Dirac operators on hypersurfaces

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Abstract. In this paper some relation among the Dirac operator on a Riemannian spin-manifold N, its projection on some embedded hypersurface M and the Dirac operator on M with respect to the induced (called standard) spin structure are given.

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1. Introduction

The Dirac operator D belongs to intensively studied operators on manifolds. It usually can be defined on a Riemannian spin-manifold and depends on its spin structure. For flat space, it is intensively studied in Clifford analysis and solutions of the equation $D\phi = 0$, called the Dirac equation (for spinor-valued or Clifford algebra-valued functions ϕ) are described in several ways. In the present paper, some relations among the Dirac operator defined on a given Riemannian spin-manifold N, its projection on some embedded hypersurface M and the Dirac operator on M with respect to the induced spin structure are given. Notations and basic facts from [5] and [9] (or [6]) are used.

2. General theory

2.1 Clifford algebras and spinors.

Let us introduce only some notations and conventions; more details can be found e.g. in [9], [2], [6]. A spinor space $\mathbf{S_n}$ is an irreducible representation of the Clifford algebra $\mathbf{R_{0,n}}$, the corresponding Spin representation is simply the restriction of action of $\mathbf{R_{0,n}}$ to the group Spin $(,\mathbf{R})(n) \subset \mathbf{R_{0,n}}$. Action of $\mathbf{R^n} \subset \mathbf{R_{0,n}}$ on $\mathbf{S_n}$ gives us a bilinear map $\tilde{\mu} : \mathbf{R^n} \times \mathbf{S} \longrightarrow \mathbf{S}$ which induces a linear map $\mu : \mathbf{R^n} \otimes \mathbf{S} \longrightarrow \mathbf{S}$.

2.2 The Dirac operator on Riemannian spin-manifolds.

Let (M, g) be an n-dimensional Riemannian oriented spin-manifold, let $P \to M$ be the principal fibre bundle of orthonormal oriented frames and let $\tilde{P} \to P$ be a spin structure on M. Let ω be an so $(,\mathbf{R})(n)$-valued 1-form on P which corresponds to the Levi-Civita connection. Then there exists a unique spin(n)-valued 1-form $\tilde{\omega}$ on $\tilde{P}$ which is a lifting of ω and gives a canonical connection on $\tilde{P}$. We have the following diagram of maps.

$$\begin{array}{ccc} T(\tilde{P}) & \xrightarrow{\tilde{\omega}} & \operatorname{spin}(n) \\ \downarrow & & \downarrow \lambda_* \\ T(P) & \xrightarrow{\omega} & \operatorname{so}(,\mathbf{R})(n)\,. \end{array}$$

The connection $\tilde{\omega}$ induces a covariant derivative ∇^s on the associated spinor bundle S over M. The Clifford multiplication $\mu : \mathbf{R^n} \otimes \mathbf{S} \longrightarrow \mathbf{S}$ induces a vector bundle homomorphism $\mu : TM \otimes S \longrightarrow S$. If h is an identification of $TM \leftrightarrow TM^*$ defined by the Riemannian metric then $\mathbf{D} := \mu \circ (h \otimes id) \circ \nabla^s$ is called the Dirac operator on M.

Let us introduce a more convenient notation, namely denote $\mu(v, \xi) := v \bullet \xi$ for $v \in \mathbf{R^n}$ and $\xi \in \mathbf{S_n}$ and $\mu(X, \xi) := X \bullet \xi$ for a vector field X and a spinor field ξ on M.

Locally the Dirac operator can be described in the following way:

Theorem 1. *Let $(e_1, e_2, ..., e_n)$ be a local orthonormal frame on an open subset $U \subset M$. Then we have*

$$\mathbf{D} = \sum_{i=1}^{n} e_i \bullet \nabla^s_{e_i}$$

on U.

2.3 Local computations.
Recall that a basis for spin$\,(n)$ is $\{\mathbf{e_i e_j},\ i < j\}$. Let $\tilde{s} : U \to \tilde{P}$ be a local section (spin frame) on U. Then we have the following trivializations of bundles on U:

$$T(U) = U \times \mathbf{R^n},\ \mathcal{C}(U) = U \times \mathbf{R_{0,n}} \ \text{ and } \ S(U) = U \times \mathbf{S}\,.$$

For a basis $(\xi_{\mathbf{1}}, \xi_{\mathbf{2}}, ..., \xi_{\mathbf{N}})$ of the spinor space $\mathbf{S}$, let us denote by $\xi_1 = [(\tilde{s}, \xi_{\mathbf{1}})], ..., \xi_N = [(\tilde{s}, \xi_{\mathbf{N}})]$ the corresponding sections of spinor bundle S on U.

It is possible to write the connection form $\tilde{\omega}$ on U in the form

$$\tilde{\omega} = \sum_{i<j} \tilde{\omega}_{ij} \mathbf{e_i e_j}\,,$$

where $\tilde{\omega}_{ij}$ are 1-forms on U. It is convenient to define $\tilde{\omega}_{ji} := -\tilde{\omega}_{ij}$ for $j > i$.

The covariant derivative ∇^S corresponding to $\tilde{\omega}$ is defined by

$$\nabla^S \xi_r = \sum_{i<j} \widetilde{\omega_{ij}} [\tilde{s}, \mathbf{e_i e_j} \bullet \xi_r] = \sum_{i<j} \widetilde{\omega_{ij}} \mathbf{e_i e_j} \bullet \xi_r\,.$$

It remains to express the forms $\widetilde{\omega}_{ij}$ using the forms ω_{ij} of the Levi-Civita connection. Let U be a simply connected domain. The connection form ω on U has the following form

$$\omega = \sum_{i<j} \omega_{ij} \mathbf{E_{ij}}\,,$$

where $\mathbf{E_{ij}}$ is a canonical basis for so$\,(, \mathbf{R})(n)$. Put again $\omega_{ij} := -\omega_{ji}$ for $i > j$. Then from the diagram 1 we get

$$\lambda_* (\sum_{i<j} \widetilde{\omega}_{ij} \mathbf{e_i e_j}) = 2 \sum_{i<j} \widetilde{\omega}_{ij} \mathbf{E_{ij}} = \sum_{i<j} \omega_{ij} \mathbf{E_{ij}}$$

and $\widetilde{\omega}_{ij} = \frac{1}{2}\omega_{ij}$.

For a local orthonormal frame $s = (e_1, ..., e_n)$ on M we have the following formulas:

$$\omega_{ij} = g(\nabla e_i, e_j)\,,$$

i.e.

$$\omega_{ij}(e_k) = g(\nabla_{e_k} e_i, e_j) := \Gamma^j_{ki}$$

and

$$\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k\,.$$

If $[e_p, e_q] = \sum_k c^r_{pq} e_r$ then

$$\Gamma^k_{ij} = \frac{1}{2}(c^i_{kj} - c^k_{ji} + c^j_{ki})\,.$$

Finally if we put for $\xi \in \Gamma(U, S)$, $\xi = \sum_r \alpha^r \xi_r$

$$X\xi = \sum_r X(\alpha^r)\xi_r\,,\ \mathbf{e_j} \bullet \xi = \sum_r \alpha^r(\mathbf{e_j} \bullet \xi_r)\,,$$

we get

a) a local expression for spinor connection

$$\nabla^S_X = X(\xi) + \frac{1}{2}\sum_{p<q} \omega_{lm}(X)\mathbf{e_p}\mathbf{e_q} \bullet \xi$$

b) a local expression for the Dirac operator

$$\mathbf{D}\xi = \sum_j \mathbf{e_j} \bullet (e_j(\xi) + \frac{1}{2}\sum_{p<q} \omega_{pq}(e_j)\mathbf{e_p}\mathbf{e_q} \bullet \xi)$$

or

$$\mathbf{D}\xi = \sum_j \mathbf{e_j} \bullet (e_j(\xi) + \frac{1}{2}\sum_{p<q} \Gamma^q_{jp}\mathbf{e_p}\mathbf{e_q} \bullet \xi)\,.$$

3. Differential operators on submanifolds

First of all recall how a differential operator (on functions) on a Riemannian manifold defines an operator (on functions) on any submanifold of N (see [7]).

Let M be a submanifold of a Riemannian manifold N, let P be a differential operator on N, i.e.

$$P : \mathcal{C}^\infty_c(N) \to \mathcal{C}^\infty_c(N)\,,$$

where $\mathcal{C}^\infty_c(N)$ denotes the space of all smooth functions on N with a compact support.

Let us define an operator $\pi_M P$ called the projection of P on M as follows:

For every point $x \in M$, let us construct geodesics in N starting in x and orthogonal to M. Taking small enough pieces of such a geodesics, we get a submanifold $V_x^\perp$ in N. Moreover for a point $y \in M$ we can take a neighborhood $U(y) \subset M$ in such a way that (after a little change if necessary) the submanifolds $V_x^\perp$ for $x \in U(y)$ do not intersect. Then we get a neighborhood $\hat{U}(y) = \cup_{x \in U(y)} V_x^\perp$ of y in N. Let us call such a neighborhood geodesic-tubular neighborhood and write shortly g-tubular neighborhood of $y \in M$ in N.

If we have a function $F \in \mathcal{C}_c^\infty(M)$ and a point $y \in M$, we can extend $F_{/U}$ to a function $\hat{F}_{/\hat{U}}$ constantly on every $V_x^\perp$ for $x \in U(y)$ and put

$$\pi_M(P)(F)(y) := P(\hat{F}_{/\hat{U}})(y) .$$

The operator P on N does not increase supports, hence the operator $\pi_M P$ is a well defined operator on M.

Let us denote the Laplace-Beltrami operator on a Riemannian manifold X by Δ_X.

Then we have:

Theorem 2 ([7]). *Let M be a submanifold of a Riemannian manifold N, then*

$$\pi_M(\Delta_N) = \Delta_M .$$

Remark 3.1. As we shall see later, we can similarly define a projection of the Dirac operator from Riemannian spin manifold N to an oriented hypersurface M (which is a spin manifold with the induced "standard" spin structure), because we are able to imbed canonically the spinor bundle on M to the spinor bundle on N. We have a possibility to extend a spinor field from the hypersurface constantly to the g-tubular neighborhood and to define the projection as explained above.

3.1 Spin structures on submanifolds.

3.1.1 Algebraic preliminaries.

For any $n \in \mathbf{Z}_+$ we can define natural imbeddings

$$\mathrm{Spin}\,(2n, \mathbf{R}) \subset \mathrm{Spin}\,(2n+1, \mathbf{R}) \subset \mathrm{Spin}\,(2n+2, \mathbf{R})$$

induced by natural imbeddings of the corresponding Clifford algebras

$$\mathbf{R}_{0,2n} \subset \mathbf{R}_{0,2n+1} \subset \mathbf{R}_{0,2n+2}$$

and we can also discuss the relations among the corresponding basic spinor spaces, considered as Clifford modules.

We have the following situation:

a) For m even, $m = 2n+2$, there is a unique spinor space $\mathbf{S}_{2n+2}$ which is irreducible as a module over ${\mathbf{R}_{0,2n+2}}^+$ but decomposes as $\mathrm{Spin}\,(2n+2, \mathbf{R})$-module as follows:

$$\mathbf{S}_{2n+2} = {\mathbf{S}_{2n+2}}^+ \oplus {\mathbf{S}_{2n+2}}^- , \tag{1}$$

where the decomposition is given by eigenspaces of the multiplication by an element $\omega := e_1 ... e_{2n+2}$ from the left.

b) For m odd $m = 2n+1$ there are two spinor spaces $\mathbf{S}_{2n+1}$ and $\widehat{\mathbf{S}_{2n+1}}$ which can be identified with the corresponding spinor spaces for dimension $2n+2$ as follows:

$$\mathbf{S}_{2n+1} := {\mathbf{S}_{2n+2}}^{+}\,,\ \widehat{\mathbf{S}_{2n+1}} := {\mathbf{S}_{2n+2}}^{-}\,.$$

c) For m even, $m = 2n$ there is again a unique spinor space $\mathbf{S}_{2n}$ which is irreducible as a module over $\mathbf{R}_{0,2n}$ and decomposes as in a) as a Spin $(2n, \mathbf{R})$ module as follows

$$\mathbf{S}_{2n} = {\mathbf{S}_{2n}}^{+} \oplus {\mathbf{S}_{2n}}^{-}\,, \tag{2}$$

we can identify again

$$\mathbf{S}_{2n} := \mathbf{S}_{2n+1} \quad \text{or} \quad \mathbf{S}_{2n} := \widehat{\mathbf{S}_{2n+1}}\,.$$

We shall use the following description of the spinor spaces and also the calculus which follows from it (see [5]). The element $\mathbf{I_{n+1}}$ is an idempotent in $\mathbf{R_{0,n}}$.

$$\mathbf{S}_{2n+2} := \Lambda \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1} = \Lambda^{ev} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1} \oplus \Lambda^{odd} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1}\,,$$

further we put

$$\mathbf{S}_{2n+1} := \Lambda^{ev} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1}\,,\ \mathbf{S}_{2\hat{n}+1} := \Lambda^{odd} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1}\,,$$

and

$$\mathbf{S}_{2n} := \Lambda^{ev} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1} = {\mathbf{S}_{2n}}^{+} \oplus {\mathbf{S}_{2n}}^{-}\,,$$

where

$$\begin{aligned} {\mathbf{S}_{2n}}^{+} &:= \{\xi \bullet \mathbf{I}_{n+1} \in \Lambda^{ev} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1} | \bar{f}_{n+1} \xi = 0\} \\ {\mathbf{S}_{2n}}^{-} &:= \{\xi \bullet \mathbf{I}_{n+1} \in \Lambda^{ev} \mathbf{W}_{n+1} \bullet \mathbf{I}_{n+1} | f_{n+1} \xi = 0\}\,. \end{aligned}$$

An intertwining map ϕ between isomorphic $\mathbf{R}_{0,2n+1}$ representations $\mathbf{S}_{2n+1}$ and $\widehat{\mathbf{S}_{2n+1}}$ is defined by:

$$\phi((s + l f_{n+1}) \mathbf{I}_{n+1}) := (-1)^n \mathbf{i} (l + s f_{n+1}) \mathbf{I}_{n+1};\ s \in \Lambda^{ev} \mathbf{W}_{n+1},\ l \in \Lambda^{odd} \mathbf{W}_{n+1}$$

and the vector e_{2n+2} acts on $\mathbf{S}_{2n+2}$ with respect to the decomposition (1) as

$$e_{2n+2}[u + \hat{v}] = (-1)^n \mathbf{i} [v + \hat{u}]\,.$$

Finally the action of e_{2n+1} on the space $\mathbf{S}_{2n} := \mathbf{S}_{2n+1}$ with respect to the decomposition (2) is the following

$$e_{2n+1}[s + l f_{2n+1}] \mathbf{I}_{n+1} = (-1)^n \mathbf{i} [s - l f_{2n+1}] \mathbf{I}_{n+1};\ s \in \Lambda^{ev} \mathbf{W}_{n+1},\ l \in \Lambda^{odd} \mathbf{W}_{n+1}\,.$$

Remark 3.2. We shall use the isomorphism of $\Lambda\mathbf{W}_n \simeq \Lambda^{ev}\mathbf{W}_{n+1}$ given by

$$s + l \to s + l f_{n+1} ,$$

where s is an even element and l is an odd element of $\mathbf{W}_n$.

3.1.2 Spin-structures on submanifolds of codimension 1.
Let M^m be an oriented submanifold of codimension 1 in the Riemannian spin manifold N^{m+1}. Then there exists a uniquely defined standard (with respect to the embedding) spin-structure on M^m induced from the spin structure on N^{m+1}. This spin-structure is defined in the following way: using the unit normal field on M^m, an embedding of $\mathcal{B}_{SO}(M)$ into $\mathcal{B}_{SO}(N)$ on M^m is defined and then we take the corresponding pull-back of $\mathcal{B}_{SO}(M)$ in $\mathcal{B}_{\mathrm{Spin}}(N)$.

Remark 3.3. Let M be a submanifold of arbitrary codimension in a Riemannian spin manifold N and let $\mathcal{N}_M$ be the normal bundle of M in N. For each spin structure on M a unique spin structure on the normal bundle $\mathcal{N}_M$ can be defined such that their direct sum is the restriction on M of given spin structure on N. Also for any spin structure on the normal bundle $\mathcal{N}_M$ there correspond spin structure on M such that their direct sum is the restriction on M of given spin structure on N. If M is an oriented hypersurface in N, the standard spin structure on M corresponds to the trivial spin-structure on $\mathcal{N}$ (unconnected 2-1 covering). Generally spin-structures on an oriented hypersurface M in N correspond to the double covering of M.

For the corresponding spinor bundles we get the following possibilities which differ in the even and odd cases:

1) For m even, $m = 2n$, there is an isomorphism of bundles

$$\mathcal{S}_{M^{2n}} \equiv \mathcal{S}_{N^{2n+1}}/M^{2n}$$

and we get a picture:

$$\begin{tikzcd}
\mathcal{S}_M^+ \oplus \mathcal{S}_M^- \arrow[d] & \\
\mathcal{S}_M \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi"] & \mathcal{S}_N \arrow[d, "\hat{\pi}"] \\
M^{2n} \arrow[r, "\iota"] & N^{2n+1} .
\end{tikzcd}$$

If ξ is a unit normal field on M^{2n} in N^{2n+1}, then for any spinor field ϕ on N^{2n+1} (resp. on a neighborhood of M^{2n} in N^{2n+1}) we have the following formulas:

a) For the action of the normal field ξ on the spinor space

$$\xi \bullet (\phi^+ + \phi^-) = \mathbf{i}(-1)^n(\phi^+ - \phi^-) .$$

b) For the covariant derivatives ∇^{S_M} and ∇^{S_N} on the corresponding spinor bundles

$$\nabla_X^{S_M}(\phi/M) = (\nabla_X^{S_N}\phi)/M + \frac{1}{2}(\nabla_X^N\xi) \bullet \xi \bullet \phi$$

for $X \in T_xM$, $x \in M$.

c) For the corresponding Dirac operators D_N on N^{2n+1} and D_M on M^{2n}:

$$D_M(\phi/M) = (D_N\phi)/M + \frac{1}{2}\sum_{j=1}^{2n} e_j \bullet (\nabla^N_{e_j}\xi) \bullet \xi \bullet \phi - \xi \bullet \nabla^{S_N}_{\xi}\phi\,.$$

d) Let e_j be principal directions on M and let λ_j be the corresponding principal curvatures, i.e.

$$\nabla^N_{e_j}\xi = -\lambda_j e_j\,.$$

Then we can write

$$D_M(\phi/M) = (D_N\phi)/M - \frac{1}{2}\sum_{j=1}^{2n} \lambda_i \xi \bullet \phi - \xi \bullet \nabla^{S_N}_{\xi}\phi$$

and

$$D_M(\phi/M) = (D_N\phi)/M - nH\xi \bullet \phi - \xi \bullet \nabla^{S_N}_{\xi}\phi\,,$$

where $H = \frac{1}{2n}\sum_{j=1}^{2n}\lambda_j$ is the mean curvature of M in N.

2) For m odd, $m = 2n+1$, there is an isomorphism of bundles $\mathcal{S}_M \oplus \widetilde{\mathcal{S}}_M$ and restriction of $\mathcal{S}_N$ on M. We have the following picture:

$$\begin{tikzcd}
\mathcal{S}_M \oplus \widetilde{\mathcal{S}}_M \arrow[d] \arrow[drr, "\equiv"] & & \\
\mathcal{S}_M \arrow[rr, "\tilde{\iota}"] \arrow[d, "\pi"] & & \mathcal{S}_N \arrow[d, "\hat{\pi}"] \\
M^{2n+1} \arrow[rr, "\iota"] & & N^{2n+2}\,.
\end{tikzcd}$$

If ξ is a unit normal field on M^{2n+1} in N^{2n+2} then for any spinor field ϕ on N^{2n+2} (resp. on a neighborhood of M^{2n+1} in N^{2n+2}) we have the following formulas:

a) For the action of the normal field ξ on the spinor space

$$\xi \bullet (\phi_1 + \widehat{\phi_2}) = \mathbf{i}(-1)^n(\phi_2 + \widehat{\phi_1})\,.$$

b) For the covariant derivatives ∇^{S_M} and ∇^{S_N} on the corresponding spinor bundles

$$(\nabla^{S_N}_X\phi)/M = \nabla^{S_M}_X(\phi_1/M) + \nabla^{\hat{S}_M}_X(\hat{\phi}_2/M) - \frac{1}{2}\nabla^N_X\xi \bullet \xi \bullet \phi\,,$$

where $X \in T_xM$, $x \in M$ and where $\phi = \phi_1 + \widehat{\phi_2}$ is a decomposition of ϕ with respect to the identification described above.

c) For the corresponding Dirac operators D_N on N^{2n+2} and D_M, $\hat{D}_M$ on M^{2n+1}:

$$(D_N\phi)/M = D_M(\phi_1/M) + \widehat{D}_M(\widehat{\phi}_2/M) - \frac{1}{2}\sum_{j=1}^{2n+1} e_j \bullet (\nabla^N_{e_j}\xi) \bullet \xi \bullet \phi + \xi \bullet \nabla^{S_N}_{\xi}\phi\,.$$

d) Let e_j be principal directions on M and let λ_j be the corresponding principal curvatures, i.e.

$$\nabla^N_{e_j}\xi = -\lambda_j e_j\,.$$

Then we can write

$$(D_N\phi)/M = D_M(\phi_1/M) + \widehat{D}_M(\widehat{\phi}_2/M) + \frac{1}{2}\sum_{j=1}^{2n+1} \lambda_j \xi \bullet \phi + \xi \bullet \nabla^{S_N}_{\xi}\phi$$

and if e.g. $\widehat{\phi}_2 \equiv 0$ we get

$$D_M(\phi_1/M) = (D_N\phi_1)/M - \frac{2n+1}{2}H\xi \bullet \phi - \xi \bullet \nabla^{S_N}_{\xi}\phi\,,$$

where $H = \frac{1}{2n}\sum_{j=1}^{2n}\lambda_j$ is the mean curvature of M in N.

Moreover for both cases (odd or even dimensional), we have the following formula for the projection of the Dirac operator from N^{k+1} to M^k:

$$\pi_M(D_N)\psi = D_M\psi + \frac{k}{2}H \cdot \xi \cdot \psi\,,$$

where ψ is a spinor field on M.

Example. The sphere Σ^m in $\mathbf{R}^{m+1}$.

There is a unique spin structure on the sphere Σ^m defined by the following diagram:

$$\begin{tikzcd} \mathrm{Spin}\,(,\mathbf{R})(m+1,\mathbf{R}) \arrow[d] \arrow[dr, "\tilde{\pi}"] & \\ SO(m+1,\mathbf{R}) \arrow[r, "\pi"] & \Sigma^m\,. \end{tikzcd}$$

The sphere Σ^m is a homogeneous space $SO(m+1,\mathbf{R})/SO(m,\mathbf{R})$ and can be represented as an orbit of the point $P := [0,\dots,0,r]$ in $\mathbf{R}^{m+1}$.

It can be also represented as a homogeneous space $\mathrm{Spin}\,(m+1,\mathbf{R})/\mathrm{Spin}\,(m,\mathbf{R})$, so the maps in the picture above are well defined, and we have a quite natural identification

$$\mathcal{B}_{SO}(\Sigma^m) \equiv SO(m+1,\mathbf{R})\,,\ \mathcal{B}_{\mathrm{Spin}}\,(\Sigma^m) \equiv \mathrm{Spin}\,(m+1,\mathbf{R})\,.$$

Moreover using the natural isomorphisms of spinor spaces in (3.1), we have the following diagram for the sphere:

$$\begin{tikzcd} \mathrm{Spin}\,(m+1,\mathbf{R}) \arrow[r, "\tilde{\iota}"] \arrow[d, "\pi"] & \mathcal{B}_{\mathrm{Spin}}\,(\mathbf{R}^{m+1}) = \mathbf{R}^{m+1} \times \mathrm{Spin}\,(m+1,\mathbf{R}) \arrow[d, "\hat{\pi}"] \\ \Sigma^m \arrow[r, "\iota"] & \mathbf{R}^{m+1}\,. \end{tikzcd}$$

The group Spin $(m+1, \mathbf{R})$ is embedded as Spin $(m, \mathbf{R})$-principal fibre subbundle into $\mathcal{B}_{\text{Spin}}(\mathbf{R}^{m+1})$ by

$$s \in \text{Spin}\,(m+1, \mathbf{R}) \longmapsto [\tilde{\pi}(s), s]\,.$$

The corresponding associated spinor bundles are also related, but there is a difference between odd and even dimensional cases: a) For m even, $m = 2n$,

$$\begin{tikzcd} \mathcal{S}_{\Sigma^{2n}} \arrow[r, "\tilde{\upsilon}"] \arrow[d, "\pi"] & \mathcal{S}_{\mathbf{R}^{2n+1}} = \mathbf{R}^{2n+1} \times \mathbf{S}_{2n+1} \arrow[d, "\pi_1"] \\ \Sigma^{2n} \arrow[r, "\iota"] & \mathbf{R}^{2n+1} \end{tikzcd}$$

and after the restriction on Σ^{2n}, we have a trivialization of $\mathcal{S}_{\Sigma^{2n}}$:

$$\mathcal{S}_{\Sigma^{2n}} \leftrightarrow \Sigma^{2n} \times \mathbf{S}_{2n+1}\,.$$

b) for m odd, $m = 2n+1$,

$$\begin{tikzcd} \mathcal{S}_{\Sigma^{2n+1}} \oplus \widetilde{\mathcal{S}_{\Sigma^{2n+1}}} \arrow[d] \arrow[dr] & \\ \mathcal{S}_{\Sigma^{2n+1}} \arrow[r, "\tilde{\upsilon}"] \arrow[d, "\pi"] & \mathcal{S}_{\mathbf{R}^{2n+2}} = \mathbf{R}^{2n+2} \times \mathbf{S}_{2n+2} \arrow[d, "\pi_1"] \\ \Sigma^{2n+1} \arrow[r, "\iota"] & \mathbf{R}^{2n+2} \end{tikzcd}$$

and after the restriction to the sphere Σ^{2n+1}, we have a trivialization

$$\mathcal{S}_{\Sigma^{2n+1}} \oplus \widetilde{\mathcal{S}_{\Sigma^{2n+1}}} \leftrightarrow \Sigma^{2n+1} \times \mathbf{S}_{2n+2}\,.$$

Let us compute the relation between the Dirac operator on $\mathbf{R}^{m+1}$ and on the sphere Σ^m under the identifications defined above.

Let $(e_1, \ldots, e_m, \xi)$ be an orthonormal frame field on an open subset of sphere, let ξ be a unit normal field to Σ^m in $\mathbf{R}^{m+1}$.

If $(x_1, \ldots, x_{m+1})$ are cartesian coordinates in $\mathbf{R}^{m+1}$ and

$$\Sigma^m := \{(x_1, \ldots, x_{m+1}) \in \mathbf{R}^{m+1} \mid \sum_{i=1}^{m+1} x_i^2 = r^2\}\,,$$

then $\xi = \frac{1}{r}\sum_{i=1}^{m+1} x_i \partial_i x$.

If $X \in \mathcal{T}_x\Sigma^m$ then the relation between covariant derivative $\tilde{\nabla}$ on $\mathbf{R}^{m+1}$ and induced covariant derivative ∇ on Σ^m is the following one:

$$\widetilde{\nabla_X} Y = \nabla_X Y + h(X,Y)\xi\,,\ \widetilde{\nabla_X}\xi = -A(X)\,,$$

where $A : \mathcal{T}_x\Sigma^m \to \mathcal{T}_x\Sigma^m$ is a linear map, $h(X,Y) = g(A(X),Y)$.

For the sphere Σ^m and for the frame field defined above we have

$$A(e_j) = -\frac{1}{r}e_j,\ \tilde{\nabla}_{e_j}\xi = -\frac{1}{r}e_j$$
$$\tilde{\nabla}_{e_i}e_j = \nabla_{e_i}e_j - \frac{1}{r}\delta_{ij}\xi$$

and furthermore for a spinor field ϕ on U

$$\tilde{D}\phi = D(\phi_{/\Sigma^m}) - \frac{1}{2}\sum_{i=1}^{m} e_i(-\frac{1}{r}e_i) \bullet \xi \bullet \phi + \xi \bullet \tilde{\nabla}_\xi \phi\,,$$

hence

$$\tilde{D}\phi = D(\phi_{/\Sigma^m}) + \xi \bullet (\xi(\phi) + \frac{m}{2r}\phi)\,.$$

Remark 3.4. There is a connection between the operator Γ on the sphere, defined in [8] and Dirac operators $\tilde{D}$ and D, namely if we change the values of operator from the spinor-valued to the Clifford algebra-valued functions and use the standard procedure, we get

$$\Gamma\phi = \xi \bullet D(\phi_{/\Sigma^m}) + \frac{m}{2r}\phi$$

and we may obtain relations as in [8].

From the relations among the operators D_N, D_M, $\phi_M(D_N)$ and Γ we can deduce relations among the elements of the corresponding kernels and further results which will be presented in the next paper.

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