Besov spaces and function series on Lie groups

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Abstract. In the paper we investigate the absolute convergence in the sup-norm of Harish-Chandra's Fourier series of functions belonging to Besov spaces defined on non-compact connected Lie groups.

Keywords: Besov spaces, Harish-Chandra-Fourier series

Classification: 46E35

1. Harish-Chandra's Fourier series.

In this section we recall the definition and basic properties of Fourier series on a Lie group introduced by Harish-Chandra. For details we refer to [2] and [13, § 4.4].

Let $\mathbf{G}$ be an n-dimensional Lie group countable at infinity and let $\mathbf{K}$ be a k-dimensional connected compact subgroup of $\mathbf{G}$. Let $\Sigma(\mathbf{K})$ denote the set of all equivalence classes of finite-dimensional irreducible representations of $\mathbf{K}$. For any $\delta \in \Sigma(\mathbf{K})$, let x_δ be the character of the class δ and $d(\delta)$ its degree. We define

$$\alpha_\delta = d(\delta)\ \overline{x}_\delta. \tag{1}$$

Let $\mathfrak{K}$ be the Lie algebra of $\mathbf{K}$. Since $\mathbf{K}$ is compact we can choose a positively-definite quadratic form Q on $\mathfrak{K}$, which is invariant under the action of the adjoint representation of $\mathbf{K}$. Let $X_1, \dots, X_k$ be a base of $\mathfrak{K}$ orthonormal with respect to Q and put

$$\Omega = I - (X_1^2 + \cdots + X_k^2). \tag{2}$$

It is well known that Ω regarded as a differential operator commutes with both left and right translations on $\mathbf{K}$, and that the functions α_δ are eigenvectors of the operator Ω with eigenvalues $c(\delta) \geq 1$,

$$\Omega(\alpha_\delta) = c(\delta)\ \alpha_\delta. \tag{3}$$

The group $\mathbf{G}$ is countable at infinity, therefore the space of smooth functions $C^\infty(\mathbf{G})$ and the space of smooth functions with compact supports $C_0^\infty(\mathbf{G})$ taken with their usual topologies are locally convex, complete and metrizable vector topological spaces. Let $\mathbf{G} \ni x \to L(x)$ be the left regular representation of $\mathbf{G}$ on $C^\infty(\mathbf{G})$

(or $C_0^\infty(\mathbf{G})$), i.e. $(L(x)f) = f(x^{-1}y)$, and $\mathbf{G} \ni x \to R(x)$ be the right regular representation of $\mathbf{G}$ on $C^\infty(\mathbf{G})$ (or $C_0^\infty(\mathbf{G})$), i.e. $(R(x)f)(y) = f(yx)$. Let f be a suitable function on $\mathbf{G}$, then the functions

$$(\alpha_\delta * f)(x) = \int_{\mathbf{K}} \alpha_\delta(y) f(y^{-1}x)\, dy, \quad x \in \mathbf{G}, \tag{4}$$

$$(f * \alpha_\delta)(x) = \int_{\mathbf{K}} \alpha_\delta(y^{-1}) f(xy)\, dy, \quad x \in \mathbf{G}, \tag{5}$$

are called a δ-Fourier component of the function f with respect to the representation $L(x)$ and $R(x)$, respectively. Here dy denotes the normalized Haar measure on $\mathbf{K}$. Identifying α_δ with an element of the space of Radon measures with compact support on $\mathbf{G}$ we can regard (4) and (5) as the convolutions on $\mathbf{G}$.

Theorem 1 (Harish-Chandra). *Let $f \in C^\infty(\mathbf{G})$ ($f \in C_0^\infty(\mathbf{K})$), then the Fourier series*

$$\sum_{\delta \in \Sigma(\mathbf{K})} \alpha_\delta * f \quad \textit{and} \quad \sum_{\delta \in \Sigma(\mathbf{K})} f * \alpha_\delta$$

converge absolutely to f in $C^\infty(\mathbf{G})$ ($C_0^\infty(\mathbf{G})$).

For every distribution $T \in \mathcal{D}'(\mathbf{G})$ on $\mathbf{G}$ its δ-Fourier component with respect to $L(x)$ and $R(x)$ can be defined respectively by

$$\alpha_\delta * T, \quad f * \alpha_\delta \qquad \text{(the convolution of distributions)}.$$

Using the notation of a contragradient representation it is not hard to see that the series

$$\sum_{\delta \in \Sigma(\mathbf{K})} \alpha_\delta * T \quad \text{and} \quad \sum_{\delta \in \Sigma(\mathbf{K})} T * \alpha_\delta$$

converge to T in $\mathcal{D}'(\mathbf{G})$ equipped with the topology of uniform convergence on bounded subsets (cf. [13, § 4.4.3]).

2. Function spaces on Lie groups.

Let e be the identity of $\mathbf{G}$ and $\mathfrak{G} = T_e\mathbf{G}$ be Lie algebra of $\mathbf{G}$ identified with the tangent space $T_e\mathbf{G}$. We equip $\mathfrak{G}$ with a scalar product g_e. For every $x \in \mathbf{G}$, a scalar product in $T_x\mathbf{G}$ is now defined by

$$g_x = d_x l(x^{-1}) g_e \qquad \text{(pull-back)}$$

where $d_x l(x^{-1})$ denotes the tangent mapping at x to $l(x^{-1}) : y \to xy$. Furnished with this Riemannian metric g the Lie group G becomes a connected complete Riemannian manifold with a positive injectivity radius and a bounded geometry (cf. [3], [9]). Furthermore, g is left invariant, that is

$$g_{yx}(d_x l(y)X, d_x l(y)Y) = g(X, Y), \quad x, y \in \mathbf{G}, \quad Y \in T_e\mathbf{G}.$$

If $r > 0$ is a sufficiently small number, then the group $\mathbf{G}$ can be covered by a countable family of sets $l(x_j)(B(r))$, $j = 1, 2, \dots$, $B(r) = \exp\{X \in \mathfrak{G} : g_e(X, Y) < r^2\}$, such that each set $l(x_j)(B(r))$ has non-empty intersection with at most N elements of the family. Moreover, there exists a resolution of unity $\{\phi_j\}$ corresponding to the above covering such that

(6) $\quad \phi_j \in C^\infty(\mathbf{G}), \quad 0 \le \phi_j \le 1, \quad \operatorname{supp} \phi_j \subseteq l(x_j)(B(r)), \quad \sum \phi_j = 1,$

(7) $\quad$ for any multi-index β there is a positive number b_β with

$$|D^\beta(\phi_j \circ l(x_j) \circ \exp)(X)| \le b_\beta, \quad j = 1, 2, \dots$$

(cf. [9, [11]]).

Definition 1 (cf. [9])**.** Let $\{\phi_j\}$ be the above resolution of unity.

(i) Let either $0 < p < \infty$, $0 < q \le \infty$ or $p = q = \infty$. Let $-\infty < s < \infty$. Then

$$F^s_{p,q}(\mathbf{G}) = \{f \in \mathcal{D}'(\mathbf{G}) : \|f \mid F^s_{p,q}(\mathbf{G})\| = \\ = (\sum_{j=1}^{\infty} \phi_j f \circ l(x_j) \circ \exp \mid F^s_{p,q}(\mathbb{R}^n)\|) < \infty\}$$

(with usual modification if $p = \infty$).

(ii) Let $0 < p \le \infty$, $0 < q \le \infty$. Let $-\infty < s_0 < s < s_1 < \infty$. Then

$$B^s_{p,q}(\mathbf{G}) = (F^{s_0}_{p,p}(\mathbf{G}), F^{s_1}_{p,p}(\mathbf{G}))_{\theta,q},$$

with $s = (1-\theta)s_0 + \theta s_1$, $0 < \theta < 1$.

Remarks. The definition of $F^s_{p,q}(\mathbf{G})$ and $B^s_{p,q}(\mathbf{G})$ is independent of the chosen resolution of unity and the chosen left invariant Riemannian metric. The group $\mathbf{G}$ can be equipped with a right invariant Riemannian metric and on this base one can introduce "right" function spaces $\overline{F}^s_{p,q}(\mathbf{G})$, $\overline{B}^s_{p,q}(\mathbf{G})$. In general the "left" and "right" spaces do not coincide, but the relation between them is not enough clear up to now. Various characterizations of the above spaces by the means of derivatives, differences and the like, can be found in [9]–[11].

A lot of properties of the scales $F^s_{p,q} - B^s_{p,q}$ on $\mathbb{R}^n$ have counterparts in the properties of the above defined function space on the Lie group $\mathbf{G}$. For example, the following embeddings hold:

(8) $\quad B^s_{p,\min(p,q)}(\mathbf{G}) \subset F^s_{p,q}(\mathbf{G}) \subset B^s_{p,\max(p,q)}(\mathbf{G})$
for $\;0 < p < \infty, \quad 0 < q \le \infty \;$ and $\; -\infty < s < \infty,$

(9) $\quad B^s_{p,q}(\mathbf{G}) \subset B^\sigma_{\infty,\infty}(\mathbf{G})\;$ for $\;0 < p, q \le \infty\;$ and $\;0 < \sigma < s - \dfrac{n}{p}.$

Moreover one can prove the following interpolation property,

$$(B^{s_0}_{p,q_0}(\mathbf{G}), B^{s_1}_{p,q_1}(\mathbf{G}))_{\theta,q} = (F^{s_0}_{p,q_0}(\mathbf{G}), F^{s_1}_{p,q_1}(\mathbf{G}))_{\theta,q} = B^s_{p,q}(\mathbf{G})$$

for $0 < p < \infty$, $0 < q_0, q_1 \le \infty$, $-\infty < s_0 < s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$.

The spaces $F^s_{p,2}(\mathbf{G})$, $1 < p < \infty$ coincide with the Bessel-potential spaces for the corresponding Beltrami-Laplace operator (cf. [11]).

3. Absolute convergence of Fourier series.

Harish-Chandra's result presented in Section 1 says that on the one hand Fourier series converge in a very strong sense for very smooth functions, on the other, the Fourier series of any distribution converges in the topology of uniform convergence on bounded sets. In this section, to fill the gap between these two convergences, we investigate the absolute convergence in the sup-norm of Fourier series of functions belonging to the Besov spaces $B^s_{p,q}(\mathbf{G})$.

Let $x\mathbf{K}$, $x \in \mathbf{K}$, denote a left-coset of $\mathbf{K}$. We will use the notation introduced in the foregoing sections. In particular, $l(x) : y \to x^{-1}y$ is an isometry of the Riemannian manifold $(\mathbf{G}, g)$ and $x\mathbf{K}$ is a compact submanifold of $\mathbf{G}$ for every $x \in \mathbf{G}$.

We need the following version of the trace theorem for the scales $F^s_{p,q}(\mathbf{G})$ – $B^s_{p,q}(\mathbf{G})$ (cf. [7]).

Lemma 1. *Let $1 \le p < \infty$, $1 \le q \le \infty$ or $p = q = \infty$ ($1 \le p$, $q \le \infty$ in the case of the $B^s_{p,q}(\mathbf{G})$-scale). Let $s > \frac{n-k}{p}$, $n = \dim \mathbf{G}$, $k = \dim \mathbf{K}$. Let $\mathcal{R}_x$, $x \in \mathbf{G}$, be the restriction operator and $\mathcal{E}_x$ the extension operator*

$$\begin{aligned} &\mathcal{R}_x : F^s_{p,q}(\mathbf{G}) \to F^{s_1}_{p,p}(x\mathbf{K}) \quad (\mathcal{R}_x : B^s_{p,q}(\mathbf{G}) \to B^{s_1}_{p,q}(x\mathbf{K})),\\ &\mathcal{E}_x : F^{s_1}_{p,p}(x\mathbf{K}) \to F^s_{p,q}(\mathbf{G}) \quad (\mathcal{E}_x : B^{s_1}_{p,q}(x\mathbf{K}) \to B^s_{p,q}(\mathbf{G}))\end{aligned}$$

described in Theorem 1 in [7], $s - s_1 = \frac{n-k}{p}$. Then there are norms $\|\cdot \mid F^s_{p,q}(x\mathbf{K})\|$ ($\|\cdot \mid B^s_{p,q}(x\mathbf{K})\|$) such that:

(i) $\|f \mid F^s_{p,q}(x\mathbf{K})\| = \|{}_xf \mid F^s_{p,q}(\mathbf{K})\|$ ($\|f \mid B^s_{p,q}(x\mathbf{K})\| = \|{}_xf \mid B^s_{p,q}(\mathbf{K})\|$), *where* ${}_xf(y) = f(x^{-1}y)$,

(ii) *there are constants $c_1 = c_1(p, q, s, \mathbf{K})$ and $c_2 = c_2(p, q, s, \mathbf{K})$ dependent on $p, q, s, \mathbf{K}$ but independent of x such that*

$$\|\mathcal{R}_x\| \le c_1 \quad \text{and} \quad \|\mathcal{E}_x\| \le c_2.$$

Remarks. The above lemma is true in a more general situation, when we replace $\mathbf{G}$ by any complete connected Riemannian manifold with bounded geometry, $\mathbf{K}$ by a compact submanifold, the translations $l(x)$ by a family of isometries and the left-cosets by images of the compact submanifold under the action of the family of the isometries. In particular one can take the right-invariant Riemannian metric on $\mathbf{G}$, the right-cosets of $\mathbf{K}$ and the spaces $\overline{F}^s_{p,q}(\mathbf{G})$, $\overline{B}^s_{p,q}(\mathbf{G})$. We recall that the operators $\mathcal{R}_x$ and $\mathcal{E}_x$ satisfy the identity $\mathcal{R}_x \circ \mathcal{E}_x = \mathrm{id}$.

Proof: The proof is similar to the proof of Theorem 1 in [7], therefore the details are omitted. The only difference is that now we have to pay some attention to the norms of the restriction and extension operators. Since the group $\mathbf{K}$ is compact, there is a finite family $\{(B(y_i, \rho_i), \Phi_i)\}_{i=1}^m$ of charts of $\mathbf{G}$ such that:

- $B(y_i, \rho_i)$ is a geodesic ball in $\mathbf{G}$ centered at $y_i \in \mathbf{K}$ with radius ρ_i, $\rho_i < i(\mathbf{G})/4$, $i(\mathbf{G})$ being the injectivity radius of $\mathbf{G}$,
- $\Phi_i(y_i) = 0$, $\Phi_i(B(y_i, \rho_i) \cap \mathbf{K}) \subset \mathbb{R}^k = \{(t_1, \dots t_n) \in \mathbb{R}^k : t_{k+1} = \dots = t_n = 0\}$,

– sets $V_i = B(y_i, \rho_i/4) \cap \mathbf{K}$, $i = 1, \dots, m$, form an open covering of $\mathbf{K}$,
– $B(y_i, \rho_i) \cap \mathbf{K} \subseteq B_{\mathbf{K}}(y, i(\mathbf{K})/2)$ for every $y \in B(y_i, \rho_i) \cap \mathbf{K}$, $B_{\mathbf{K}}(y, r)$ being a geodesic ball in $\mathbf{K}$,

(cf. [7, § 4.2]).

The sets ${}_xV_i = B(xy_i, \rho_i/4) \cap x\mathbf{K}$, $i = 1, \dots, m$, cover $x\mathbf{K}$. Let $\{\psi_i\}_{i=1}^m$ be a smooth resolution of unity corresponding to the covering $\{V_i\}_{i=1}^m$. Then the functions ${}_x\psi_i(y) = \psi_i(x^{-1}y)$, $i = 1, \dots, m$, form a resolution of unity corresponding to the covering $\{{}_xV_i\}$ of $x\mathbf{K}$. The expression

$$\|f \mid F^s_{p,q}(x\mathbf{K})\| = \sum_{i=1}^m \|({}_x\psi_i f) \circ l(x^{-1}) \circ \Phi_i^{-1} \mid F^s_{p,q}(\mathbf{G})\| \tag{10}$$

is a norm in $F^s_{p,q}(x\mathbf{K})$. It should be clear that

$$\|f \mid F^s_{p,q}(x\mathbf{K})\| = \|{}_xf \mid F^s_{p,q}(\mathbf{K})\|,$$

where $\|\cdot \mid F^s_{p,q}(\mathbf{K})\|$ is given by (10) with $x = e$.

Let a function $\beta_i \in C^\infty(\mathbf{G})$ be such that $\operatorname{supp} \beta \subseteq B(y_i, \rho_i)$, $0 \le \beta_i \le 1$, $\beta_i(B(y_i, \rho_i/4)) = \{1\}$. Let $f \in F^s_{p,q}(\mathbf{G})$, then we define the restriction of f on $x\mathbf{K}$ by

$$\mathcal{R}_x(f)(y) = \sum_{i=1}^m ({}_x\beta_i f_i \circ \Phi_i \circ l(x))(y)$$

where f_i is the restriction of the function $(\beta_i f) \circ l(x^{-1}) \circ \Phi^{-1}$ on $\mathbb{R}^k$ (cf. [12, Theorem 2.7.2]). Theorem 1 in [7] asserted that the operator $\mathcal{R}_x$ is a continuous linear operator from $F^s_{p,q}(\mathbf{G})$ onto $F^{s_1}_{p,q}(x\mathbf{K})$, $s_1 = s - \frac{n-k}{p}$. Moreover, according to the proof of this theorem, the norm of $\mathcal{R}_x$ depends on:

– the number m,
– the norm of the restriction operator $\mathcal{R}_x : F^s_{p,q}(\mathbb{R}^n) \to F^{s_1}_{p,p}(\mathbb{R}^k)$,
– the cardinal number of the sets ${}_xJ_i$, $x \in \mathbf{G}$, $i = 1, \dots, m$,

$${}_xJ_i = \{j : l(x_j)(B(r)) \cap B(xy_i, \rho_i) \neq \emptyset\},$$

– the norms of the pointwise multiplier operator in $F^s_{p,p}(\mathbb{R}^k)$ defined by the functions ${}_x\psi_i \circ l(x^{-1}) \circ \Phi_i^{-1}$ and ${}_x\beta_i \circ l(x^{-1}) \circ \Phi_i^{-1}$,
– the norms of the isomorphism of $F^s_{p,q}(\mathbb{R}^n)$ defined by the diffeomorphisms $(\Phi_j \circ l(x)) \circ (l(x^{-1}) \circ \Phi_j)$ and $\exp_{x_j}^{-1} \circ l(x^{-1}) \circ \Phi_i^{-1}$, $\exp_y$ being the Riemannian exponential mapping at a point y, (cf. [7, § 4.1 and 4.2]). But,
– there is a constant C such that for every x and i $\operatorname{card}({}_xJ_i) \le C$ (cf. [7, Lemma 2]),
– the norms of the pointwise multiplier operators are bounded by the constant independent of x because

$$\|{}_x\psi_i \circ l(x^{-1}) \circ \Phi_i^{-1} \mid C^m(\mathbb{R}^k)\| = \|\psi_i \circ \Phi_i^{-1} \mid C^m(\mathbb{R}^k)\|$$

and

$$\|_x\beta_i \circ l(x^{-1}) \circ \Phi_i^{-1} \mid C^m(\mathbb{R}^k)\| = \|\beta_i \circ \Phi_i^{-1} \mid C^m(\mathbb{R}^k)\|$$

(cf. [8, Theorem 1]),

– a norm of an isomorphism of $F^s_{p,q}(\mathbb{R}^n)$ given by a diffeomorphism of $\mathbb{R}^n$ depends on a constant which bounds from below a determinant of a Jacobian matrix of the given diffeomorphism. We have

$$\exp_{x_j}^{-1} \circ l(x^{-1}) \circ \Phi_i^{-1} = (\exp_{x_j}^{-1} \circ \exp_{xy_i}) \circ (\exp_{xy_i}^{-1} \circ l(x^{-1}) \circ \Phi_i^{-1}) \,.$$

But $l(x^{-1})$ is an isometry of the Riemannian manifold $\mathbf{G}$, therefore $\exp_{xy_i} \circ l(x^{-1}) \circ \Phi_i^{-1} = d_{xy_i} l(x^{-1}) \circ \exp_{y_i} \circ \exp_{y_i} \circ \Phi_i^{-1}$. Thus, $\det(\exp_{xy_i} \circ \Phi_i^{-1}) = \det(\exp_{y_i} \circ \Phi_i^{-1})$ by the properties of normal coordinates. In consequence, we can find a common below bound for the determinants of the Jacobian matrices of the diffeomorphism we are interested in (cf. [8, Theorem 2], [11, § 4.1]).

This proves the lemma for the $F^s_{p,q}$-scale and the restriction operator. The proof for the extension operator is similar. The statement for the $B^s_{p,q}$-scale follows from the properties of real interpolation method. □

It was proved by Harish-Chandra that for sufficiently large m

$$\sum_{\delta \in \Sigma(\mathbf{K})} d(\delta)^2 c(\delta)^{-m} < \infty \qquad \text{(cf. [4, Lemma 7])}.$$

Thus for every $r \in \mathbb{R}$, $0 < r \leq 2$, there is the smallest non-negative number m_r such that

$$\sup_{\delta \in \Sigma(\mathbf{K})} d(\delta)^r c(\delta)^{-m} < \infty, \quad \text{for every } \ m > m_r. \tag{11}$$

Proposition 1. *Let $1 \leq r \leq 2$, $w \in \mathbb{R}$, $w + m_r + \frac{k}{2}(1 - \frac{r}{2}) > 0$. Let $s > \frac{2}{r}(w + m_r) + k(\frac{1}{r} - \frac{1}{2}) + \frac{n-k}{2}$. Then there is a positive number C such that*

$$\sum_{\delta \in \Sigma(\mathbf{K})} c(\delta)^w \|\alpha_\delta * f\| \leq C \|f \mid \overline{B}^s_{2,r}(\mathbf{G})\|^r \tag{12}$$

holds for all $f \in \overline{B}^s_{2,r}(\mathbf{K})$.

Proof: First we show that there is a constant C such that

$$\sum_{\delta \in \Sigma(\mathbf{K})} c(\delta)^v |\langle \overline{x}_\delta, f \rangle|^r \leq C \|f \mid B^{s_0}_{2,r}(\mathbf{K})\|^r \tag{13}$$

holds for $1 \leq r \leq 2$, $v \in \mathbb{R}$, $v + \frac{k}{2}(1 - \frac{r}{2}) > 0$, $s_0 = 2\frac{v}{r} + k(\frac{1}{r} - \frac{1}{2})$ and all $f \in B^{s_0}_{p,q}(\mathbf{K})$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L_2(\mathbf{K})$.

The above inequality is known for abstract Besov spaces (cf. [6, Theorem 6.5.3]). So, the only thing we have to do is to show that for a compact Lie group the Riemannian approach and the abstract approach lead to the same space.

First let us recall that in case of compact manifolds the definition of the scales $F^s_{p,q} - B^s_{p,q}$ is independent of a chosen Riemannian metric. In particular the "left" scales $F^s_{p,q} - B^s_{p,q}$ and the "right" scales $\overline{F}^s_{p,q} - \overline{B}^s_{p,q}$ on compact Lie group coincide. The operator Ω (cf. (2)) with the domain $D(\Omega) = \{f \in L_2(\mathbf{K}) : \Omega f \in L_2(\mathbf{K})\}$ is a self-adjoint, positively defined operator in $L_2(\mathbf{K})$ with pure point spectrum. Therefore, there are abstract potential spaces $D(\Omega^s)$, $D(\Omega^s) \subset L_2(\mathbf{K})$, and abstract Besov spaces B^s_q, $B^s_q \subset L_2(\mathbf{K})$, $s > 0$, $1 \le q \le \infty$, corresponding to the operator Ω — cf. [6, Definition 6.2.2 and 6.3.1] — the relevant properties of the Riesz means follow easily from the spectral theorem. The operator Ω coincides with the Laplace-Beltrami operator corresponding to the bi-invariant Riemannian metric on $\mathbf{K}$. Therefore Theorem 4 in [11] assures us that $D(\Omega^s) = F^{2s}_{2,2}(\mathbf{K})$, $s > 0$. Now the identity

$$B^s_q = B^{2s}_{2,q}(\mathbf{K}), \quad s > 0, \quad 1 \le q \le \infty,$$

follows by interpolation (cf. [6, Theorem 6.3.2] and [11, Theorem 5]). Moreover, we have the following estimate for the number $N(\lambda)$ of eigenvalues of Ω, counted with multiplicity, $\le \lambda : N(\lambda) \le \lambda^{k/2}$. The last inequality is nothing more than the famous Weyl asymptotic formula. Thus (13) follows from Theorem 6.4.3 in [6], because the functions $\overline{x}_\delta$ form an orthogonal system of eigenvectors of the operator Ω with eigenvalues $c(\delta)$ (cf. (1), (3)).

From (11) and (13) it follows that

$$\begin{aligned}
&\sum_{\delta\in\Sigma(\mathbf{K})} c(\delta)^w |\langle \alpha_\delta, f\rangle|^r \le \\
&\le \sup_{\delta\in\Sigma(\mathbf{K})} (d(\delta)^r c(\delta)^{-m}) \sum_{\delta\in\Sigma(\mathbf{K})} c(\delta)^{w+m} |\langle \alpha_\delta, f\rangle|^r \le \\
&\le C\|f \mid B^{s_1}_{2,r}(\mathbf{K})\|^r, \qquad s_1 > \frac{2}{r}(w+m_r) + k(\frac{1}{r} - \frac{1}{2}).
\end{aligned} \tag{14}$$

Now, let $f \in \overline{B}^s_{2,r}(\mathbf{G})$. Let f_x, $x \in \mathbf{G}$, denote the function $f_x(y) = f(xy)$. Then

$$\begin{aligned}
(\alpha_\delta * f)(x) &= \int_{\mathbf{K}} \alpha_\delta(y) f(y^{-1}x)\,dy = \int_{\mathbf{K}} \alpha_\delta(y) f_x(y^{-1})\,dy = \\
&= \int_{\mathbf{K}} \overline{\alpha_\delta(y)} f_x(y)\,dy = \langle \mathcal{R}_e(f_x), \alpha_\delta\rangle,
\end{aligned} \tag{15}$$

because $x_\delta(y^{-1}) = \overline{x_\delta(y)}$. Applying Lemma 1 to the spaces $B^s_{2,r}(\mathbf{G})$ we have

$$\|\mathcal{R}_e(f_x) \mid B^{s_2}_{2,r}(\mathbf{K})\| = \|\mathcal{R}_x(f) \mid B^{s_1(\mathbf{K}x)}_{2,r}\| \le C\|f \mid \overline{B}^s_{2,r}(\mathbf{G})\|, \tag{16}$$

$s - s_1 = \frac{n-k}{2}$, C being independent of x.

It follows from (14)–(16) that for every $x \in \mathbf{G}$

$$\sum_{\delta \in \Sigma(\mathbf{K})} c(\delta)^w |(\alpha_\delta * f)(x)|^r \leq C \| f \mid \overline{B}^s_{2,r}(\mathbf{G})\|^r,$$

which completes the proof. □

Let $C(\mathbf{G})$ denote the Banach space of bounded continuous functions on $\mathbf{G}$ with the standard norm.

Theorem 2. *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $s > 2m_1 + \frac{n}{p} + k \max(0, \frac{1}{2} - \frac{1}{p})$. Let $f \in \overline{B}^s_{p,q}(\mathbf{G})$. Then the Fourier series $\sum_{\delta \in \Sigma(\mathbf{K})} \alpha_\delta * f$ converges absolutely in $C(\mathbf{G})$ to the function f. Moreover, there is a constant C such that*

$$\sum_{\delta \in \Sigma(\mathbf{K})} \|\alpha_\delta * f\|_\infty \leq C \|f \mid \overline{B}^s_{p,q}(\mathbf{G})\|.$$

Proof: Let $h \in B^{2m+k/2}_{2,1}(\mathbf{K})$. The formula (13) with $w = 0$ and $r = 1$ reads as follows:

$$\sum_{\delta \in \Sigma(\mathbf{K})} |\langle h, \alpha_\delta \rangle| \leq C \|h \mid B^{2m+k/2}_{2,1}(\mathbf{K})\|. \tag{17}$$

Since

$$\|(\psi_i h) \circ \Phi_i^{-1} \mid F^s_{p,q}(\mathbb{R}^k)\| = \|(\psi_i h) \circ \Phi_i^{-1} \mid F^s_{p,q}(U_i)\|, \quad U_i = \Phi_i(B(y_i, \delta_i) \cap \mathbf{K}),$$

it follows from (10) that if $-\infty < s_1 < s_0 < \infty$, $s_0 - \frac{k}{p_0} > s_1 - \frac{k}{p_1}$, $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$ or $p_0 = q_0 = \infty$, $p_1 = q_1 = \infty$ then

$$F^{s_0}_{p_0,q_0}(\mathbf{K}) \subset F^{s_1}_{p_1,q_1}(\mathbf{K}) \qquad \text{(cf. [12, Theorem 3.3.1])}.$$

By the real method of interpolation we have

$$B^{s_0}_{p_0,q_0}(\mathbf{K}) \subset B^{s_1}_{p_1,q_1}(\mathbf{K}), \tag{18}$$

for $-\infty < s_1 < s_0 < \infty$, $s_0 - \frac{k}{p_0} > s_1 - \frac{k}{p_1}$, $1 \leq p_0, p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$.

Let $f \in B^s_{p,q}(\mathbf{G})$. Then $f \in C(\mathbf{G})$ (cf. [11, Theorem 5]), $\alpha_\delta * f \in C(\mathbf{G})$ for any $\delta \in \Sigma(\mathbf{K})$ and $\mathcal{R}_e(f_x) \in B^{s_0}_{p,q}(\mathbf{K})$, $s_0 = s - \frac{n-k}{p}$. But (18) and the inequality $s_0 > 2m_1 + \max(\frac{k}{2}, \frac{k}{p})$ imply

$$\|\mathcal{R}_e(f_x) \mid B^{2m+k/2}_{2,1}(\mathbf{K})\| \leq C \|\mathcal{R}_e(f_x) \mid B^{s_0}_{p,q}(\mathbf{K})\|.$$

Thus,

$$\sum_{\delta\in\Sigma(\mathbf{K})} |\langle \mathcal{R}_e(f_x), \alpha_\delta\rangle| \le C\|\mathcal{R}_e(f_x) \mid B^{s_0}_{p,q}(\mathbf{K})\| \qquad \text{cf. (17).}$$

Now, by Lemma 1

$$\sum_{\delta\in\Sigma(\mathbf{K})} \|\alpha_\delta * F\|_\infty \le C\|f \mid \overline{B}^{s}_{p,q}(\mathbf{G})\|.$$

So the Fourier series of f converges in $C(\mathbf{G})$. But this series converges to f in $\mathcal{D}'(\mathbf{G})$ in the topology of uniform convergence on bounded sets, therefore it converges to f in $C(\mathbf{G})$. □

Remarks.

1. Since $f * \alpha_\delta(x) = \langle {}_x f, \alpha_\delta\rangle$ both Proposition 1 and Theorem 2 are true if we replace $\alpha_\delta * f$ by $f * \alpha_\delta$ and $\overline{B}^{s}_{p,q}(\mathbf{G})$ by $B^{s}_{p,q}(\mathbf{G})$.
2. One also can regard the series

$$\sum_{\delta_1,\delta_2\in\Sigma(\mathbf{K})} \alpha_{\delta_1} * f * \alpha_{\delta_2}\,.$$

It was proved by Harish-Chandra that the above series converges to f absolutely in $C^\infty(\mathbf{G})$ ($C^\infty_0(\mathbf{G})$) if $f \in C^\infty(\mathbf{G})$ ($f \in C^\infty_0(\mathbf{G})$). One can ask whether the series converges absolutely to f in the sup-norm if $f \in B^{s}_{p,q}(\mathbf{G}) \cap \overline{B}^{s}_{p,q}(\mathbf{G})$.

References

[1] Chavel I., *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.
[2] Harish-Chandra, *Discrete series for semi-simple Lie groups II*, Acta Math. **116** (1966), 1–111.
[3] Price J.F., *Lie Groups and Compact Groups*, University Press, Cambridge, 1979.
[4] Robinson D.W., *Lie groups and Lipschitz spaces*, Duke J. Math. **57** (1988), 357–395.
[5] ———, *Lipschitz operators*, J. Funct. Anal. **85** (1989), 179–211.
[6] Schmeisser H.J., Triebel H., *Topics in Fourier Analysis and Function Spaces*, Wiley, Chichester, 1987.
[7] Skrzypczak L., *Traces of function spaces $F^s_{p,q} - B^s_{p,q}$ type on submanifolds*, Math. Nachr. **146** (1990), 137–147.
[8] Triebel H., *Diffeomorphism properties and pointwise multiplier for function spaces*, in: Function Spaces, Proc. Inter. Conf. Poznań 1986, Teubner-Texte, Leipzig, 1988, 75–84.
[9] ———, *Function spaces on Lie groups, the Riemannian approach*, J. London Math. Soc. **35** (1987), 327–338.
[10] ———, *How to measure smoothness of distributions on Riemannian symmetric manifolds and Lie groups?*, Zeitsch. Anal. ihre Anwend. **7** (1988), 471–480.
[11] ———, *Spaces of Besov-Hardy-Sobolev type on complete Riemannian manifolds*, Ark. Mat. **24** (1986), 299–337.
[12] ———, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.
[13] Warner B., *Harmonic Analysis on Semi-simple Lie groups I*, Springer-Verlag, Berlin, 1972.

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(*Received* May 29, 1992)