ_2 are positive constants such that $\sigma_1 < 4/N$. In particular, we are interested in the question if $\lambda = 0$ is a bifurcation point for the equation (1.1).

Since the problem (1.1) is considered in $\mathbb{R}^N$, the linearization $- \triangle$ has no eigenvalues and $\lambda = 0$ is the infimum of the spectrum of $- \triangle$. In case that $r \equiv 0$, this problem has been studied by many authors. See for instance [5]–[7], [9], [13]–[18] and the literature quoted therein. In case that $r \not\equiv 0$, we only know some existence results for the equation (1.1) (see [1], [2], [8] and [12]), but no bifurcation results. In the following, we will close this gap by presenting some bifurcation results for the general case.

We always assume that the functions q and r satisfy the subsequent conditions:

(A) The functions $q, r : \mathbb{R}^N \to \mathbb{R}$ are measurable and r fulfills $r(x) \geq 0$ for almost all $x \in \mathbb{R}^N$.

(B) There exist a constant $0 < a \leq 2 - (\sigma_1 N/2)$ and an open ball $B \subset \mathbb{R}^N$, satisfying $B \neq \emptyset$ and $0 \notin \bar{B}$ ($\bar{B}$ is the closure of B), such that $q(x) \geq f(x)|x|^{-a}$ holds for almost all $x \in \zeta$, where $\zeta = \{tx; t \geq 1, x \in B\}$ and $f : \zeta \to [0, \infty)$ is a measurable function satisfying $f(x) \to \infty$ as $|x| \to \infty$.

Moreover, we assume that there exists a constant $\mathcal{K}$ such that

$$r(x) \leq \mathcal{K}|x|^b \ \text{ holds for almost all } \ x \in \zeta,$$

where b is defined by $b = (2-a)(\sigma_2/\sigma_1) - 2$.

(C) The functions r and $q_- = \min(q,0)$ are locally integrable.

(D) The function $q_+ = \max(q,0)$ can be written as $q_+ = q_1 + q_2$, where

(D1) the function q_1 satisfies $0 \le q_1 \in L^\infty$, and $q_1(x)$ tends uniformly to zero as $|x| \to \infty$,

(D2) and the function q_2 satisfies $0 \le q_2 \in L^{p_0}$ for some constant

$$2N/(4-\sigma_1 N) < p_0 < \infty.$$

We want to point out that the above assumptions allow the function q to decay exponentially to $-\infty$ or faster in some direction, and allow the function r to increase exponentially to $+\infty$ or faster in some direction.

Theorem 1.1. *Suppose that the functions q and r satisfy the assumptions* (A)–(D) *and that the constant a is defined as in condition* (B). *Then, there exists a constant $\mu_a \in (0,\infty]$, depending on a, such that for each $\mu \in (0,\mu_a)$ there exists a nonpositive constant $\lambda(\mu)$ and a nontrivial nonnegative function $u_\mu \in H^1 \cap L^\infty$ which solves equation* (1.1) *in the sense of distributions. In case that $a = 2 - (\sigma_1 N/2)$, we have $\mu_a = \infty$. Moreover, it follows that $\lambda(\mu) \to 0$, $\|u_\mu\|_{H^1} \to 0$ and, if $p \in [2,\infty]$, that $\|u_\mu\|_p \to 0$ as $\mu \to 0$. Hence, $\lambda = 0$ is a bifurcation point for equation* (1.1) *in H^1 and in L^p for $p \in [2,\infty]$.*

Corollary 1.2. (a) *If $q_-, r \in L^p_{\mathrm{loc}}$ holds for some constant $p > N/2$, then u_μ is positive and locally Hölder continuous.*

(b) *If q and r are locally Hölder continuous, then we have $u_\mu \in C^2$ and the equation* (1.1) *holds in the classical sense.*

Corollary 1.3. *Suppose in addition to* (A)–(D) *that $p_0 \ge 2$ and that $q, r \in L^\infty + L^2$. Then, it follows that $u_\mu \in H^2$ and that $\|u_\mu\|_{H^2} \to 0$ as $\mu \to 0$. Thus, $\lambda = 0$ is a bifurcation point for* (1.1) *in H^2.*

Remark 1.4. In case that $r \equiv 0$, Corollary 1.3 improves Theorem 2.6 (c) in [13]. In [13] it is assumed that q is nonnegative, that $q = q_+$ satisfies condition (D) and that $p_0 \ge 2$. Moreover, it is assumed

(i) that there exist constants $A > 0$ and $0 \le t < 2 - (\sigma_1 N/2)$ such that $q(x) \ge A(1+|x|)^{-t}$ holds a.e. in $\mathbb{R}^N$. In case that $N \ge 3$ the author requires additionally

(ii) that $\sigma_1 < 2/(N-2)$ and $p_0 > 2N/(2 - \sigma_1(N-2))$. Hence, Corollary 1.3 shows that the condition (i) can be weakened considerably and that condition (ii) is superfluous.

The solutions of the equation (1.1) supply standing waves for nonlinear Klein-Gordon and Schrödinger equations. So, from the standpoint of physics it is an interesting question if the solutions of (1.1) decay exponentially to 0 at infinity.

For the proof of the exponential decay to 0 we need an additional assumption:

(E) There exists a constant $R_0 > 0$ such that q_2 satisfies

$$q_2(x) = 0 \quad \text{for almost all } |x| \ge R_0.$$

Theorem 1.5. *Suppose that $\sigma_2 \leq \sigma_1$ and that the functions q and r satisfy the assumptions* (A)–(E). *Then, for each $\mu \in (0, \mu_a)$ the function u_μ decays exponentially to* 0 *at infinity.*

Theorem 1.6. *Suppose that $\sigma_1 < \sigma_2$ and that the functions q and r satisfy the assumptions* (A)–(E). *Then, there exists a decreasing sequence $(\mu_n) \subset (0, \mu_a)$ such that $\lim_{n\to\infty} \mu_n = 0$ and u_{μ_n} decays exponentially to* 0 *at infinity.*

The proofs for Theorem 1.5–1.6 can be found in § 4.

2. Some preliminaries.

For $p \in [1, \infty]$, $L^p = L^p(\mathbb{R}^N)$ and $L^p_{\text{loc}} = L^p_{\text{loc}}(\mathbb{R}^N)$ are the usual Lebesgue spaces and $\|\cdot\|_p$ is the norm on L^p. If $1 < p < \infty$, then the dual index p' of p is defined by $p' = p/(p-1)$. Furthermore, H^k $(k = 1, 2)$ is the Hilbert space $H^k(\mathbb{R}^N) = W^{k,2}(\mathbb{R}^N)$. The norm on H^1 is given by $\|u\|_{H^1} = (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}$ and the norm on H^2 by $\|u\|_{H^2} = (\| \triangle u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}$. Finally, $C_0^\infty = C_0^\infty(\mathbb{R}^N)$ denotes the set of all functions which have compact support and derivatives of any order.

If $N = 2$, then it follows from the Sobolev imbedding theorem that for each $p \in [2, \infty)$ there exists a constant A_p such that

$$\|u\|_p \leq A_p \|u\|_{H^1} \text{ holds for all } u \in H^1. \tag{2.1}$$

In case that $N \geq 3$, we define $2^* = 2N/(N-2)$. Then, there exists a constant C_0 such that

$$\|u\|_{2^*} \leq C_0 \|\nabla u\|_2 \text{ holds for all } u \in H^1. \tag{2.2}$$

In particular we see that for each $p \in [2, 2^*]$ there exists a constant B_p such that

$$\|u\|_p \leq B_p \|u\|_{H^1} \text{ holds for all } u \in H^1. \tag{2.3}$$

Let F be one of the Banach spaces H^1, H^2 or L^p. Then a real number λ is called a bifurcation point for the equation (1.1) in F if and only if there exists a sequence $(\lambda_n, u_n) \subset \mathbb{R} \times F$ such that $u_n \not\equiv 0$, $\lambda_n \to \lambda$, $\|u_n\|_F \to 0$ $(n \to \infty)$ and

$$\int \nabla u_n \nabla \varphi \, dx - \int q|u_n|^{\sigma_1} u_n \varphi \, dx + \int r|u_n|^{\sigma_2} u_n \varphi \, dx = \lambda_n \int u_n \varphi \, dx$$

holds for all $\varphi \in C_0^\infty$ and $n \in \mathbb{N}$.

When the domain of integration is not indicated, it is understood to be $\mathbb{R}^N$.

Lemma 2.1. *Let $v \in H^1$ be a nonnegative function. Then, there exists a sequence (φ_n) of nonnegative functions $\varphi_n \in C_0^\infty$ such that*

$$\varphi_n \to v \ \text{ in } \ H^1.$$

Proof: The functions η_n $(n \in \mathbb{N})$ may be chosen such that $\eta_n \in C_0^\infty$, $0 \leq \eta_n \leq 1$, $\eta_n(x) = 1$ holds for $|x| \leq n$, $\eta_n(x) = 0$ if $|x| \geq n+1$ and $\|\nabla \eta_n\|_\infty \leq C$, where the constant C is independent of n. Then $\eta_n v \to v$ in H^1.

For a function $u \in L^1_{\text{loc}}$, the regularization u_ε may be defined as in [3, p. 147]. Then, we can find a sequence (ε_n) of positive numbers ε_n, satisfying $\varepsilon_n \to 0$, such that $\varphi_n = (\eta_n v)_{\varepsilon_n} \to v$ in H^1. □

Lemma 2.2. *Let $v \in H^1$ be a nonnegative function and, for $t > 0$, v_t may be defined by $v_t = \min(v, t)$. Then it follows that $v_t \in H^1$, $\partial_i v_t = \partial_i v$ holds almost everywhere in $\{x; v(x) \le t\}$ and $\partial_i v_t = 0$ holds almost everywhere in $\{x; v(x) > t\}$. Moreover, for each $s \in [1, \infty)$, we have $0 \le v_t^s \in H^1 \cap L^\infty$ and $\partial_i v_t^s = s v_t^{s-1} \partial_i v_t$ $(i = 1, \dots N)$.*

PROOF: The first part of the lemma follows from Lemma 1.1 in [10] and Theorem 7.8 in [3]. The functions η_n and the regularizations u_ε may be defined as in the proof of Lemma 2.1. Then, there exists a sequence of positive numbers (ε_n) such that $\varepsilon_n \to 0$ and

$$\varphi_n = (\eta_n v_t)_{\varepsilon_n} \longrightarrow v_t \quad \text{in} \quad H^1.$$

Here, the functions φ_n satisfy $\varphi_n \in C_0^\infty$ and $0 \le \varphi_n \le t$. Since $\varphi_n \to v_t$ in L^2, we can find a subsequence $(\varphi_{n(k)})$ of (φ_n) such that $\varphi_{n(k)}(x) \to v_t(x)$ for almost all $x \in \mathbb{R}^N$.

Now, suppose that $s > 1$. Then it follows that $\varphi_{n(k)}^s \in C_0^1$ and that

$$\partial_i \varphi_{n(k)}^s = s \varphi_{n(k)}^{s-1} \partial_i \varphi_{n(k)} \,.$$

Moreover, since $|v_t^s - \varphi_{n(k)}^s| \le s|v_t - \varphi_{n(k)}| t^{s-1}$, we see that $\varphi_{n(k)}^s \to v_t^s$ in L^2. Hence, we obtain: $\partial_i v_t^s = s v_t^{s-1} \partial_i v_t$. □

The following lemma can be found in [11, p. 93].

Lemma 2.3. *Suppose that $\varphi(t)$ $(t \in [t_0, \infty))$ is a nonnegative and nonincreasing function such that $\varphi(h) \le C(h - t)^{-\gamma} \varphi(t)^\delta$ holds for all $h > t \ge t_0$. The constants γ and C are assumed to be positive and δ may satisfy $\delta > 1$. Then, for $d = C^{1/\gamma} \varphi(t_0)^{(\delta-1)/\gamma} 2^{\delta/(\delta-1)}$ it follows that $\varphi(t_0 + d) = 0$.*

3. Proof of the main results.

In the present paragraph, we will prove Theorem 1.1 and Corollary 1.2–1.3. We start with

Lemma 3.1. *There exist positive constants α and β, and for each $\varepsilon > 0$ a constant $K_\varepsilon > 0$, such that*

$$(2 + \sigma_1)^{-1} \int q_+ |u|^{2+\sigma_1} \, dx \le \varepsilon \|\nabla u\|_2^2 + K_\varepsilon \Big(\|u\|_2^{2+\alpha} + \|u\|_2^{2+\beta} \Big)$$

holds for all $u \in H^1$.

PROOF: For $\varepsilon = \frac{1}{4}$, the proof can be found in [5, pp. 568–569]. For general $\varepsilon > 0$, the proof proceeds quite similarly. □

The nonlinear functional ξ may be defined by

$$\xi(u) = \frac{1}{2} \int |\nabla u|^2 \, dx - (2 + \sigma_1)^{-1} \int q |u|^{2+\sigma_2} \, dx \\ + (2 + \sigma_2)^{-1} \int r |u|^{2+\sigma_2} \, dx.$$

By D, we denote the set

$$D = \{u \in H^1;\ \int |q_-||u|^{2+\sigma_1}\,dx < \infty \ \text{ and } \ \int r|u|^{2+\sigma_2}\,dx < \infty\}.$$

Moreover, for $\mu \geq 0$, we define $D_\mu = \{u \in D;\ \|u\|_2 \leq \mu\}$. Then, according to Lemma 3.1, we see that $I(\mu) = \inf_{u \in D_\mu} \xi(u)$ is a well defined real number.

Lemma 3.2. (a) *Suppose that the constant a in condition* (B) *satisfies $a = 2 - (\sigma_1 N/2)$. Then it follows that $I(\mu) < 0$ holds for all $\mu > 0$.*

(b) *Suppose that $a < 2 - (\sigma_1 N/2)$. Then, there exists a constant $\mu_a > 0$ such that $I(\mu) < 0$ holds for all $\mu \in (0, \mu_a)$.*

Remark 3.3. In the following, we define $\mu_a = \infty$ if $a = 2 - (\sigma_1 N/2)$.

Proof of Lemma 3.2: The ball B may be defined as in condition (B) and ν may be a positive constant. Then, the function $\varphi_0 \in C_0^\infty$ may be chosen such that $\operatorname{supp}\varphi_0 \subset B$ and $\|\varphi_0\|_2 = \nu$. Moreover, for each $t \geq 1$, we define $\varphi_t(x) = t^k \varphi_0(t^{-1}x)$, where $k = (a-2)/\sigma_1$. Since $\|\varphi_t\|_2 = \nu t^{k+(N/2)}$, we see that $\varphi_t \in D_{\nu t^{k+(N/2)}}$ and that

$$\begin{aligned}
I\Big(\nu t^{k+(N/2)}\Big) \leq \xi(\varphi_t) = t^{2k+N-2}\Big(&\frac{1}{2}\int |\nabla\varphi_0(x)|^2\,dx \\
&- t^{2+k\sigma_1}(2+\sigma_1)^{-1}\int_B q(tx)|\varphi_0(x)|^{2+\sigma_1}\,dx \\
&+ t^{2+k\sigma_2}(2+\sigma_2)^{-1}\int_B r(tx)|\varphi_0(x)|^{2+\sigma_2}\,dx\Big) \\
\leq t^{2k+N-2}\Big(&\frac{1}{2}\int |\nabla\varphi_0(x)|^2\,dx \\
&- \inf_{x\in B} f(tx)(2+\sigma_1)^{-1}\int_B |x|^{-a}|\varphi_0(x)|^{2+\sigma_1}\,dx \\
&+ \mathcal{K}(2+\sigma_2)^{-1}\int_B |x|^b|\varphi_0(x)|^{2+\sigma_2}\,dx\Big).
\end{aligned}$$

Since $\inf_{x\in B} f(tx) \to \infty$ as $t \to \infty$, we can find a constant $t_0 \geq 1$ such that

$$I\Big(\nu t^{k+(N/2)}\Big) < 0 \ \text{ holds for all } \ t > t_0. \tag{3.1}$$

Now, suppose that $a = 2 - (\sigma_1 N/2)$. Then, we have $k + (N/2) = 0$. Hence, the part (a) of the lemma follows from (3.1) for $\nu = \mu$. In case that $a < 2 - (\sigma_1 N/2)$, we have $k + (N/2) < 0$. Then, the assertion of the part (b) follows from (3.1) if we define $\nu = 1$, $\mu_a = t_0^{k+(N/2)}$ and $\mu = t^{k+(N/2)}$. □

Lemma 3.4. *For each $\mu \in (0,\mu_a)$ there exists a function $u_\mu \in D_\mu$ such that $u_\mu \geq 0$, $\|u_\mu\|_2 > 0$ and $\xi(u_\mu) = I(\mu)$.*

PROOF: Let $\mu \in (0,\mu_a)$, and $(v_n) \subset D$ may be a sequence such that $\xi(v_n) \to I(\mu)$. Then, we may assume without restriction that $\xi(v_n) \leq 0$ and that $v_n \geq 0$ holds for all n. Hence, we obtain from Lemma 3.1:

$$\begin{aligned} \frac{1}{4}\|\nabla v_n\|_2^2 + (2+\sigma_1)^{-1} \int |q_-||v_n|^{2+\sigma_1}\,dx \\ + (2+\sigma_2)^{-1} \int r|v_n|^{2+\sigma_1}\,dx \leq K_{1/4}(\mu^{2+\alpha} + \mu^{2+\beta}). \end{aligned} \tag{3.2}$$

Since (v_n) is bounded in H^1, we can find a subsequence of (v_n), still denoted by (v_n), and a $u_\mu \in H^1$ such that $v_n \underset{w}{\longrightarrow} u_\mu$ in H^1 and $v_n(x) \to u_\mu(x)$ for almost all $x \in \mathbb{R}^N$. Then, it follows from the uniform boundedness principle, (3.2) and Fatou's lemma that $\|u_\mu\|_2 \leq \mu$, $\|\nabla u_\mu\|_2 \leq \liminf \|\nabla u_n\|_2$,

$$\int |q_-||u_\mu|^{2+\sigma_1}\,dx \leq \liminf \int |q_-||v_n|^{2+\sigma_1}\,dx < \infty$$

and

$$\int r|u_\mu|^{2+\sigma_2}\,dx \leq \liminf \int r|v_n|^{2+\sigma_2}\,dx < \infty.$$

Moreover, we see that $u_\mu \geq 0$. Since the imbedding $H^1(G) \to L^{(2+\sigma_1)p_0'}(G)$ is compact for all bounded balls G and $q_1(x) \to 0$ as $|x| \to \infty$, it follows that

$$\int q_+|v_n|^{2+\sigma_1}\,dx \longrightarrow \int q_+|u_\mu|^{2+\sigma_1}\,dx \quad \text{(see [5, p. 570])}.$$

Moreover, we obtain

$$I(\mu) \leq \xi(u_\mu) \leq \liminf \xi(v_n) = I(\mu) < 0$$

and consequently that $\xi(u_\mu) = I(\mu)$ and $\|u_\mu\|_2 > 0$. □

Lemma 3.5. *For $\mu \in (0,\mu_a)$, the function u_μ may be chosen as in Lemma 3.4. Then, it follows that*

$$\int \nabla u_\mu \nabla\varphi\,dx - \int q|u_\mu|^{\sigma_1}u_\mu\varphi\,dx + \int r|u_\mu|^{\sigma_2}u_\mu\varphi\,dx = \lambda(\mu)\int u_\mu\varphi\,dx$$

holds for all functions $\varphi \in C_0^\infty$, where

$$\lambda(\mu) = \|u_\mu\|_2^{-2}\Big(\|\nabla u_\mu\|_2^2 - \int q|u_\mu|^{2+\sigma_1}\,dx + \int r|u_\mu|^{2+\sigma_2}\,dx\Big).$$

PROOF: Let $\varphi \in C_0^\infty$. Then $d\xi(\|u_\mu\|_2\|u_\mu + \varepsilon\varphi\|_2^{-1}(u_\mu + \varepsilon\varphi))/d\varepsilon \mid_{\varepsilon=0} = 0$ implies the assertion. □

Lemma 3.6. *The constant $\lambda(\mu)$ may be defined as in Lemma 3.5. Then, we have $\lambda(\mu) \le 0$.*

Proof: For all $t \in (0,1]$, we have

$$\xi(u_\mu) = I(\mu) \le I(t\mu) \le \xi(tu_\mu).$$

Hence $\lambda(\mu) = \|u_\mu\|_2^{-2} d\xi(tu_\mu)/dt\mid_{t=1} \le 0$ implies the assertion. □

Proposition 3.7. *The constants α and β may be chosen as in Lemma 3.1. Then, there exists a constant C such that*

$$|\lambda(\mu)| \le C(\mu^\alpha + \mu^\beta) \quad \text{and} \quad \|\nabla u_\mu\|_2^2 \le C(\mu^{2+\alpha} + \mu^{2+\beta})$$

holds for all $\mu \in (0, \mu_a)$. Hence, $\lambda = 0$ is a bifurcation point for the equation (1.1) in H^1.

Proof: Since $\xi(u_\mu) < 0$, we obtain from Lemma 3.1 that

$$\|\nabla u_\mu\|_2^2 \le 4K_{1/4}(\|u_\mu\|_2^{2+\alpha} + \|u_\mu\|_2^{2+\beta}) \le 4K_{1/4}(\mu^{2+\alpha} + \mu^{2+\beta}). \tag{3.3}$$

Moreover, since $\lambda(\mu) \le 0$, it follows from (3.3) and Lemma 3.1 that

$$\begin{aligned} |\lambda(\mu)| = -\lambda(\mu) &\le \|u_\mu\|_2^{-2} \int q_+ |u_\mu|^{2+\sigma_1}\, dx \\ &\le (2+\sigma_1)(4K_{1/4} + K_1)\Big(\|u_\mu\|_2^\alpha + \|u_\mu\|_2^\beta\Big) \le C\big(\mu^\alpha + \mu^\beta\big). \end{aligned}$$

□

Lemma 3.8. *For all nonnegative functions $v \in H^1$ we obtain*

$$\int \nabla u_\mu \nabla v\, dx \le \lambda(\mu) \int u_\mu v\, dx + \int q_+ u_\mu^{1+\sigma_1} v\, dx \tag{3.4}$$

and, according to Lemma 3.6, that

$$\int \nabla u_\mu \nabla v\, dx \le \int q_+ u_\mu^{1+\sigma_1} v\, dx. \tag{3.5}$$

Proof: Clearly, the assertion holds for all nonnegative functions $v \in C_0^\infty$. Hence, the result follows from Lemma 2.1. □

Lemma 3.9. *Suppose that $N \ge 3$ and that $\int q_+ u_\mu^{1+\sigma_1+s}\, dx < \infty$ holds for some constant $s > 1$. Then, it follows that $u_\mu \in L^{2^*(s+1)/2}$.*

Proof: For $t > 0$, the function v_t may be defined by $v_t = \min(u_\mu, t)$. Then, according to Lemma 2.2, we see that $0 \le v_t^s \in H^1$. Inserting v_t^s in (3.5) shows that

$$4s(s+1)^{-2} \int |\nabla v_t^{(s+1)/2}|^2\, dx \le \int q_+ u_\mu^{1+\sigma_1+s}\, dx.$$

Hence, using (2.2) and letting $t \to \infty$, we obtain the assertion by Fatou's lemma. □

Lemma 3.10. *For each $p \in [2,\infty)$, we have $u_\mu \in L^p$.*

PROOF: For $N=2$ and for $p \in [2,2^*]$, if $N \geq 3$, the assertion follows from the Sobolev imbedding theorem. Now, suppose that $N \geq 3$ and that the constants r_n and s_n are defined by $r_n = 2^*(1+\varepsilon_0)^n$ and $s_n = (r_n/p_0') - 1 - \sigma_1$, where $\varepsilon_0 = (2^*/2p_0') - (\sigma_1/2) - 1$. Here, the constant p_0 is defined as in condition (D2). Since $p_0 > 2N/(4-\sigma_1 N + 2\sigma_1)$ and $r_n \geq 2^*$, it follows that $\varepsilon_0 > 0$ and $s_n > 1$.

Now, assume that $u_\mu \in L^{r_n}$ holds for some $n \in \mathbb{N}_0$. Then $2 \leq 1+\sigma_1+s_n < (1+\sigma_1+s_n)p_0' = r_n$ implies that

$$\int q_+ u_\mu^{1+\sigma_1+s_n}\,dx < \infty.$$

Hence, we obtain from Lemma 3.9 that $u_\mu \in L^{2^*(s_n+1)/2}$. But

$$\begin{aligned}(2^*/2)(s_n+1) &= (2^*/2)((r_n/p_0') - \sigma_1)\\ &\geq (2^*/2)(r_n/p_0') - (r_n/2)\sigma_1\\ &= r_n(1+\varepsilon_0) = r_{n+1}\end{aligned}$$

implies that $u_\mu \in L^{r_{n+1}}$. Hence, we see that $u_\mu \in L^p$ holds for all $p \in [2^*,\infty)$. $\square$

Lemma 3.11. *For each $\mu \in (0,\mu_a)$, we have $u_\mu \in L^\infty$.*

PROOF: For $t > 0$, we define the function U_t by $U_t = (u_\mu - t)_+$ and the set $A(t)$ by $A(t) = \{x;\ u_\mu(x) \geq t\}$. Then, we obtain from (3.5) that

$$\int \nabla u_\mu \nabla U_t\,dx \leq \int_{A(t)} q_+ u_\mu^{2+\sigma_1}\,dx. \tag{3.6}$$

The constant p_1 may be defined by $p_1 = 2N/(4-\sigma_1 N)$. Since $p_0 > p_1$, we can find a constant $p_2 \in (1,\infty)$ such that $1/p_0' \cdot 1/p_2' = 1/p_1'$. Then, the inequality (3.6) implies

$$\int |\nabla U_t|^2\,dx \leq C(\mu)(\operatorname{meas} A(t))^{1/p_1'} \tag{3.7}$$

for all $t > 0$, where $C(\mu)$ is defined by

$$C(\mu) = \|q_1\|_\infty \Big(\int u_\mu^{(2+\sigma_1)p_1}\,dx\Big)^{1/p_1} + \|q_2\|_{p_0}\Big(\int u_\mu^{(2+\sigma_1)p_0'p_2}\,dx\Big)^{1/(p_0'p_2)}. \tag{3.8}$$

Now, let us assume that $N \geq 3$. Then, it follows from (2.2) and (3.7) that

$$\Big(\int_{A(t)} (u_\mu - t)^{2^*}\,dx\Big)^{2/2^*} \leq C_0^2 C(\mu)(\operatorname{meas} A(t))^{1/p_1'}. \tag{3.9}$$

Moreover, for each $h > t$, we have

$$\begin{aligned}(3.10)\qquad \Big(\int_{A(t)} (u_\mu - t)^{2^*}\,dx\Big)^{2/2^*} &\geq \Big(\int_{A(h)} (u_\mu - t)^{2^*}\,dx\Big)^{2/2^*} \\ &\geq (h-t)^2(\operatorname{meas} A(h))^{2/2^*}.\end{aligned}$$

Combining (3.9) and (3.10) yields

$$\operatorname{meas} A(h) \leq (C_0^2 C(\mu))^{2^*/2}(h-t)^{-2^*}(\operatorname{meas} A(t))^{2^*/2p_1'}$$

for all $h > t > 0$. Since $2^*/(2p_1') = 1 + (\sigma_1 N)/2(N-2) > 1$, it follows from Lemma 2.3 that u_μ is essentially bounded. Moreover, for each $t_0 > 0$, we have

$$\|u_\mu\|_\infty \leq d + t_0,$$

where $d = C_0 C(\mu)^{1/2}(\operatorname{meas} A(t_0))^{\sigma_1/4} 2^{1+(2(N-2)/\sigma_1 N)}$. For $t_0 = \|u_\mu\|_2$, it follows that

$$\operatorname{meas} A(t_0) \leq \|u_\mu\|_2^{-2}\int_{A(t_0)} u_\mu^2\,dx \leq 1.$$

Hence, we obtain that

$$(3.11)\qquad \|u_\mu\|_\infty \leq C_0 C(\mu)^{1/2} 2^{1+(2(N-2)/\sigma_1 N)} + \mu.$$

Finally, we consider the case that $N = 2$. Here, we obtain for all $t > 0$:

$$\begin{aligned}(3.12)\qquad \int U_t^2\,dx &\leq \int_{A(t)} u_\mu^2\,dx \\ &\leq \Big(\int_{A(t)} u_\mu^{2p_1}\,dx\Big)^{1/p_1}(\operatorname{meas} A(t))^{1/p_1'}.\end{aligned}$$

Combining (3.7) and (3.12) yields

$$\|U_t\|_{H^1}^2 \leq C^*(\mu)(\operatorname{meas} A(t))^{1/p_1'}$$

for all $t > 0$, where

$$(3.13)\qquad C^*(\mu) = C(\mu) + \Big(\int u_\mu^{2p_1}\,dx\Big)^{1/p_1}.$$

Hence, (2.1) implies

$$\Big(\int_{A(t)} (u_\mu - t)^p\,dx\Big)^{2/p} \leq C_p^2 C^*(\mu)(\operatorname{meas} A(t))^{1/p_1'}$$

for all $t > 0$ and $p \in [2,\infty)$. Then, proceeding as in the case that $N \geq 3$, one can show that

$$\operatorname{meas} A(h) \leq C_p^p C^*(\mu)^{p/2}(h-t)^{-p}(\operatorname{meas} A(t))^{p/(2p_1')}$$

holds for all $h > t > 0$ and $p \in [2,\infty)$. Hence, according to Lemma 2.3, we see that u is essentially bounded and that

$$(3.14)\qquad \|u_\mu\|_\infty \leq C_p C^*(\mu)^{1/2} 2^{(p/(2p_1'))((p/2p_1')-1)} + \mu$$

if $p > 2p_1'$. □

Lemma 3.12. *For all $p \in [2,\infty)$ we have $\|u_\mu\|_p \to 0$ as $\mu \to 0$.*

Proof: We start with the case that $N = 2$. Then, according to (2.1), we obtain:

$$\|u_\mu\|_p \le C_p \|u_\mu\|_{H^1} \quad \text{for all} \quad \mu \in (0, \mu_a).$$

Hence, the assertion follows from Proposition 3.7. In case that $N \ge 3$ and $p \in [2, 2^*]$, the assertion is obtained by (2.3) and Proposition 3.7. Now, assume that $N \ge 3$ and that $p \in (2^*, \infty)$. Then, we can find a constant $t > 0$ such that $p = (1 + (t/2))2^*$. Thus, by the Sobolev inequality (2.2), we see that

$$\begin{aligned} \|u_\mu\|_p^{2+t} &= \|u_\mu^{1+(t/2)}\|_{2^*}^2 \le C_0^2 \|\nabla u_\mu^{1+(t/2)}\|_2^2 \\ &= C_0^2 (1 + (t/2))^2 (1+t)^{-1} \int \nabla u_\mu \nabla u_\mu^{1+t}\, dx. \end{aligned} \tag{3.15}$$

The right hand side of (3.15) is well defined since u_μ is bounded. From (3.5), we conclude that

$$\begin{aligned} \int \nabla u_\mu \nabla u_\mu^{1+t}\, dx &\le \int q_+ u_\mu^{2+\sigma_1+t}\, dx \\ &\le \|q_1\|_\infty \int u_\mu^{2+\sigma_1+t}\, dx + \|q_2\|_{p_0} \Big(\int u_\mu^{(2+\sigma_1+t)p_0'}\, dx \Big)^{1/p_0'}. \end{aligned} \tag{3.16}$$

Since

$$\begin{aligned} p_0' &< 2N/(2(N-2) + \sigma_1 N) < 2N/(2(N-2) + \sigma_1(N-2)) \\ &\le (2N + tN)/((2+\sigma_1)(N-2) + t(N-2)) \\ &= (2 + \sigma_1 + t)^{-1} \cdot (2N + tN)/(N-2) \\ &= (2 + \sigma_1 + t)^{-1} p, \end{aligned}$$

we see that there is a constant $\tau \in (0,1)$ such that

$$(2 + \sigma_1 + t)p_0' = \tau p + (1-\tau)2.$$

Hence, by Hölder's inequality, we obtain

$$\Big(\int u_\mu^{(2+\sigma_1+t)p_0'}\, dx \Big)^{1/p_0'} \le \|u_\mu\|_p^{p\tau/p_0'} \|u_\mu\|_2^{2(1-\tau)/p_0'}.$$

Then, using again the fact that $p_0' < 2N/(2(N-2) + \sigma_1 N)$, it is not difficult to show that $p\tau/p_0' < 2 + t$.

Quite similarly, one can prove that there exist constants $c_1 \in (0, 2+t)$ and $c_2 > 0$ such that $\int u_\mu^{2+\sigma_1+t}\, dx \le \|u_\mu\|_p^{c_1} \|u_\mu\|_2^{c_2}$. Hence, we conclude from (3.15), (3.16) and Young's inequality that $\|u_\mu\|_p \to 0$ as $\mu \to 0$. □

Lemma 3.13. *We have* $\|u_\mu\|_\infty \to 0$ *as* $\mu \to 0$.

Proof: The constants $C(\mu)$ and $C^*(\mu)$ may be defined as in (3.8) and (3.13). Then, according to Lemma 3.12, it follows that $C(\mu) \to 0$ and $C^*(\mu) \to 0$ as $\mu \to 0$. Hence, the assertion follows from (3.11) and (3.14). □

Proof of Corollary 1.2: Suppose that the assumptions of part (a) are fulfilled. Then, according to Lemma 3.5, we see that

$$-\triangle u_\mu + c(x)u_\mu = 0 \text{ holds in } \mathcal{D}'(\mathbb{R}^N),$$

where $c(x) = -q(x)u_\mu^{\sigma_1}(x) + r(x)u_\mu^{\sigma_2}(x) - \lambda(\mu)$. Since $p_0 > N/2$ and $u_\mu \in L^\infty$, we see that $c \in L^{p_1}_{\text{loc}}$, where $p_1 = \min(p_0, p)$ satisfies $p_1 > N/2$. Now, the assertion follows from Theorem 7.1 and Corollary 8.1 in [10].

Next, we suppose that the assumptions of the part (b) are fulfilled. Then, it follows from part (a) that u is locally Hölder continuous. Hence, the distribution $\triangle u_\mu$ can be represented by a locally Hölder continuous function. Thus, the assertion of the part (b) follows by a well known result from the regularity theory of elliptic differential equations. □

Proof of Corollary 1.3: According to Lemma 3.5, we see that

$$-\triangle u_\mu = \lambda(\mu)u_\mu + qu_\mu^{1+\sigma_1} - ru_\mu^{1+\sigma_2} \text{ holds in } \mathcal{D}'(\mathbb{R}^N). \tag{3.17}$$

Then, it follows from the assumptions and from Lemma 3.10 – Lemma 3.13 that the right hand side of (3.17) defines a function $F_\mu \in L^2$ such that $\|F_\mu\|_2 \to 0$ as $\mu \to 0$. Consequently, we see that $u_\mu \in H^2$ and that $\|u_\mu\|_{H^2} \to 0$ as $\mu \to 0$. □

4. Exponential decay.

Lemma 4.1. *Suppose that the functions q and r satisfy the assumptions* (A)–(E) *and that for $\mu \in (0, \mu_a)$ the function u_μ and the constant $\lambda(\mu)$ are defined as in Lemma 3.4 resp. Lemma 3.5. Moreover, we assume that $\lambda(\mu) < 0$ holds for some $\mu \in (0, \mu_a)$. Then, for each $c \in (0, -\lambda(\mu))$ there exists a constant A_c such that*

$$u_\mu(x) \le A_c \exp(-(-\lambda(\mu) - c)^{1/2}|x|)$$

holds for almost all $x \in \mathbb{R}^N$.

Proof: Using the fact that u_μ is bounded, we conclude from (D1) and (E) that there exists a constant $R_c > R_0$ such that

$$q_+(x)u_\mu^{\sigma_1}(x) \le c \text{ holds for almost all } x \in \{y;\ |y| > R_c\}. \tag{4.1}$$

The function ψ may be defined by

$$\psi(x) = A_c \exp(-(-\lambda(\mu) - c)^{1/2}|x|) \quad (x \in \mathbb{R}^N).$$

Here, the constant A_c may be chosen such that

$$\psi(x) \geq u_\mu(x) \text{ holds for almost all } x \in \{y;\ |y| \leq R_c\}. \tag{4.2}$$

Then it follows that $\psi \in H^1$ and that

$$\int \nabla\psi\nabla v\, dx \geq (\lambda(\mu) + c)\int \psi v\, dx \tag{4.3}$$

holds for all nonnegative functions $v \in H^1$.

Inequality (4.2) shows that $(u_\mu - \psi)_+$ is a nonnegative function on H^1 satisfying $(u_\mu - \psi)_+(x) = 0$ for almost all $x \in \{y;\ |y| \leq R_c\}$. Hence, we obtain from (3.4), (4.1) and (4.3) that

$$\begin{aligned}
\|\nabla(u_\mu - \psi)_+\|_2^2 &= \int \nabla(u_\mu - \psi)\nabla(u_\mu - \psi)_+\, dx \\
&\leq \lambda(\mu)\int u_\mu(u_\mu - \psi)_+\, dx + c\int u_\mu(u_\mu - \psi)_+\, dx \\
&\qquad - (\lambda(\mu) + c)\int \psi(u_\mu - \psi)_+\, dx \\
&= (\lambda(\mu) + c)\|(u_\mu - \psi)_+\|_2^2 \leq 0
\end{aligned}$$

and consequently that $u_\mu \leq \psi$. □

Lemma 4.2. *Let q and r satisfy the assumptions* (A)–(D) *and suppose that $\sigma_2 \leq \sigma_1$. Then $\lambda(\mu) < 0$ holds for all $\mu \in (0, \mu_a)$.*

PROOF: Since $\xi(u_\mu) < 0$, we see that

$$\int r|u_\mu|^{2+\sigma_2}\, dx < -((2 + \sigma_2)/2)\|\nabla u_\mu\|_2^2 + ((2 + \sigma_2)/(2 + \sigma_1))\int q|u_\mu|^{2+\sigma_1}\, dx$$

and that

$$\lambda(\mu) < \|u_\mu\|_2^{-2}\Big(-(\sigma_2/2)\|\nabla u_\mu\|_2^2 + ((\sigma_2 - \sigma_1)/(2 + \sigma_1))\int q|u_\mu|^{2+\sigma_1}\, dx\Big).$$

Then using the fact that

$$\int q|u_\mu|^{2+\sigma_1}\, dx > -(2 + \sigma_1)\xi(u_\mu) > 0,$$

we obtain the assertion. □

Now, we consider the case that $\sigma_1 < \sigma_2$. Since $I(\cdot)$ is a monotone decreasing function on $[0, \mu_a)$, we can find a measurable subset $\mathcal{M}$ of $[0, \mu_a)$ such that $[0, \mu_a)\backslash\mathcal{M}$ has measure zero and $I(\cdot)$ is differentiable on $\mathcal{M}$ (see [4, Theorem 17.12]). Then, we see that

$$I'(\mu) \leq 0 \text{ holds for all } \mu \in \mathcal{M}. \tag{4.4}$$

Lemma 4.3. *The function $I(\cdot)$ is Lipschitz continuous on $[0, \mu_a)$ and for all $\mu \in \mathcal{M}$ we have $I'(\mu) \geq \mu^{-1} \|u_\mu\|_2^2 \lambda(\mu)$.*

Proof: Let $0 \leq \nu < \mu < \mu_a$. Then, we obtain

$$I(\nu) \leq \xi((\nu/\mu)u_\mu)$$

and therefore that

$$\begin{aligned} I(\nu) - I(\mu) \leq \frac{1}{2}((\nu/\mu)^2 - 1) \int |\nabla u_\mu|^2 \, dx \\ - (2+\sigma_1)^{-1}((\nu/\mu)^{2+\sigma_1} - 1) \int q |u_\mu|^{2+\sigma_1} \, dx \\ + (2+\sigma_2)^{-1}((\nu/\mu)^{2+\sigma_2} - 1) \int r |u_\mu|^{2+\sigma_2} \, dx. \end{aligned} \tag{4.5}$$

Thus, (4.5) implies for $\mu \in \mathcal{M}$: $I'(\mu) \geq \mu^{-1} \|u_\mu\|_2^2 \lambda(\mu)$. Moreover, we obtain

$$\begin{aligned} |I(\mu) - I(\nu)| |\mu - \nu|^{-1} &= (I(\nu) - I(\mu))(\mu - \nu)^{-1} \\ &\leq (2+\sigma_1)^{-1}(1 - (\nu/\mu)^{2+\sigma_1})(\mu - \nu)^{-1} \int q_+ |u_\mu|^{2+\sigma_1} \, dx \\ &\leq (1 - (\nu/\mu))(\mu - \nu)^{-1} \int q_+ |u_\mu|^{2+\sigma_1} \, dx \\ &= \mu^{-1} \int q_+ |u_\mu|^{2+\sigma_1} \, dx. \end{aligned}$$

Hence, Lemma 3.1 and Proposition 3.7 show that

$$|I(\mu) - I(\nu)| \leq C(\mu^{1+\alpha} + \mu^{1+\beta}) |\mu - \nu|.$$

□

Lemma 4.4. *There exists a monotone decreasing sequence $(\mu_n) \subset (0, \mu_a)$ such that $\lim_{n \to \infty} \mu_n = 0$ and $\lambda(\mu_n) < 0$ holds for all n.*

Proof: Suppose that $\lambda(\mu) \geq 0$ holds for all $\mu \in (0, \mu_a)$. Then, according to Lemma 3.6, we see that $\lambda(\mu) = 0$ holds for all $\mu \in (0, \mu_a)$. Furthermore, (4.4) and Lemma 4.3 would imply that $I'(\mu) = 0$ for all $\mu \in \mathcal{M}$ and consequently that $I(\cdot)$ is constant on $[0, \mu_a)$ (see [4, Theorem 18.15]). In particular, we would obtain that

$$0 = I(0) = I(\min((\mu_a/2), 1)) < 0.$$

Hence, there exists a constant $\mu_1 \in (0, \mu_a)$ such that $\lambda(\mu_1) < 0$. Now, repeating this procedure, we can find a $\mu_2 \in (0, \min(\mu_1, 1/2))$ such that $\lambda(\mu_2) < 0$. Moreover, by induction we can show that for each n there is a constant $\mu_n \in (0, \min(\mu_{n-1}, 1/n))$ so that $\lambda(\mu_n) < 0$. □

Finally, we see that Lemma 4.1 and Lemma 4.2 imply Theorem 1.5 and that Theorem 1.6 is obtained by Lemma 4.1 and Lemma 4.4.

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Department of Mathematics, University of Bayreuth, P.O.B. 101251,
W-8580 Bayreuth, Germany

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