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Copure injective resolutions, flat resolvents and dimensions

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Abstract. In this paper, we show the existence of copure injective preenvelopes over noetherian rings and copure flat preenvelopes over commutative artinian rings. We use this to characterize n-Gorenstein rings. As a consequence, if the full subcategory of strongly copure injective (respectively flat) modules over a left and right noetherian ring R has cokernels (respectively kernels), then R is 2-Gorenstein.

Keywords: preenvelopes, copure injective, copure flat, n-Gorenstein, resolutions

Classification: 18G05, 18G10, 18G15

1. Introduction.

R will denote an associative ring with a unit element and all modules are unitary. Module and noetherian will mean left R-module and left noetherian respectively.

A submodule A of B is said to be a *copure submodule* if B/A is injective, and a module M is said to be *copure injective* if it is injective with respect to all copure submodules. These modules were introduced and studied in Enochs-Jenda [6]. It is easy to see that copure injective modules are precisely those modules M such that $\mathrm{Ext}^1(E, M) = 0$ for all injective R-modules E. Therefore, over principal ideal domains, copure injective modules are precisely injective modules. We will say that M is *strongly copure injective* if $\mathrm{Ext}^i(E, M) = 0$ for all injective R-modules E for all $i \geq 1$.

A module M is said to be *copure flat* if it is flat with respect to all copure submodules. It is again easy to see that M is copure flat if and only if $\mathrm{Tor}_1(E, M) = 0$ for all injective right R-modules E. We will say that M is *strongly copure flat* if $\mathrm{Tor}_i(E, M) = 0$ for all injective right R-modules E for all $i \geq 1$. A submodule $A \subset B$ is said to be *E-pure* if $0 \to E \otimes A \to E \otimes B$ is exact for all injectives E.

A map $\psi : M \to C$ where C is strongly copure injective (respectively flat) is said to be a *copure injective* (respectively *flat*) *preenvelope* if any diagram

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d] & C \arrow[dl, dotted] \\ C' & \end{tikzcd}$$

with C' strongly copure injective (respectively flat) can be completed. If furthermore, the diagram

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, "\psi"'] & C \arrow[dl, dotted] \\ C & \end{tikzcd}$$

can only be completed by automorphisms of C, then the map $\psi : M \to C$ is said to be a *copure injective* (respectively *flat*) *envelope*.

In this paper, we show that if R is noetherian, then every R-module has a copure injective preenvelope (Theorem 2.2). We note that any copure injective preenvelope $M \to C$ is a monomorphism since the injective envelope of M is strongly copure injective.

An exact sequence $0 \to M \to C^\circ \to C^1 \to \cdots$ where $M \to C^\circ$, $Coker(M \to C^\circ) \to C^1$, $Coker(C^{n-1} \to C^n) \to C^{n+1}$ for $n \geq 1$ are copure injective preenvelopes is called a *copure injective resolution* of M. If there is a copure injective resolution $0 \to M \to C^\circ \to C^1 \to \cdots \to C^n \to 0$, we say that M has *copure injective dimension* (cid) $\leq n$. In Theorem 2.5, we show the existence of copure flat preenvelopes over commutative artinian rings. In this case, we define a *copure flat resolvent* (see Enochs [3]) of an R-module M to be a sequence (not necessarily exact) $0 \to M \to T^\circ \to T^1 \to \cdots$ where $M \to T^\circ$, $Coker(M \to T^\circ) \to T^1$, $Coker(T^{n-1} \to T^n) \to T^{n+1}$ for $n \geq 1$ are copure flat preenvelopes. The meaning of *copure flat resolvent dimension* of M is now clear.

We define *copure flat dimension* (cfd) of an R-module M for any ring R to be the largest positive integer n such that $\mathrm{Tor}_n(E, M) \neq 0$ for some injective right R-module E. If M is strongly copure flat, then we set $cfdM = 0$.

In Section 3, we study copure injective and flat dimensions. The results of this section are then used to characterize n-Gorenstein rings (i.e. R is left and right noetherian, $id_R R \leq n$ and $idR_R \leq n$, (see Iwanaga [8])) in terms of copure injective, copure flat, and copure flat resolvent dimensions.

$\mathrm{Ext}^i(N, M), \mathrm{Tor}_i(N, M)$ will denote $\mathrm{Ext}^i_R(N, M), \mathrm{Tor}^R_i(N, M)$ respectively and the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will be denoted by M^+.

2. Copure injective and flat preenvelopes.

We start with the following

Lemma 2.1. *Let R be noetherian and let $\aleph_\alpha$ be an infinite cardinal. Then there is an infinite cardinal $\aleph_\beta$ such that if C is a strongly copure injective R-module and $S \subset C$ is a submodule with $CardS \leq \aleph_\alpha$, then there is a strongly copure injective submodule D of C containing S with $CardD \leq \aleph_\beta$.*

Proof: Since R is noetherian, there is a representative set $\{X_k\}$ of injective R-modules (see Gabriel [7]). Therefore, to show that an R-module M is strongly copure injective, it suffices to show that $\mathrm{Ext}^i(\oplus X_k, M) = 0$ for all $i \geq 1$. So let $\cdots \to P_1 \to P_0 \to \oplus X_k \to 0$ be a projective resolution of $\oplus X_k$. We note that for each syzygy K_j, $\mathrm{Ext}^i(K_j, C) = 0$ for all $i \geq 1$ since C is strongly copure injective. So for each $j \geq 1$, the diagram

$$\begin{tikzcd} K_j \arrow[r] \arrow[d] & P_{j-1} \arrow[dl, dotted] \\ C & \end{tikzcd}$$

can be completed into a commutative diagram.

We now construct D in steps. First, we set $S = S_0$ and consider the maps $\sigma : K_j \to S_0$. Then σ can be extended to $\sigma' : P_{j-1} \to C$. Let $S_1 = S_0 + \sum \sigma'(P_{j-1})$, sum over $\sigma \in \mathrm{Hom}(K_j, S_0), j \geq 1$. Then we construct S_2 from S_1 in the same way and so on. Now define $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ if α is a limit ordinal. Let λ be the least limit ordinal which is not cofinal with any ordinal whose cardinality is less or equal to $\sum_j Card K_j$. Hence if $\sigma \in \mathrm{Hom}(K_j, S_\lambda)$, then $\sigma(K_j) \subset S_\beta$ for $\beta < \lambda$ for each $j \geq 1$. Then σ can be extended to $\sigma' : P_{j-1} \to S_{\beta+1}$ with $\beta + 1 < \lambda$. So let $D = S_\lambda$. Then we see from the construction that $S \subset D \subset C$, D is strongly copure injective, and $Card D \leq \aleph_\beta$. $\square$

Theorem 2.2. *Let R be noetherian. Then every R-module has a copure injective preenvelope.*

Proof: The lemma above shows that if M is an R-module with $Card M \leq \aleph_\alpha$, then for any homomorphism $M \to C$ with C strongly copure injective, there is a homomorphism $M \to D, D \subset C$, D strongly copure injective, and $Card D \leq \aleph_\beta$ which agrees with the map $M \to C$. Now we set two such homomorphisms $M \to D, M \to D'$ equivalent if the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & D \arrow[d, dotted] \\ & D' \end{tikzcd}$$

can be completed by an isomorphism. If X is the set of representatives of such homomorphisms $M \to D$, then $M \to \prod_{D \in X} D$ is a strongly copure injective preenvelope.

$\square$

Remark. It follows from the proof of Proposition 6.1 of Enochs [3] that if R is noetherian and every projective limit of strongly copure injective R-modules is strongly copure injective, then every R-module has a copure injective envelope.

Lemma 2.3. *Let R be right noetherian, $\{X_k\}$ be a representative set of injective right R-modules on $N = \oplus X_k$. If $\aleph_\alpha$ is an infinite cardinal, then there is an infinite cardinal $\aleph_\beta$ such that if M is a strongly copure flat R-module and $S \subset M$*

is a submodule with Card $S \leq \aleph_\alpha$, then there is a T with $S \subset T \subset M$ and Card $T \leq \aleph_\beta$ such that $S \subset T$ induces the zero map

$$\mathrm{Tor}_i(N,S) \to \mathrm{Tor}_i(N,T) \text{ for all } i \geq 1.$$

PROOF: We first note that M is strongly copure flat if and only if $\mathrm{Tor}_i(N,M) = 0$ for all $i \geq 1$.

Let $\cdots \to P_1 \to P_0 \to N \to 0$ be a projective resolution of N. Then Card $\mathrm{Tor}_i(N,S) \leq$ Card $(P_i \otimes N)$. Hence given $\aleph_\alpha$, there exists an infinite cardinal $\aleph_\delta \geq \aleph_\alpha$ such that if Card $S \leq \aleph_\alpha$ then Card $\mathrm{Tor}_i(N,S) \leq \aleph_\delta$ for all $i \geq 1$.

Now note that $0 = \mathrm{Tor}_i(N,M) \cong \lim_\to \mathrm{Tor}_i(N,S')$ $(i \geq 1)$ where the limit is over all S' with $S \subset S' \subset M$ and with S'/S finitely generated. Hence for $z \in \mathrm{Tor}_i(N,S)$, there is an S' such that $\mathrm{Tor}_i(N,S) \to \mathrm{Tor}_i(N,S')$ maps z to 0. Choosing one such S' for each z and letting T be the sum of the S''s chosen, we see that

$$\mathrm{Tor}_i(N,S) \to \mathrm{Tor}_i(N,T)$$

is the zero map for all $i \geq 1$.

Now it is easy to see that we can choose $\aleph_\beta \geq \aleph_\alpha$ so that Card $T \leq \aleph_\beta$ whenever Card $S \leq \aleph_\alpha$ (no matter what choice of S''s we make). □

Lemma 2.4. *Let R be right noetherian and let $\aleph_\alpha$ be an infinite cardinal. Then there is an infinite cardinal $\aleph_\beta$ such that if M is a strongly copure flat R-module and $S \subset M$ is a submodule with Card $S \leq \aleph_\alpha$, then there is a strongly copure flat submodule T of M containing S with Card $T \leq \aleph_\beta$.*

PROOF: We use the notation in Lemma 2.3 above. Let $\alpha_0 = \alpha$ and $\alpha_1 = \beta$. Let α_1 play the role of α and α_2 be the new β guaranteed by the lemma. Repeating the procedure, we get $\aleph_{\alpha_0}, \aleph_{\alpha_1} \ldots$ with obvious properties. Then given $S \subset M$ with Card $S \leq \aleph_\alpha = \aleph_{\alpha_0}$, let $S_0 = S$ and find $S_1, S_2, \ldots$ with $S_1 \subset S_2 \subset \cdots \subset M$ such that Card $S_j \leq \aleph_{\alpha_j}$ and such that $S_j \subset S_{j+1}$ induces the zero map

$$\mathrm{Tor}_i(N,S_j) \to \mathrm{Tor}_i(N,S_{j+1}) \text{ for all } i \geq 1.$$

So let $T = \bigcup_{j=1}^{\infty} S_j$. Then

$$\mathrm{Tor}_i(N,T) = \lim_{\to j} \mathrm{Tor}_i(N,S_j) = 0 \text{ for all } i \geq 1.$$

Thus T is strongly copure flat. Furthermore, if we set $\aleph_\beta = \sup_j \aleph_{\alpha_j}$, then Card $T \leq \aleph_\beta$. □

Theorem 2.5. *If R is a commutative artinian ring, then every R-module has a copure flat preenvelope.*

PROOF: $E(R/m)$ is finitely generated for each $m \in mSpecR$ since R is commutative artinian. So the product of strongly copure flat R-modules is strongly copure flat by induction on Theorem 2.2 of Lenzing [11]. Then the theorem follows from Lemma 2.4 as in the proof of Theorem 2.2. □

3. Dimensions.

We start with the following

Lemma 3.1. *Let R be noetherian. Then the following are equivalent for an R-module M.*

1) $cidM \leq n$;
2) $\mathrm{Ext}^i(E, M) = 0$ *for all injective R-modules E for all* $i \geq n+1$;
3) *Every n^{th} cosyzygy of M is strongly copure injective.*

Proof: $1 \Leftrightarrow 2$. Let $0 \to M \to C^\circ \to C^1 \to \cdots \to C^{n-1} \to D \to 0$ be an exact sequence with $C^\circ, C^1, \cdots, C^{n-1}$ strongly copure injective. Then $\mathrm{Ext}^{j+n}(E, M) \cong \mathrm{Ext}^j(E, D)$. Thus the result follows. The same proof gives $2 \Leftrightarrow 3$. □

Remark. We note that the copure injective dimension of an R-module M can be considered as the largest positive integer n such that $\mathrm{Ext}^n(E, M) \neq 0$ for some injective module E. Taking this as a definition of copure injective dimension, one may drop the noetherian condition in Lemma 3.1 above.

Corollary 3.2. *If $idM < \infty$, then $cidM = idM$.*

Proof: $cidM \leq idM$ follows from the lemma above since injectives are copure injectives. Now suppose $cidM = n$. Then $\mathrm{Ext}^i(E, M) = 0$ for all injectives E and for all $i \geq n+1$. But $idM < \infty$. So $idM \leq n$ by Lemma 2.2 of Jenda [9]. □

We also have the following

Lemma 3.3. *The following are equivalent for a module M over any ring R.*

1) $cfdM \leq n$;
2) $\mathrm{Tor}_i(E, M) = 0$ *for all injective right R-modules E for all* $i \geq n+1$;
3) *Every n^{th} syzygy of M is strongly copure flat.*

Proof: $1 \Leftrightarrow 2$ by definition.

$2 \Leftrightarrow 3$. We simply note that if K_n is an n^{th} syzygy of M, then $\mathrm{Tor}_{j+n}(E, M) \cong \mathrm{Tor}_j(E, K_n)$. So the result follows. □

Remark. If R is noetherian and M is a finitely generated R-module, then

$$\mathrm{Tor}_i(E, M) \cong \mathrm{Tor}_i(\mathrm{Hom}(R, E), M) \cong \mathrm{Hom}(\mathrm{Ext}^i(M, R), E)$$

for any injective right R-module E. So in this case, copure flat dimension coincides with Gorenstein dimension ($G - dim$) if the latter is finite (see Auslander-Bridger [1, p. 95]).

Lemma 3.4. *Let M be an R-module. Then*

$$cfdM = cidM^+.$$

Proof: This follows from the standard isomorphism $\mathrm{Tor}_i(E, M)^+ \cong \mathrm{Ext}^i(E, M^+)$. □

Proposition 3.5. *Let $0 \to M \to C^\circ \to C^1 \to \cdots$ be a copure injective resolution of an R-module M. Then the sequence $0 \to T \otimes M \to T \otimes C^\circ \to T \otimes C^1 \to \cdots$ is exact for all right strongly copure flat R-modules T.*

Proof: Let $S^i(M) = Ker(C^i \to C^{i+1}), i \geq 1$ and $S^\circ(M) = M$. We consider the short exact sequence $0 \to S^i(M) \to C^i \to S^{i+1}(M) \to 0$. Then $0 \to T \otimes S^i(M) \to T \otimes C^i \to T \otimes S^{i+1}(M) \to 0$ is exact if and only if $0 \to (T \otimes S^{i+1}(M))^+ \to (T \otimes C^i)^+ \to (T \otimes S^i(M))^+ \to 0$ is exact. But the latter is equivalent to $0 \to \mathrm{Hom}(S^{i+1}(M), T^+) \to \mathrm{Hom}(C^i, T^+) \to \mathrm{Hom}(S^i(M), T^+) \to 0$ being exact. So the result follows since T^+ is strongly copure injective by Lemma 3.4. $\square$

Lemma 3.6. *Let R be a commutative artinian ring. Then*

$$cidM = cfdM^+$$

Proof: We note that if E is injective, then

$$\begin{aligned} \mathrm{Tor}_i(E, M^+) &\cong \mathrm{Tor}_i(E(R/m), M^+), \text{ over } m \in mSpecR \\ &\cong \mathrm{Ext}^i(E(R/m), M)^+ \text{ since } E(R/m) \end{aligned}$$

is finitely generated.

On the other hand, $\mathrm{Ext}^i(E, M) \cong \prod \mathrm{Ext}^i(E(R/m), M)$. So $\mathrm{Tor}_i(E, M^+) = 0$ if and only if $\mathrm{Ext}^i(E, M) = 0$. $\square$

Proposition 3.7. *Let R be a commutative artinian ring and $0 \to M \to T^\circ \to T^1 \to \cdots$ be a copure flat resolvent of M. Then $0 \to M \otimes C \to T^\circ \otimes C \to T^1 \otimes C \to \cdots$ is exact for all strongly copure injective R-modules C.*

Proof: The result follows as in Proposition 3.5 above since C^+ is strongly copure flat by Lemma 3.6. $\square$

4. Gorenstein rings.

We are now in a position to prove the following

Theorem 4.1. *The following are equivalent for a left and right noetherian ring R.*

1) *R is n-Gorenstein.*
2) *$cidM \leq n$ for all R-modules (left and right) M.*
3) *Every n^{th} cosyzygy of an R-module (left and right) is strongly copure injective.*
4) *$cfdM \leq n$ for all R-modules (left and right) M.*
5) *$cfdM \leq n$ for all finitely generated R-modules (left and right) M.*
6) *Every n^{th} syzygy of an R-module (left and right) is strongly copure flat.*
7) *Projective resolutions of the n^{th} syzygies of finitely generated R-modules (left and right) are projective resolvents.*

Furthermore, if R is commutative artinian, then the above statements are equivalent to

8) *Copure flat resolvent dimension of each R-module is at most n.*

PROOF: $1 \Leftrightarrow 2 \Leftrightarrow 3$. R n-Gorenstein means $pdE \leq n$ for all injective R-modules (left and right) by Jensen [10, Theorem 5.9]. So the equivalences follow from Lemma 3.1.

$1 \Leftrightarrow 4 \Leftrightarrow 6$. This easily follows from Lemma 3.3 as above and Enochs-Jenda [4, Theorem 4.4].

$4 \Leftrightarrow 5$ is trivial since Tor commutes with direct limits.

$5 \Leftrightarrow 7$. If M is finitely generated, then $\mathrm{Ext}^i(M,-)$ commutes with direct sums (see Lenzing [11, Theorem 2]). So $\mathrm{Ext}^i(M,R) = 0$ if and only if $\mathrm{Ext}^i(M,P) = 0$ for all projectives P. Therefore by the remark after Lemma 3.3, $\mathrm{Tor}_i(E,M) = 0$ for all injectives E if and only if $\mathrm{Ext}^i(M,P) = 0$ for all projectives P, So the result follows as in Lemma 2.1 of Enochs-Jenda [5].

$2 \Leftrightarrow 8$. Propositions 3.5 and 3.7 show that if R is a commutative artinian ring and M, N are R-modules, then $M \otimes N$ is right balanced by *copure inj* $\times$ *copure flat* in the language of Enochs-Jenda [4] and so right derived functors $\mathrm{Tor}^i(M,N)$ can be computed using either copure injective resolutions of M or copure flat resolvents of N. So $cidM \leq n$ if and only if copure flat resolvent dimension of N is at most n. $\square$

Corollary 4.2. *Let R be left and right noetheiran. Then the following are equivalent.*

1) *R is 1-Gorenstein.*
2) *Every copure injective R-module (left and right) is strongly copure injective.*
3) *Every copure flat R-module (left and right) is strongly copure flat.*
4) *Every homomorphic image of a copure injective R-module (left and right) is copure injective.*
5) *Every submodule of a copure flat R-module (left and right) is copure flat.*

In this case, the full subcategories of copure injective modules (left and right) and copure flat modules (left and right) have cokernels and kernels respectively.

PROOF: $1 \Rightarrow 2, 3$ follow from the theorem above.

$2 \Rightarrow 1$. Let M be an R-module. Consider the short injective resolution $0 \to M \to E^\circ \to D \to 0$. Let $E_\circ \to D$ be an injective precover and $K = Ker(E_\circ \to D)$. Then $E_\circ \to D$ is surjective and so $\mathrm{Ext}^{i+1}(E,K) \cong \mathrm{Ext}^i(E,D)$ for $i \geq 1$. But K is copure injective (see Enochs-Jenda [6, Lemma 2.1]). So D is strongly copure injective. Thus every first cosyzygy is strongly copure injective. So the result follows from the theorem.

$3 \Rightarrow 1$. Now we consider the short projective resolution $0 \to K \to P_\circ \to M \to 0$. Let $K \to F^\circ$ be a flat preenvelope. Then $K \to F^\circ$ is $1{-}1$ and so $\mathrm{Tor}_{i+1}(E, F^\circ/K) \cong \mathrm{Tor}_i(E,K)$ for $i \geq 1$. But F°/K is copure flat by Enochs-Jenda [4, Proposition 3.3]. So K is strongly copure flat. Thus the result follows from the theorem.

$1 \Leftrightarrow 4$. The same proof as the commutative version in Enochs-Jenda [6, Corollary 3.4] noting that R is 1-Gorenstein if and only if $pdE \leq 1$ for all injective modules (left and right) E.

$1 \Rightarrow 5$. Let S be a submodule of a copure flat R-module M and E be an injective right R-mdoule. We have an exact sequence

$$\cdots \to \mathrm{Tor}_2(E, M) \to \mathrm{Tor}_2(E, M/S) \to \mathrm{Tor}_1(E, S) \to \mathrm{Tor}_1(E, M) = 0.$$

But $\mathrm{Tor}_2(E, M/S) = 0$ since $pdE_R \leq 1$. So S is copure flat. Similarly for right R-modules.

$5 \Rightarrow 1$. Consider a short projective resolution $0 \to K \to P_\circ \to M \to 0$ of M. Then $\mathrm{Tor}_2(E, M) \cong \mathrm{Tor}_1(E, K)$. So if every submodule of copure flat is copure flat, then $\mathrm{Tor}_2(E, M) = 0$ for all R-modules M and for all injective right R-modules E. Therefore $pdE_R \leq 1$ for all such E and so $id_R R \leq 1$. Similarly $idR_R \leq 1$.

The remaining statements of the corollary easily follow from the parts (4) and (5). $\square$

The following is analogous to a result of Bernecker [2].

Corollary 4.3. *Let R be left and right noetherian. If the full subcategory of strongly copure injective (respectively flat) R-modules has cokernels (respectively kernels), then R is 2-Gorenstein.*

Proof: Let $C \to D$ be a cokernel of $C' \to C$ with C', C strongly copure injective and E be an injective generator. Then $0 \to \mathrm{Hom}(D, E) \to \mathrm{Hom}(C, E) \to \mathrm{Hom}(C', E)$ is exact implies that $C \to D$ is onto. So cokernels are surjective. Therefore, every 2^{nd} cosyzygy is strongly copure injective by assumption. Thus R is 2-Gorenstein by the theorem above. Similarly for strongly copure flat modules. $\square$

Remark. It is easy to check that if the full subcategory of strongly copure injective (respectively flat) R-modules (left and right) has kernels (respectively cokernels), then it has cokernels (respectively kernels) and thus R is 2-Gorenstein.

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