Complexity of the axioms of the alternative set theory

A. Sochor

Abstract. If $\mathbf{T}$ is a complete theory stronger than $\mathbf{ZF}_{\mathrm{Fin}}$ such that axiom of extensionality for classes + $\mathbf{T}$ + $(\exists X)\Phi_i$ is consistent for $1\leq i \leq k$ (each alone), where Φ_i are normal formulae then we show $\mathbf{AST} + (\exists X)\Phi_1 + \cdots + (\exists X)\Phi_k$ + scheme of choice is consistent. As a consequence we get: there is no proper Δ_1-formula in $\mathbf{AST}$ + scheme of choice. Moreover the complexity of the axioms of $\mathbf{AST}$ is studied, e.g. we show axiom of extensionality is Π_1-formula, but not Σ_1-formula and furthermore prolongation axiom, axioms of choice and cardinalities are Π_2-formulae, but not Π_1-formulae in $\mathbf{AST}$ without the axiom in question.

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According to the philosophical point of view understanding the alternative set theory as a description of "the human approach to the real world" (see [V1979]), we grasp this theory as open to the addition of new principles — the formal theory **AST** (cf. [S1979]) repeated below summarizes only the **basic** ideas of the alternative set theory. There are two viewpoints which we should respect if we add new axioms — from the formal standpoint the most important requirement is the consistency of the obtained system and from the philosophical point of view the choice of supplementary axioms has to be in harmony with the philosophical interpretation of the alternative set theory. In fact, the philosophical motivation makes it sometimes possible to decide whether we should prefer to add a formula or its negation as a new axiom even in the case that the formula in question is independent w.r.t. the system of axioms accepted so far. Two types of objects in the alternative set theory are philosophically distinguished so that sets are considered as formal counterparts of collections in the "real world" and classes are held as descriptions of our idealizations and generalizations. Accepting this standpoint we have to treat the system of sets as given and we are able to select only the system of classes — we can decide how large a system of ideas we take into account. If it is not excluded from particular reasons, it is evidently convenient to start with the largest system of classes ("human ideas") as possible, because a smaller system of classes can be investigated in the framework of a larger one. According to this approach, we have to accept axioms which guarantee the rise of the largest system of classes as possible (under the assumption of the consistency of the theory arising in this way, of course). The principle "to investigate a system of classes as rich as possible" forces us to prefer formulae of the form $(\exists X)\Phi(X)$ where Φ is a normal formula (i.e. a formula in which only set-variables are quantified), because axioms of this type enrich the system of classes without doubt. However, it is more difficult to solve the problem which formulae with more changes of quantifiers for classes should be preferred — both such a formula and its negation postulate existence of some classes. Up to now, a general philosophically justifiable principle for such a choice of formulae has not been formulated, but for formulae with one change of quantifiers the principle "to add as many classes as possible" gives us at least a partial guideline (formulated by P. Vopěnka).

Let Φ be a normal formula with two free variables and let us imagine a game in which the first player wants to put the formula $(\exists X)(\forall Y)\Phi(X,Y)$ across and the second one does the same with the formula $(\forall Z)(\exists Q)\neg\Phi(Z,Q)$, i.e. an infinite game in which at odd steps the first player chooses a class X with $(\forall Y)\Phi(X,Y)$ and at even steps the second player completes the system of classes so that for every class Z (in particular even for the class X chosen by the first player at the previous step) there exists Q such that $\neg\Phi(Z,Q)$. Under the assumption that at every step such a choice is possible, the second player wins the game because he is able to invalidate all witnesses of the formula $(\exists X)(\forall Y)\Phi(X,Y)$ suggested by the first player in the previous step. Thus it is more acceptable to add as additional axioms the formulae of the form $(\forall Z)(\exists Q)\Phi(Z,Q)$, where Φ is a normal formula. The suggested instruction for the choice of new axioms does not determine quite precisely our choice, since the ordering in which formulae are taken into account can have an influence on the preference of formulae, in fact, we shall see that there are formulae which can be written simultaneously in the form $(\exists X)(\forall Y)\Phi(X,Y)$ and in the form $(\forall Z)(\exists Q)\Psi(Z,Q)$, where both formulae Φ and Ψ are normal (and hence if we use only the principle for the choice of new axioms described above, the decision whether we choose the formula in question or its negation depends only on the fact which of these formulae is taken into account as the first; by "can be written" we mean, of course, the formulae in question are equivalent in the theory **AST** or in its investigated strengthening).

As usual we consider normal formulae both as Σ_0-formulae and as Π_0-formulae. If Σ_k-formulae and Π_k-formulae are specified, then we define as Π_{k+1}-formulae (as Σ_{k+1}-formulae respectively) all formulae of the form $(\forall X)\Phi$, where Φ is a Σ_k-formula (of the form $(\exists X)\Phi$, where Φ is a Π_k-formula respectively). The above described principle can be expressed in this terminology as the preference of Π_2-formulae over Σ_2-formulae.

The acceptance of the principle in question puts before us a series of questions and we are going to give answers to some of them in this paper. In particular, we show that all axioms of the theory **AST** are in harmony with the described principle except the axiom of extensionality — more precisely we shall see that the axioms A5, A6 and A7 are (equivalent to) Π_2-formulae and the axiom A4 is even a Σ_1-formula, however the axiom of extensionality is a Π_1-formula (the axioms A3 and A8 are set-formulae — i.e. formulae in which only set-variables occur — as for A2, the complexity of a given formula determines the complexity of the instance of the scheme of existence of classes corresponding to this formula, nevertheless, neglecting the increase of the complexity produced by the complexity of the original formula, one can consider also instances of the scheme of existence of classes as formulae of the preferred form). We show, moreover, the axioms A5–A7 bring further ideas because they can be equivalently formulated as Σ_2-formulae (and they cannot be described as Π_1-formulae; furthermore, we are going to show the axioms A5 and A7 cannot be formulated neither as Σ_1-formulae). In other words the axioms A5–A7 carry out the above formulated principle for the choice of axioms, however, their reception is not forced by it.

Another problem is to describe which Σ_1-formulae are excluded as additional axioms by the acceptance of the axioms of **AST** — as an example of such formula can serve the negation of the axiom of extensionality which can be consistently joined to every (consistent) strengthening of $\mathbf{ZF}_{\mathrm{Fin}}$ (i.e. of Zermelo-Fraenkel set theory in which the axiom of infinity is replaced by its negation) obtained by the addition of set-formulae (and the acceptance of the axiom of extensionality automatically excludes the consistent extension by its negation). On the other side, we shall see that the axiom of extensionality is the sole axiom of **AST** which is able to prevent the reception of some Σ_1-formula as a supplementary axiom — moreover, any enrichment of the theory **AST** by a Σ_1-formula does not exclude the possibility to add to the enriched theory any Σ_1-formula as an additional axiom except the case that such a possibility is excluded by a set-formula together with the axiom of extensionality.

We have claimed that the axiom of extensionality is the sole axiom among axioms of **AST** which is no set-formula and which can make it impossible to join Σ_1-formulae. Let us realize that this axiom is an assumption restricting our possibility even in a much deeper sense because it expresses the restriction of considered properties to properties expressible in the language of set theory (if objects are not distinguishable by their elements, they are identified) and one accepts the axiom of extensionality **since** it formalizes this restriction.

In the end of the paper we show that set-formulae are the only formulae which are in **AST** equivalent both to a Σ_1-formula and to a Π_1-formula.

In the article we deal only with one principle determining the choice of formulae as supplementary axioms to **AST**; a different philosophical view justifies the addition of the scheme of choice (see below) and the scheme of dependent choices both discussed in [S1983] and accepted as principles of the alternative set theory in [V1989] (the second principle is called there “the second axiom of the way to the horizon”) and also the supplementation of the principle of reflection (see [S-V1981]) to **AST**. It is necessary to emphasize that there is no conflict among these principles and the principle investigated in this paper.

The theory AST has one sort of variables $X, Y, \ldots$ for classes, predicates $=$ and $\in$; sets are defined in the same way as in the classical theories — as elements i.e. $Set(X) \equiv (\exists Y)\ X \in Y$ and we use variables $x, y \ldots$ for sets. Since we want to investigate the complexity of the axioms, we repeat — for the reader’s convenience — eight axioms (the second one is actually a scheme of axioms) constituting the axiomatic system **AST** and simultaneously we count the complexity of formulae in question; we use the usual set-theoretical notation, e.g. V denotes the universal class, the predicate Fnc expresses the property “to be a function” etc.; let us recall the usual definition of the universal class is done by a set-formula and further predicates Fnc, "to be a linear ordering" and "to be an equivalence" can be written as normal formulae.

A1 axiom of extensionality is the Π_1-formula

$$(\forall X, Y)[(\forall z)(z \in X \equiv z \in Y) \equiv X = Y].$$

A2 scheme of existence of classes. For **every** formula $\Phi(Z_0, \dots, Z_k)$ we accept the axiom

$$(\forall X_1, \dots, X_k)(\exists Y)(\forall y)(y \in Y \equiv \Phi(y, X_1, \dots, X_k)).$$

A3 axiom of existence of sets is the set-formula

$$(\exists x)(\forall y)(y \notin x) \ \& \ (\forall x, y)(\exists z)(\forall q)(q \in z \equiv [q \in x \vee q = y]).$$

We denote the theory with the axioms A1–A3 by **TC** (the theory of classes); in **TC** the set-constant 0 can be defined by a set-formula. We put $FN = \{x; (\forall Z)([0 \in Z \ \& \ (\forall y)(y \in Z \to y \cup \{y\} \in Z)] \to x \in Z)\}$; as usual the symbol N denotes the class of natural numbers. According to the definition, the class FN is the smallest class containing 0 and closed under the successor operation. Let us recall elements of the class N are separated by a set-formula, however, the formula marking off the elements of the class FN is a Π_1-formula (and there is no Σ_1-formula selecting the elements of the class FN even in the whole **AST** because if $*$ is a shifting of the horizon (see [S-V1983]) then every set is a $*$-set and therefore for every normal formula Φ with two free variables one has $(\forall \alpha \in N)([(\exists X)\Phi(\alpha, X)]^* \to (\exists X)\Phi(\alpha, X))$, however, we have $FN \subset FN^*$).

To simplify the notation let us introduce the predicate "the pair Z_1, Z_2 codes a system of classes which is closed under Bernays-Gödel's operations" putting

$$\begin{gathered} GB(Z_1, Z_2) \equiv (\forall x, y \in Z_1)(\exists z_1, \dots, z_6 \in Z_1)(E = Z_2''\{z_1\} \ \& \\ \& \ dom(Z_2''\{x\}) = Z_2''\{z_2\} \ \& \ (Z_2''\{x\})^{-1} = Z_2''\{z_3\} \ \& \\ \& \ \{\langle u, v, w\rangle; \langle v, w, u\rangle \in Z_2''\{x\}\} = Z_2''\{z_4\} \ \& \ Z_2''\{x\} - Z_2''\{y\} = Z_2''\{z_5\} \ \& \\ \& \ Z_2''\{x\} \times Z_2''\{y\} = Z_2''\{z_6\}); \end{gathered}$$

(see [B] or [G]). Note that the formula defining the predicate GB is a normal one.

A4 axiom of induction is the Σ_1-formula

$$\begin{gathered} (\exists K, S)[GB(K, S) \ \& \ (\forall u)(\exists z \in K)(u = S''\{z\}) \ \& \\ (\forall x \in K)([0 \in S''\{x\} \ \& \ (\forall w, v)(w \in S''\{x\} \to w \cup \{v\} \in S''\{x\})] \to S''\{x\} = V)]. \end{gathered}$$

Let us note that the previous axiom expresses existence of a pair of classes coding the system of classes which contains all sets, is closed under Bernays-Gödel's operations and in which the universal class is the unique class W satisfying

$[0 \in W \ \& \ (\forall w, v)(w \in W \rightarrow w \cup \{v\} \in W)]$ (and that if we replace the last formula by the formula $[0 \in W \ \& \ (\forall w, v \in W) \ w \cup \{v\} \in W]$ then we get the formula equivalent to A4 together with A8). In the theory **TC** it is possible to define the satisfaction relation for all formal set-formulae (the codes of) which are elements of the class FN and the axiom A4 is equivalent to the induction (i.e. to the formula $V \models ([\varphi(0) \ \& \ (\forall x, y)(\varphi(x) \rightarrow \varphi(x \cup \{y\}))] \rightarrow (\forall x)\varphi(x)))$ for all formal set-formulae which are elements of the class FN.

The predicate $We(X, R)$ is defined by the Π_1-formula

$$\begin{aligned}&(\forall Y)[\text{“}R \text{ is a linear ordering of the class } X\text{” } \&\\ &\& \ [(\exists y)(y \in Y) \rightarrow (\exists y \in Y)(\forall z \in Y)\langle y, z\rangle \in R]]\end{aligned}$$

and guarantees that every nonempty sub**class** of X has the first element.

The possibility to contract variables is a consequence of the scheme of existence of classes and thus this scheme essentially simplifies the calculation of the complexity of formulae, e.g. the conjunction of two Σ_1-formulae is (in **TC** equivalent to) a Σ_1-formula. Furthermore, in the theory **TC**, the formulae $(\exists Y) \ X \in Y$ and $(\exists y) \ X \in y$ are equivalent and thence the predicate Set can be considered as normal. The formula

$$\begin{aligned}&(\forall Y \subseteq X)(Set(Y) \ \& \ (\forall Z)[((\forall z \in Z)(z \subseteq Y) \ \& \ (\exists z)(z \in Z)) \rightarrow\\ &\rightarrow (\exists z \in Z)(\forall y)(z \subset y \rightarrow y \notin Z)])\end{aligned}$$

defining the predicate $Fin(X)$ is then a Π_1-formula and therefore $\neg Fin(X)$ is a Σ_1-formula. Let us remark that the predicate $Fin(X)$ is defined in [V1979] by the Π_1-formula $(\forall Y \subseteq X)Set(Y)$ only and that in A1–A4 the last-named formula is equivalent to our definition (which formalizes finiteness even in the case that the theory in question admits infinite sets). Furthermore, let us note that the equality $FN = \{x \in N; Fin(x)\}$ is trivial in A1–A4 and thus the predicate Fin cannot be equivalently written as a Σ_1-formula even in the whole **AST**; on the other hand in [Sg1986] it is proved that $Fin(X)$ is equivalent to a normal formula in the theory A1–A4 + $We(V, R)$, where R is a new constant.

The predicate $Count(X)$ i.e. the formula

$$\neg Fin(X) \ \& \ (\exists R)(\text{“}R \text{ is a linear ordering of the class } X\text{” } \& \ (\forall q) \ Fin(R''\{q\}))$$

is — according to the previous considerations — (in **TC** equivalent to) a Σ_2-formula. On the other hand using the countability of the class FN we are able to write the formula “X is at most countable” by

$$(\forall Y)([0 \in Y \ \& \ (\forall y \in Y) \ y \cup \{y\} \in Y] \rightarrow (\exists G)[Fnc(G) \ \& \ X \subseteq G''Y])$$

i.e. the predicate $Count(X)$ is in **TC** equivalent both to a Π_2-formula and to a Σ_2-formula.

A5 prolongation axiom is the Π_2-formula

$$(\forall F)([Fnc(F) \ \& \ Count(F)] \rightarrow (\exists f)[Fnc(f) \ \& \ F \subseteq f]).$$

It is useful to realize that the acceptance of the prolongation axiom still reduces the complexity of the predicate "to be at most countable" since it becomes equivalent (in A1–A5) to the Π_1-formula

$$(\forall Y)([0 \in Y \ \&\ (\forall y \in Y)\ y \cup \{y\} \in Y] \to (\exists g)[Fnc(g)\ \&\ X \subseteq g''Y]).$$

A6 axiom of choice is the Σ_2-formula

$$(\exists R) We(V, R).$$

A7 axiom of cardinalities is the Π_2-formula

$$(\forall X)[Fin(X) \vee Count(X) \vee \\ \vee (\exists F)(Fnc(F)\ \&\ Fnc(F^{-1})\ \&\ dom(F) = X\ \&\ rng(F) = V)].$$

A8 axiom of regularity can be written (in A1–A4, see [S1982]) as the set-formula

$$(\forall x)[(\exists y)(y \in x) \to (\exists y \in x)\ y \cap x = 0]\ \&\ (\forall x)(\exists y)[x \in y\ \&\ (\forall z \in y)(\forall q \in z)\ q \in y].$$

We denote by the symbol $\mathbf{AST}_{-\mathrm{i}}$ the theory $\mathbf{AST}$ without the axiom Ai.

The axiom of extensionality was formulated as a Π_1-formula, furthermore, let us realize this axiom cannot be in $\mathbf{AST}_{-1}$ simultaneously expressed as a Σ_1-formula (under the assumption of the consistency of $\mathbf{AST}$, of course), because such a reformulation is excluded by the possibility "to add a new copy" of a proper class (having the same elements as the original one but different from it) — such an extension does not change the validity of Σ_1-formulae.

The prolongation axiom is (in $\mathbf{TC}$) equivalent also to the Σ_2-formula

$$(\exists Y)(\forall F, G)([0 \in Y\ \&\ (\forall y \in Y)\ y \cup \{y\} \in Y]\ \&\ [(Fnc(F) \\ \&\ Fnc(G)\ \&\ F \subseteq G''Y) \to (\exists f)(Fnc(f)\ \&\ F \subseteq f)]),$$

because the equivalence

$$(\forall Y)([0 \in Y\ \&\ (\forall y \in Y)\ y \cup \{y\} \in Y] \to ((\exists G)[Fnc(G)\ \&\ F \subseteq G''Y] \equiv \\ \equiv (\exists G)[Fnc(G)\ \&\ F \subseteq G''FN]))$$

is provable in $\mathbf{TC}$ with the constant FN.

The axiom of choice is a Σ_2-formula and in [V1979] it is proved that it is equivalent to the axiom of extensional coding i.e. to the Π_2-formula

$$(\forall R)(\text{“}R \text{ is an equivalence”} \to (\exists X)[(\forall x \in dom(R))(\exists y \in X)(\langle x, y\rangle \in R)\ \& \\ \&\ (\forall x, y \in X)(x \neq y \to \langle x, y\rangle \notin R)]).$$

One can unfortunately show this equivalence (at the present, at least) only using the axiom of cardinalities (i.e. in the theory A1–A5, A7); luckily, the axiom of extensional coding together with the Π_2-formula

$$(\forall Y)(\exists R)(\text{“}R \text{ is a linear ordering of the universal class”} \ \& \\ \& \ [(0 \in Y \ \& \ (\forall y \in Y)\ y \cup \{y\} \in Y) \rightarrow \\ \rightarrow (\forall f)(Y \subseteq dom(f) \rightarrow (\exists z \in Y)\ \langle f(z \cup \{z\}), f(z)\rangle \notin R)])$$

is equivalent to the axiom of choice already in the theory A1–A5: let R be a linear ordering of the universal class with the property $(\forall f)(FN \subseteq dom(f) \rightarrow (\exists n \in FN)\langle f(n+1), f(n)\rangle \notin R)$ and let us define the equivalence S as the class $\{\langle\langle x,q\rangle,\langle y,q\rangle\rangle; \langle x,q\rangle \in R \ \& \ \langle y,q\rangle \in R \ \& \ x \neq q \neq y\}$, every selector of S chooses from each nonempty $\{x; \langle x,q\rangle \in R \ \& \ x \neq q\}$ one element and thus assuming $0 \neq X$ has no first element in the ordering R we can construct a function F on FN with $(\forall n \in FN)\langle F(n+1), F(n)\rangle \in R$ and by the prolongation axiom there is a function f with $(\forall n \in FN)\langle f(n+1), f(n)\rangle \in R$ which contradicts the stated property of the ordering R.

The axiom of cardinalities is in the theory A1–A6 equivalent to the formula

$$(\exists R)(We(V,R) \ \& \ (\forall x)[Fin(R''\{x\}) \vee Count(R''\{x\})])$$

which can be considered as a Σ_2-formula, because we have seen the formula $Fin(X) \vee Count(X)$ is equivalent to a Π_1-formula in the mentioned theory.

The order of the axioms A1–A7 is not quite random — the first three axioms express the principles accepted in all set theories with classes (in spite of the fact that in different theories a different strength of the second principle is required) and the axioms A4 and A5 can be considered as the most important axioms distinguishing **AST** from classical set theories — the first of these axioms guarantees that all sets are formally finite and the second one represents the human approach to the horizon as to a crossable (mentally, at least) boundary; on the other hand, the main objections appeared against the acceptance of the axiom A7, because it restricts infinite cardinalities to two types only (nevertheless from a different point of view A7 is no restriction since it guarantees many one-one mappings and we have seen that from the standpoint discussed in this paper both A7 and $\neg$A7 give the same chance for the rise of a rich system of classes; furthermore, there are important theorems of the alternative set theory the proofs of which essentially use A7). The above stated results seem to confirm the correctness of the chosen order of axioms — e.g. we found a Π_2-formula equivalent to the axiom of choice using the prolongation axiom and for the proof of the equivalence of the axiom of cardinalities and the Σ_2-formula in question we used both axioms A5 and A6. However, the question whether even a different order of axioms has similar properties has not been seriously studied up to now, e.g. it is not known if the axiom A7 can be rewritten as a Σ_2-formula in the theory A1–A4.

Let us note that the scheme of choice i.e. the assumption

$$(\forall x)(\exists Y)\Phi(x,Y) \rightarrow (\exists Z)(\forall x)\Phi(x, Z''\{x\})$$

required for every formula Φ simplifies the calculation of the complexity of formulae because it guarantees for each Σ_k-formula $\Phi(Z_0,\dots,Z_n)$ that the formula $(\forall z)\Phi(z,Z_1,\dots,Z_n)$ is (equivalent to) a Σ_k-formula, too.

We have shown that the choice of the axioms of the theory **AST** fulfils the principle "to investigate a system of classes as rich as possible"; in the following we show that the axioms of **AST** — except the axiom of extensionality — do not exclude a consistent addition of any Σ_1-formula (which is consistent with $\mathbf{ZF}_{\mathrm{Fin}}$, of course) as a supplementary axiom; in the following considerations we assume metamathematics is sufficiently strong (e.g. Zermelo-Fraenkel set theory, but the fourth order arithmetic is sufficient, too).

If $\mathbf{T}$ is a theory containing as axioms only set-formulae then we denote by the symbol A1 + $\mathbf{T}$ the theory with classes in which sets are defined as elements and the axioms of which are the axiom A1 and axioms of the theory $\mathbf{T}$. If $\mathfrak{A}$ is a model of a theory with the language containing classes and sets then the symbol $\mathbb{A}$ denotes its restriction to sets only — we define

$$\mathbb{A}=(\{x;\mathfrak{A}\models Set(x)\},\{\langle x,y\rangle;\mathfrak{A}\models[x\in y\ \&\ Set(x)\ \&\\ \&\ Set(y)]\},\{\langle x,y\rangle;\mathfrak{A}\models[x=y\ \&\ Set(x)\ \&\ Set(y)]\}).$$

Metatheorem. *Let* $\mathbf{T}$ *be a complete theory (containing set-formulae only) stronger (or equal) than* $\mathbf{ZF}_{\mathrm{Fin}}$ *and let* $\Phi_1(Z),\dots,\Phi_k(Z)$ *be normal formulae with one free variable. If theories* A1 + $\mathbf{T}$ + $(\exists Z)\Phi_i(Z)$ *are consistent for* $1\le i\le k$, *then the theory* $\mathbf{AST}+\mathbf{T}+(\exists Z)\Phi_1(Z)+\dots+(\exists Z)\Phi_k(Z)$ + scheme of choice *is consistent, too.*

Demonstration: Without loss of generality we can suppose the generalized continuum hypothesis; the theory $\mathbf{T}$ is assumed to be consistent and thus there is a countable model $\mathfrak{T}$ of it. Let $\mathfrak{S}=(S,\widetilde{E},Id)$ be the ultraproduct (with absolute equality, say) of the model $\mathfrak{T}$ over a nontrivial ultrafilter on ω and let $\mathfrak{A}$ be the expansion of $\mathfrak{S}$ by "all" subsets of its field (more precisely we put $q=\{x\subseteq S;\neg(\exists y\in S)\ x=\{z;\mathfrak{S}\models z\in y\}\}$ and assuming S and q are disjoint we define $\mathfrak{A}=(S\cup q,\widetilde{E}\cup E|q,Id))$. Then $\mathfrak{A}$ is a model of **AST** + scheme of choice (cf. [S1982]); the prolongation axiom is satisfied in $\mathfrak{A}$ because $\mathfrak{S}$ is saturated and the axiom of cardinalities and the scheme of choice in $\mathfrak{A}$ are consequences of the continuum hypothesis. By Loš' theorem $\mathfrak{A}\models\mathbf{T}$, moreover the class FN in the sense of $\mathfrak{A}$ is isomorphic to ω (in the following we neglect this isomorphism) and there is t with $[\mathfrak{A}\models Set(t)]\ \&\ \mathbf{T}=\{\Psi;\mathfrak{A}\models\Psi\in FN\cap t\}$. Therefore for $1\le i\le k$ we get $\mathfrak{A}\models$ "there is no proof of inconsistency of the theory A1 + $\mathbf{T}$ + $(\exists Z)\Phi_i(Z)$ (the code of) which is an element of FN", because for these i the theories A1 + $\mathbf{T}$ + $(\exists Z)\Phi_i(Z)$ are assumed to be consistent. Let us fix i so that $1\le i\le k$; the following considerations (cf. [P-S1984]) are done in the model $\mathfrak{A}$. Since we suppose that there is no proof of inconsistency of the theory A1 + $\mathbf{T}$ + $(\exists Z)\Phi_i(Z)$ (the code of) which is an element of FN, there is a model $\mathfrak{B}\models$ A1 + $\mathbf{T}$ + $(\exists Z)\Phi_i(Z)$ (with the absolute equality, say) and we are able to choose a revealment $\mathfrak{C}$ of $\mathfrak{B}$ (see [S-V1980]); according to the definition of revealment, the structure $\mathfrak{C}=(C,\overline{E},Id)$ is a model of the same theory. Therefore the structure $\mathbb{C}$ is a model of the theory $\mathbf{T}$ and hence it is elementarily equivalent to the structure (V, E, Id), thus there is an isomorphism F of these structures (see [S1982]; both structures are fully revealed). The isomorphism F determines the system $\mathfrak{M}$ of the images of classes of the model $\mathfrak{C}$ by F (in more detail: $\mathfrak{M} = \{F''Z; (\exists X \in C)\ Z = \{y; \mathfrak{C} \models y \in X\}\}$) and let $*$ denote the interpretation determined by this system (i.e. we define $Cls^*(X) \equiv X \in \mathfrak{M}$ and $\in$ and $=$ are left absolute). Then $[(\exists Z)\Phi_i(Z)]^*$, because using $\mathfrak{C} \models$ A1 one is able to show, by induction w.r.t. the complexity of a formula Ψ, the equivalence $[\mathfrak{C} \models \Psi(X_1, \ldots, X_m)] \equiv \Psi^*(F''\{q; \mathfrak{C} \models q \in X_1\}, \ldots, F''\{q; \mathfrak{C} \models q \in X_m\})$ for each $X_1, \ldots, X_m \in C$ (for $F(y) = F''\{q; \mathfrak{C} \models q \in y\}$ is satisfied by an arbitrary y with $\mathfrak{C} \models Set(y)$). According to the normality of the formula Φ_i we get $(\forall X \in \mathfrak{M})(\Phi_i^*(X) \equiv \Phi_i(X))$, and hence there must be a class X satisfying $\Phi_i(X)$. Summarizing we have proved $\mathfrak{A} \models \mathbf{AST} + \mathbf{T} + (\exists Z)\Phi_1(Z) + \cdots + (\exists Z)\Phi_k(Z) +$ scheme of choice. $\square$

As an evident consequence of the previous metatheorem we get the impossibility to express any of the axioms A5–A7 as a Π_1-formula in the theory arising from $\mathbf{AST}$ + the scheme of choice by the omission of the axiom in question. In fact, if for some $i = 5, 6, 7$ there would be a normal formula $\Phi(Z)$ with one free variable so that $\mathbf{AST}_{-\mathrm{i}}$ + the scheme of choice $\vdash$ Ai $\equiv (\forall Z)\Phi(Z)$, then A1 + $\mathbf{ZF}_{\mathrm{Fin}}$ + $(\exists Z)\neg\Phi(Z)$ would be consistent (the theory $\mathbf{AST}_{\mathrm{i}}$ + $\neg$Ai is consistent by [S1983] and [Ve1984]) and thus it would be possible to construct a complete theory $\mathbf{T}$ stronger than $\mathbf{ZF}_{\mathrm{Fin}}$ such that the theory A1 + $\mathbf{T}$ + $(\exists Z)\neg\Phi(Z)$ would be consistent and using the previous metatheorem we would obtain also the consistency of the theory $\mathbf{AST}$ + $\neg$Ai + the scheme of choice, which is absurd.

It is not difficult to show the axioms A5 and A7 are not expressible by Σ_1-formulae — moreover there are no closed Σ_1-formulae Θ_1, Θ_2 which could be consistently added to $\mathbf{AST}$ and such that Θ_1 would imply the axiom A5 in $\mathbf{AST}_{-5}$ + the scheme of choice or Θ_2 would imply A7 in $\mathbf{AST}_{-7}$ + the scheme of choice. Investigating the complexity of the prolongation axiom it is sufficient to choose (in ZF + CH, say) a countable model $\mathfrak{A} \models \mathbf{AST} + \Theta_1$ and define the structure $\mathfrak{B}$ as the structure arising by expansion of “all” subsets of the model $\mathbb{A}$ as classes. It is evident that $\mathfrak{B} \models \mathbf{AST}_{-5}$ + the scheme of choice and moreover $\mathfrak{B} \models \neg$A5 because $\mathfrak{B} \models (\exists F)$“$F$ is a one-one mapping of FN onto V”. Our assumption that Θ_1 is a Σ_1-formula and $\mathfrak{A} \models \Theta_1$ imply $\mathfrak{B} \models \Theta_1$, which contradicts $\mathfrak{B} \models \neg$A5 and the assumption $\mathbf{AST}_{-5}$ + the scheme of choice $\models \Theta_1 \rightarrow$ A5. Dealing with the axiom A7 we fix $\aleph_1$-saturated model $\mathfrak{A} \models \mathbf{AST} + \Theta_2$ such that $card(\{x; \mathfrak{A} \models Set(x)\}) = \aleph_2$ (one can construct such a model using the compactness theorem to the construction of a model $\mathfrak{C} \models \mathbf{AST} + \Theta_2$ for which $card(\{x; \mathfrak{C} \models Set(x)\}) = \aleph_2$ and define the model $\mathfrak{A}$ as the ultraproduct of $\mathfrak{C}$ over a nontrivial ultrafilter over ω). The structure $\mathfrak{B}$ arising by addition of “all” parts of the model $\mathbb{A}$ is a model of $\mathbf{AST}_{-7} + \neg$A7 + the scheme of choice and we can finish the proof as in the previous case.

The complexity of the axiom of choice was not seriously investigated up to now; it remains as an open problem whether there is a normal formula $\Phi(Z)$ with one free variable such that $\mathbf{AST}_{-6}$ + the scheme of choice $\vdash$ A6 $\equiv (\exists Z)\Phi(Z)$. In particular it is not known if there is a model $\mathfrak{A} \models \mathbf{AST}$ and its supermodel $\mathfrak{B}$ which is a model of the theory $\mathbf{AST}_{-6} + \neg$ A6 + the scheme of choice and such that $\mathbb{A} = \mathbb{B}$. Let us note that models fulfilling these conditions must satisfy $FN_{\mathfrak{B}} \subset FN_{\mathfrak{A}}$, because the axiom A6 is implied by the Σ_1-formula

$$(\exists R)[\text{"}R \text{ is a linear ordering of the universal class"} \ \& \\ \& \ (\forall f)(FN \subseteq dom(f) \rightarrow (\exists n \in FN)\langle f(n+1), f(n)\rangle \notin R)]$$

in the theory obtained from $\mathbf{AST}_{-6}$ by the extension of the language by the constant FN and by the extension of the axiomatic system by the formula defining the constant FN.

We have mentioned that there are formulae which are simultaneously equivalent in $\mathbf{AST}$ to a Π_2-formula and a Σ_2-formula. As an example of such formula can serve the formula expressing elementary equivalence of models (FN, E, Id) and (N, E, Id); this property is considered as a possible candidate for an axiom of the alternative set theory (the theory $\mathbf{AST}$ with this axiom is more similar to Robinson's nonstandard methods than $\mathbf{AST}$ itself), however the theory $\mathbf{AST}$ with the negation of the axiom of elementary equivalence is also very interesting — imagine e.g. the theory $\mathbf{AST}$ with "there is a proof of inconsistency of the formalization of $\mathbf{ZF}_{\mathrm{Fin}}$" (which is consistent relatively to $\mathbf{AST}$, but $\mathbf{AST} \vdash$ "there is no proof of inconsistency of the formalization of $\mathbf{ZF}_{\mathrm{Fin}}$ the code of which is an element of FN"). The property of elementary equivalence of models (FN, E, Id) and (N, E, Id) can be expressed (see [S1983]) by the Π_2-formula

$$(\forall K, R)[(GB(K, R) \ \& \ (\forall \widetilde{K}, \widetilde{R})(GB(\widetilde{K}, \widetilde{R}) \rightarrow \\ \rightarrow (\forall z \in \widetilde{K})(\exists q \in K) \ \widetilde{R}''\{z\} = R''\{q\})) \rightarrow \\ \rightarrow (\forall x \in K)(\forall y \in N)(R''\{x\} = \{y\} \rightarrow y \in FN)]$$

and also as a Σ_2-formula

$$(\exists K, R)[GB(K, R) \ \& \ (\forall x \in K)(\forall y \in N)(R''\{x\} = \{y\} \rightarrow y \in FN)]$$

(let us remind the formula $y \in FN$ can be considered as a Π_1-formula according to the definition of this class).

Let $\mathbf{S}$ be a theory stronger or equal to A1 + $\mathbf{ZF}_{\mathrm{Fin}}$ and weaker or equal to $\mathbf{AST}$ + the scheme of choice. The following result shows that there are no proper Δ_1-formulae in $\mathbf{S}$, because each Δ_1-formula in $\mathbf{S}$ is equivalent to a set-formula in this theory. Let us note that the result we are going to formulate can be essentially generalized to theories quite different from $\mathbf{AST}$, however, such a generalization exceeds the problems of this paper.

Metatheorem. *For every pair of normal formulae $\Phi_1(Z)$ and $\Phi_2(Z)$ with one free variable such that*

$$\mathbf{AST} + \text{scheme of choice} \ \vdash (\exists Z)\Phi_1(Z) \equiv (\forall Z)\Phi_2(Z)$$

there is a set-formula Ψ *with*

$$\mathrm{A1} + \mathbf{ZF}_{\mathrm{Fin}} \vdash ([(\exists Z)\Phi_1(Z) \rightarrow \Psi] \ \& \ [\Psi \rightarrow (\forall Z)\Phi_2(Z)]).$$

Demonstration: Let the symbol [0] denote the negation and [1] denote the empty symbol. We are going to demonstrate our result in two steps.

(a) At first let us assume that there are closed set-formulae $\Psi_1, \ldots, \Psi_k$ such that for each k-tuple of zeros and ones f (at least) one of the theories

$$\mathrm{A1} + \mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + (\exists Z)\Phi_1(Z)$$

and

$$\mathrm{A1} + \mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + \neg(\forall Z)\Phi_2(Z)$$

is inconsistent. Let Ψ denote the disjunction of all formulae of the form $[f(1)]\Psi_1$ & $\ldots$ & $[f(k)]\Psi_k$ where f is a k-tuple of zeros and ones such that the theory A1 + $\mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + \neg(\forall Z)\Phi_2(Z)$ is inconsistent. The negation of the formula Ψ is equivalent (in the predicate calculus) to the disjunction of (some) formulae of the form $[f(1)]\Psi_1$ & $\ldots$ & $[f(k)]\Psi_k$ where f is a k-tuple of zeros and ones such that the theory A1 + $\mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + (\exists Z)\Phi_1(Z)$ is inconsistent. If the theory A1 + $\mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + \Theta$ is inconsistent then A1 + $\mathbf{ZF}_{\mathrm{Fin}} \vdash [f(1)]\Psi_1$ & $\ldots$ & $[f(k)]\Psi_k \rightarrow \neg\Theta$ and therefore using the form of formulae the disjunction of which are the formulae Ψ and $\neg\Psi$ we get

$$\mathrm{A1} + \mathbf{ZF}_{\mathrm{Fin}} \vdash ([\neg\Psi \rightarrow \neg(\exists Z)\Phi_1(Z)] \ \& \ [\Psi \rightarrow \neg\neg(\forall Z)\Phi_2(Z)]).$$

(b) To obtain a contradiction let us suppose that there is no k-tuple of set-formulae with the property we dealt with in the previous step. Let us enumerate all set-formulae and let us construct by induction a complete theory $\mathbf{T}$ (consisting of set-formulae only) stronger than $\mathbf{ZF}_{\mathrm{Fin}}$ and such that both theories A1 + $\mathbf{T}$ + $(\exists Z)\Phi_1(Z)$ and A1 + $\mathbf{T} + (\exists Z)\neg\Phi_2(Z)$ are consistent. In more detail: we deal with the tree of formulae of the form $[f(1)]\Psi_1$ & $\ldots$ & $[f(k)]\Psi_k$, where $\Psi_1, \ldots, \Psi_k$ are first formulae in the chosen enumeration and f is a k-tuple of zeros and ones such that both theories A1 + $\mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + (\exists Z)\Phi_1(Z)$ and A1 + $\mathbf{ZF}_{\mathrm{Fin}} + [f(1)]\Psi_1 + \cdots + [f(k)]\Psi_k + (\exists Z)\neg\Phi_2(Z)$ are consistent and the ordering of which is defined so that $[f(1)]\Psi_1$ & $\ldots$ & $[f(i)]\Psi_i$ is before $[f(1)]\Psi_1$ & $\ldots$ & $[f(k)]\Psi_k$ for $1 \leq i \leq k$. Each level of the tree is finite (because the set of all k-tuples of zeros and ones is finite) and assuming the negation of the assumption mentioned in the case (a) the investigated tree is infinite. Hence there must exist an infinite branch of this tree and it gives us a complete theory $\mathbf{T}$ stronger than $\mathbf{ZF}_{\mathrm{Fin}}$ such that for each finite subtheory $\mathbf{S}$ of $\mathbf{T}$ both theories A1 + $\mathbf{ZF}_{\mathrm{Fin}} + \mathbf{S} + (\exists Z)\Phi_1(Z)$ and A1 + $\mathbf{ZF}_{\mathrm{Fin}} + \mathbf{S} + (\exists Z)\neg\Phi_2(Z)$ are consistent; the use of the compactness theorem gets the consistency of both theories A1 + $\mathbf{T} + (\exists Z)\Phi_1(Z)$ and A1 + $\mathbf{T}$ + $(\exists Z)\neg\Phi_2(Z)$. If $\mathbf{T}$ is a theory with the required properties, then the theory $\mathbf{AST}$ + $\mathbf{T} + (\exists Z)\Phi_1(Z) + (\exists Z)\neg\Phi_2(Z)$ + scheme of choice is consistent according to the previous metatheorem, which contradicts the assumption **AST** + scheme of choice $\vdash (\exists Z)\Phi_1(Z) \equiv (\forall Z)\Phi_2(Z)$. □

The metatheorems proved in the paper can be even strengthened — e.g. one can replace "scheme of choice" by the stronger "axiom of constructibility" introduced in [S1985]. We have indicated in some cases a theory containing the axiom A4 as a theory sufficient for the provability of an equivalence, however, in all such cases it is enough to use only particular set-formulae which are consequences of A4 (the negation of the axiom of infinity, the powerset axiom and so on).

In the end let us summarize some open problems in the area we dealt with:

(a) Is the equivalence of the axioms of choice and extensional coding provable in the theory A1–A5 and A8? (Is there a Σ_2-formula equivalent to the axiom of extensional coding in the theory A1–A5 (or even in A1–A4)?)

(b) Is there a Π_2-formula equivalent to the axiom of choice in the theory A1–A4?

(c) Is there a Σ_1-formula which is equivalent to the axiom of choice in the theory $\mathbf{AST}_{-6}$?

(d) Is the axiom of cardinalities equivalent to a Σ_2-formula in the theory A1–A5 (or even in A1–A4)?

(e) The axiom of constructibility was formulated as a Σ_3-formula; is it possible to express it by a formula with lower complexity?

References

[B] Bernays P., *A system of axiomatic set theory*, JSL **2** (1937), 65–77.

[G] Gödel Kl., *The consistency of the axiom of choice and of the general continuum hypothesis*, Ann. of Math. Studies, Princeton, 1940.

[P-S1984] Pudlák P., Sochor A., *Models of the alternative set theory*, JSL **49** (1984), 570–585.

[S1979] Sochor A., *Metamathematics of the alternative set theory I*, Comment. Math. Univ. Carolinae **20** (1979), 697–722.

[S1982] ———, *Metamathematics of the alternative set theory II*, Comment. Math. Univ. Carolinae **23** (1982), 55–79.

[S1983] ———, *Metamathematics of the alternative set theory III*, Comment. Math. Univ. Carolinae **24** (1983), 137–154.

[S1985] ———, *Constructibility and shiftings of view*, Comment. Math. Univ. Carolinae **26** (1985), 477–498.

[S-V1980] Sochor A., Vopěnka P., *Revealments*, Comment. Math. Univ. Carolinae **21** (1980), 97–118.

[S-V1981] ———, *The axiom of reflection*, Comment. Math. Univ. Carolinae **22** (1981), 87–111.

[S-V1983] ———, *Shiftings of the horizon*, Comment. Math. Univ. Carolinae **24** (1983), 127–136.

[Sg1986] Sgall J., *Construction of the class FN*, Comment. Math. Univ. Carolinae **27** (1986), 435–436.

[Ve1984] Vencovská A., *Independence of the axiom of choice in the alternative set theory*, Open days in model theory and set theory, Proceedings 1981 Jadwisin (Leeds 1984).

[V1979] Vopěnka P., *Mathematics in the Alternative Set Theory*, TEUBNER TEXTE, Leipzig, 1979.

[V1989] ______, *Úvod do matematiky v alternatívnej teórii množín* (in Slovak), ALFA Bratislava, 1989.

Mathematical Institute, Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic

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