On p-sequential p-compact spaces

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Abstract. It is shown that a space X is $L({}^{\mu}p)$-Weakly Fréchet-Urysohn for $p \in \omega^*$ iff it is $L({}^{\nu}p)$-Weakly Fréchet-Urysohn for arbitrary $\mu, \nu < \omega_1$, where ${}^{\mu}p$ is the μ-th left power of p and $L(q) = \{{}^{\mu}q : \mu < \omega_1\}$ for $q \in \omega^*$. We also prove that for p-compact spaces, p-sequentiality and the property of being a $L({}^{\nu}p)$-Weakly Fréchet-Urysohn space with $\nu < \omega_1$, are equivalent; consequently if X is p-compact and $\nu < \omega_1$, then X is p-sequential iff X is ${}^{\nu}p$-sequential (Boldjiev and Malyhin gave, for each P-point $p \in \omega^*$, an example of a compact space X_p which is ${}^{2}p$-Fréchet-Urysohn and it is not p-Fréchet-Urysohn. The question whether such an example exists in ZFC remains unsolved).

Keywords: p-compact, p-sequential, FU(p)-space, Rudin-Keisler order, tensor product of ultrafilters, left power of ultrafilters, SMU(M)-space, WFU(M)-space

Classification: 04A20, 54A25, 54D55

0. Introduction.

In [BM], Boldjiev and Malyhin gave an example of a compact Franklin space X_p which is a FU(p^2)-space but not a FU(p)-space, for each P-point $p \in \omega^*$. We prove in this article that this is not the case when we consider p-sequentiality; that is, every compact ${}^{2}p$-sequential space is p-sequential for every $p \in \omega^*$ (3.9). In order to obtain this result we introduce, in the first section, the left exponentiation ${}^{\nu}p$ of $p \in \omega^*$ for each $\nu < \omega_1$, and we study its basic properties and its relation with the power p^{ν} defined by Booth in [Bo]. In Section 2, we analyze the concepts of M-Weakly Fréchet-Urysohn space (WFU(M)-space) and M-Strongly Fréchet-Urysohn space (SFU(M)-space) for $M \subset \omega^*$. In the last section, we prove that if X is a p-compact space, then X is p-sequential iff X is a WFU($L({}^{\nu}p)$)-space, where $L(q) = \{{}^{\mu}q : \mu < \omega_1\}$ with $q \in \omega^*$ (3.7 and 3.8). As a consequence, in the class of p-compact spaces we have that p-sequentiality and ${}^{\nu}p$-sequentiality coincide.

1. Preliminaries.

We restrict our attention throughout this paper to Tychonoff spaces. For $A \subset X$, the closure and interior of A in X are denoted by $\mathrm{Cl}_X(A)$ (or simply $\mathrm{Cl}(A)$) and $\mathrm{In}_X(A)$, respectively. For $x \in X$, $\mathcal{N}(x)$ will be the set of all neighborhoods of x. The Stone-Čech compactification $\beta(\omega)$ of the natural numbers is identified with the set of all ultrafilters on ω, where a basic clopen subset of $\beta(\omega)$ is $\widehat{A} = \mathrm{Cl}_{\beta(\omega)}(A) = \{p \in \beta(\omega) : A \in p\}$ for $A \subset \omega$. The remainder of $\beta(\omega)$ is $\omega^* = \beta(\omega) \setminus \omega$ and, for $A \subset \omega$, we let $A^* = \widehat{A} \cap \omega^*$. If $f : \omega \to \omega$ is a function, then $\overline{f} : \beta(\omega) \to \beta(\omega)$ denotes the Stone-Čech extension of f. The <u>Rudin-Keisler</u> (pre-)order on ω^* is defined by $p \leq_{\mathrm{RK}} q$ if there is a surjection $f : \omega \to \omega$ such that $\overline{f}(q) = p$, for $p, q \in \omega^*$. If

$p, q \in \omega^*$ satisfy $p \leq_{\mathrm{RK}} q$ and $q \leq_{\mathrm{RK}} p$, then we say that p and q are RK-equivalent and write $p \simeq_{\mathrm{RK}} q$. It is not difficult to verify that $p \simeq_{\mathrm{RK}} q$ iff there is a permutation σ of ω such that $\overline{\sigma}(p) = q$. The type of $p \in \omega^*$ is $T(p) = \{q \in \omega^* : p \simeq_{\mathrm{RK}} q\}$.

Now we recall the definition of p-limit, for $p \in \omega^*$, introduced and studied by Bernstein in [Be].

Definition 1.1. Let $(x_n)_{n<\omega}$ be a sequence in a space X and $p \in \omega^*$. An element x of X is a p-limit point of $(x_n)_{n<\omega}$ (in symbols, $x = p\text{-}\lim_{n\to\infty} x_n$) if for each $V \in \mathcal{N}(x)$, $\{n < \omega : x_n \in V\} \in p$.

If $p \leq_{\mathrm{RK}} q$, then every p-limit point is also a q-limit point as stated in the next lemma, the proof of which is easy.

Lemma 1.2. *Let $(x_n)_{n<\omega}$ be a sequence in a space X such that $p\text{-}\lim_{n\to\infty} x_n = x \in X$. If $f : \omega \to \omega$ is a function such that $\overline{f}(q) = p$, then $x = q\text{-}\lim_{n\to\infty} x_{f(n)}$.*

In [Be] the author also considered the following notion.

Definition 1.3. Let $p \in \omega^*$. A space X is p-compact if every sequence $(x_n)_{n<\omega}$ of points of X has a p-limit point in X.

The sum of a countable set of ultrafilters on ω with respect to an ultrafilter on ω has been studied by Frolík [F]; for the general case, arbitrary filters on arbitrary sets, by Vopěnka [V] and Katětov [K].

Definition 1.4. Let $p \in \omega^*$ and $\{p_n : n < \omega\} \subseteq \omega^*$. The sum of $\{p_n : n < \omega\}$ with respect to p, denoted $\Sigma_p p_n$, is the set

$$\{A \subseteq \omega \times \omega : \{n < \omega : \{m < \omega : (n, m) \in A\} \in p_n\} \in p\}.$$

It is evident that $\Sigma_p p_n$ is an ultrafilter on $\omega \times \omega$ and can be viewed as an ultrafilter on ω via a bijection between $\omega \times \omega$ and ω. If $p, q \in \omega^*$ and $p_n = q$ for each $n < \omega$ then $\Sigma_p p_n$ is the usual tensor product $p \otimes q$ of p and q. It is not hard to see that $\otimes$ is not a commutative operation on ω^*. However, Booth [Bo] showed that $\otimes$ induces a semigroup structure on the set of types of ω^*.

We also have that the sum and tensor product satisfy:

Lemma 1.5. *Let $(p_n)_{n<\omega}$, $(q_n)_{n<\omega}$ be two sequences in ω^* and $p, s, q, r \in \omega^*$. Then*

(1) (Blass [Bl]) *if $\{n < \omega : p_n \leq_{\mathrm{RK}} q_n\} \in p$, then $\Sigma_p p_n \leq_{\mathrm{RK}} \Sigma_p q_n$; and $\Sigma_p p_n <_{\mathrm{RK}} \Sigma_p q_n$ if $\{n < \omega : p_n < q_n\} \in p$.*
(2) (Kunen, see [Bo, 2.21]) *if $(r_n)_{n<\omega}$ is a discrete sequence in ω^* and $r_n \simeq_{\mathrm{RK}} \Sigma_{q_n} p_k$ for all $n < \omega$, then $\Sigma_p r_n \simeq_{\mathrm{RK}} \Sigma_{\Sigma_p q_n} p_n$;*
(3) (folklore) *$r <_{\mathrm{RK}} p \otimes r$ and $r <_{\mathrm{RK}} r \otimes p$;*
(4) *if $p \leq_{\mathrm{RK}} s$ and $q \leq_{\mathrm{RK}} r$, then $p \otimes q \leq_{\mathrm{RK}} s \otimes r$.*
(5) (Blass [Bl]) *If $f : \omega \to \omega$ is a function satisfying $\overline{f}(q) = p$, and $p_n \leq_{\mathrm{RK}} q_n$ for all $n < \omega$, then $\Sigma_p p_n \leq_{\mathrm{RK}} \Sigma_q q_{f(n)}$.*

Throughout this paper, for each $2 \leq \nu < \omega_1$ we fix an increasing sequence $(\nu(n))_{n<\omega}$ of ordinals in ω_1 so that

(1) if $2 \leq \nu < \omega$, $\nu(n) = \nu - 1$;
(2) $\omega(n) = n$ for $n < \omega$;
(3) if ν is a limit ordinal, then $\nu(n) \nearrow \nu$;
(4) if $\nu = \mu + m$ where μ is a limit ordinal and $m < \omega$, then $\nu(n) = \mu(n) + m$ for each $n < \omega$.

In [Bo], the power (or the right power) $T(p)^\nu$ is defined for each $0 < \nu < \omega_1$ and for $p \in \omega^*$. For our convenience, if $0 < \nu < \omega_1$ and $p \in \omega^*$, then p^ν stands for an arbitrary point in $T(p)^\nu$. The basic properties of Booth's powers of ultrafilters are summarized in the following lemma.

Lemma 1.6. *Let $p, q \in \omega^*$. Then*

(1) (Booth [Bo]) *if* $1 < \nu < \omega_1$, *then* $p^\nu \simeq_{\mathrm{RK}} \Sigma_p p^{\nu(n)}$;
(2) (Booth [Bo]) *if* $0 < \mu < \nu < \omega_1$, *then* $p^\mu <_{\mathrm{RK}} p^\nu$;
(3) *if* $p \leq_{\mathrm{RK}} q$, *then* $p^\nu \leq_{\mathrm{RK}} q^\nu$ *for all* $0 < \nu < \omega_1$;
(4) ([G-F$_2$, 2.29]) *if* $0 < \nu < \omega_1$ *is a limit ordinal and* $\omega \leq \mu < \nu$, *then* $p \otimes p^\mu \leq_{\mathrm{RK}} p^\nu$;
(5) $\forall\, 0 < \mu, \nu < \omega_1 \exists\, \theta < \omega_1$ $(p^\mu \otimes p^\nu \leq_{\mathrm{RK}} p^\theta)$;
(6) $\forall\, 0 < \mu, \nu < \omega_1 \exists\, \theta < \omega_1$ $((p^\mu)^\nu \leq_{\mathrm{RK}} p^\theta)$.

Proof: The proofs of (3), (5) and (6) are similar to those given for 1.7 (3′), 1.7 (5′) and 1.7 (6′) below, respectively, and we omit them. □

We can also define a left exponentiation which will play an important role in the next section:

${}^2T(p) = T(p \otimes p)$ and ${}^{n+1}T(p) = T(p) \otimes {}^nT(p)$ for $n < \omega$. If ${}^\mu T(p)$ has been defined for all $0 < \mu < \nu < \omega_1$ and ν is a limit ordinal, then ${}^\nu T(p) = T(\overline{e}(p))$, where $e : \omega \to \omega^*$ is an embedding with $e(n) \in {}^{\nu(n)}T(p)$ for $n < \omega$. If $\nu = \mu + 1$, then ${}^\nu T(p) = T(p) \otimes {}^\mu T(p)$ (the basic difference between the left power and Booth's power is that in [Bo] $T(p)^{\mu+1}$ is defined by $T(p)^\mu \otimes T(p)$). As above, if $0 < \nu < \omega_1$ and $p \in \omega^*$, then ${}^\nu p$ stands for an arbitrary point in ${}^\nu T(p)$. Observe that, because of associativity of $\otimes$ on the set of types, ${}^nT(p) = T(p)^n$ for every $n < \omega$, and therefore ${}^\omega T(p) = T(p)^\omega$. It is proved in [Bo, Corollary 2.23] that $T(p)^{\omega+1} <_{\mathrm{RK}} {}^{\omega+1}T(p)$.

Some properties of the left power of ultrafilters and its relations with the right power are given in the next lemma.

Lemma 1.7. *Let $p, q \in \omega^*$. Then*

(2′) *if* $0 < \mu < \nu < \omega_1$, *then* ${}^\mu p <_{\mathrm{RK}} {}^\nu p$;
(3′) *if* $p \leq_{\mathrm{RK}} q$, *then* ${}^\nu p \leq_{\mathrm{RK}} {}^\nu q$ *for all* $0 < \nu < \omega_1$;
(4′), (5′) $\forall\, 0 < \mu, \nu < \omega_1 \exists\, \theta < \omega_1$ $({}^\mu p \otimes {}^\nu p \leq_{\mathrm{RK}} {}^\theta p)$;
(6′) $\forall\, 0 < \mu, \nu < \omega_1 \exists\, \theta < \omega_1$ $({}^\mu({}^\nu p) \leq_{\mathrm{RK}} {}^\theta p)$;
(7) $\forall\, 0 < \mu < \omega \exists\, \theta, \tau < \omega_1$ $(p^\mu \leq_{\mathrm{RK}} {}^\theta p$ *and* ${}^\mu p \leq_{\mathrm{RK}} p^\tau)$.

Proof: (2′) Since ${}^\mu p \simeq_{\mathrm{RK}} p^\mu$ for every $0 < \mu \leq \omega$, then by 1.6 (2) we have: ${}^\mu p <_{\mathrm{RK}} {}^\nu p$ for all $0 < \mu < \nu \leq \omega$. Suppose that for every $\mu < \lambda < \nu < \omega_1$ the inequality ${}^\mu p <_{\mathrm{RK}} {}^\lambda p$ holds. If $\nu = \lambda + 1$, then, by 1.5 (3), ${}^\mu p \leq_{\mathrm{RK}} {}^\lambda p <_{\mathrm{RK}} p \otimes {}^\lambda p$

$\simeq_{\mathrm{RK}} {}^{\nu}p$. Now, assume that ν is a limit ordinal. Then there is $N < \omega$ such that $\mu < \nu(n)$ for every $n > N$. By induction hypothesis we have that ${}^{\mu}p <_{\mathrm{RK}} {}^{\nu(n)}p$ for every $n > N$. So, $\{n < \omega : {}^{\mu}p <_{\mathrm{RK}} {}^{\nu(n)}p\} \in p$. Therefore, by 1.5 (1), we obtain that ${}^{\mu}p <_{\mathrm{RK}} {}^{\mu+1}p \simeq_{\mathrm{RK}} \Sigma_p {}^{\mu}p <_{\mathrm{RK}} \Sigma_p {}^{\nu(n)}p \simeq_{\mathrm{RK}} {}^{\nu}p$.

(3′) First we shall show that there is $g : \omega \to \omega$ onto such that $g(m) \leq m$ for all $m < \omega$ and $\overline{g}(q) = p$. We consider the following two cases:

I. There is no finite-to-one function $f : \omega \to \omega$ for which $\overline{f}(q) = p$. Let $g : \omega \to \omega$ be onto such that $\overline{g}(q) = p$. Assume that $A = \{m < \omega : m < g(m)\} \in q$. Then, there is $N \in B = g[A]$ such that $|g^{-1}(N) \cap A| = \omega$. If $m > N$ and $m \in g^{-1}(N) \cap A$, then $g(m) = N < m$, which is a contradiction. Therefore, $\{m < \omega : g(m) \leq m\} \in q$. We may assume that $g(m) \leq m$ for all $m < \omega$.

II. There is a finite-to-one function $f : \omega \to \omega$ such that $\overline{f}(q) = p$. Then, for each $n < \omega$ we have that $f^{-1}(n) = \{k_0^n, \ldots, k_{r_n}^n\}$. Define $h : \omega \to \omega$ by $h(n) = \min\{k_0^n, \ldots, k_{r_n}^n\}$. Notice that h is one-to-one. Put $g = h \circ f$. If $m < \omega$ and $f(m) = n$, then $g(m) = h(f(m)) = h(n) \leq m$ since $m \in \{k_0^n, \ldots, k_{r_n}^n\}$. Since h is one-to-one, by [CN, 9.2 (b)], $\overline{g}(q) = \overline{h}(\overline{f}(q)) = \overline{h}(p) \simeq_{\mathrm{RK}} p$. This proves our claim.

We now proceed by induction. By 1.5 (4) we have that ${}^{n}p \leq_{\mathrm{RK}} {}^{n}q$ for all $1 \leq n < \omega$. Assume that ${}^{\mu}p \leq_{\mathrm{RK}} {}^{\mu}q$ for all $\mu < \nu < \omega_1$. If $\nu = \mu + 1$, by 1.5 (4), we have that ${}^{\nu}p \simeq_{\mathrm{RK}} p \otimes {}^{\mu}p \leq_{\mathrm{RK}} q \otimes {}^{\mu}q \simeq_{\mathrm{RK}} {}^{\nu}q$. Suppose that ν is a limit ordinal. Let $g : \omega \to \omega$ be such that $g(n) \leq n$ for all $n < \omega$ and $\overline{g}(q) = p$. By assumption, and using (2′), we have that ${}^{\nu(n)}p \leq_{\mathrm{RK}} {}^{\nu(n)}q$ and ${}^{\nu(g(n))}q \leq_{\mathrm{RK}} {}^{\nu(n)}q$. From 1.5 (5) and 1.5 (1) it follows that ${}^{\nu}p \simeq_{\mathrm{RK}} \Sigma_p {}^{\nu(n)}p \leq_{\mathrm{RK}} \Sigma_q {}^{\nu(g(n))}q \leq_{\mathrm{RK}} \Sigma_q {}^{\nu(n)}q \simeq_{\mathrm{RK}} {}^{\nu}q$.

(4′), (5′) We proceed by induction on μ. By definition we have that $p \otimes {}^{\nu}p \leq_{\mathrm{RK}} {}^{\nu+1}p$ for every $\nu < \omega_1$. Assume that for each $\nu < \omega_1$ and each $\lambda < \mu < \omega_1$, there is $\theta < \omega_1$ for which ${}^{\lambda}p \otimes {}^{\nu}p \leq_{\mathrm{RK}} {}^{\theta}p$. First, suppose that $\mu = \lambda + 1$, then by induction hypothesis there exists $\theta < \omega_1$ such that ${}^{\mu}p \otimes {}^{\nu}p \simeq_{\mathrm{RK}} p \otimes ({}^{\lambda}p \otimes {}^{\nu}p) \leq_{\mathrm{RK}} p \otimes {}^{\theta}p \simeq_{\mathrm{RK}} {}^{\theta+1}p$. Now, assume that ${}^{\mu}p \simeq_{\mathrm{RK}} \Sigma_p {}^{\mu(n)}p$. By assumption, for each $n < \omega$, there is $\lambda_n < \omega_1$ such that ${}^{\mu(n)}p \otimes {}^{\nu}p \leq_{\mathrm{RK}} {}^{\lambda_n}p$. Set $\lambda = \sup\{\lambda_n : n < \omega\}$. Then, ${}^{\mu(n)}p \otimes {}^{\nu}p \leq_{\mathrm{RK}} {}^{\lambda}p$ for all $n < \omega$. Hence, by 1.5 (2) and 1.5 (1), ${}^{\mu}p \otimes {}^{\nu}p \simeq_{\mathrm{RK}} (\Sigma_p {}^{\mu(n)}p) \otimes {}^{\nu}p \simeq_{\mathrm{RK}} \Sigma_p ({}^{\mu(n)}p \otimes {}^{\nu}p) \leq_{\mathrm{RK}} p \otimes {}^{\lambda}p \simeq_{\mathrm{RK}} {}^{\lambda+1}p$.

(6′) The proof is by induction on μ. Suppose that for each $\nu < \omega_1$ and each $\lambda < \mu < \omega_1$ there is θ for which ${}^{\lambda}({}^{\nu}p) \leq_{\mathrm{RK}} {}^{\theta}p$. If $\mu = \lambda + 1$, then by induction hypothesis there exists $\delta < \omega_1$ such that ${}^{\lambda+1}({}^{\nu}p) \simeq_{\mathrm{RK}} {}^{\nu}p \otimes {}^{\lambda}({}^{\nu}p) \leq_{\mathrm{RK}} {}^{\nu}p \otimes {}^{\delta}p$. Because of (5′) we can find $\theta < \omega_1$ for which ${}^{\mu}({}^{\nu}p) \leq_{\mathrm{RK}} {}^{\nu}p \otimes {}^{\delta}p \leq_{\mathrm{RK}} {}^{\theta}p$. If μ is a limit ordinal we have that ${}^{\mu}({}^{\nu}p) \simeq_{\mathrm{RK}} \Sigma_q {}^{\mu(n)}q$, where $q = {}^{\nu}p$. By assumption, for each $n < \omega$ there is λ_n such that ${}^{\mu(n)}q \leq_{\mathrm{RK}} {}^{\lambda_n}p$. If we put $\lambda = \sup\{\lambda_n : n < \omega\}$, then ${}^{\mu(n)}q \leq_{\mathrm{RK}} {}^{\lambda}p$ and so $\Sigma_q {}^{\mu(n)}q \leq_{\mathrm{RK}} q \otimes {}^{\lambda}p$. Applying (5′) there is $\theta < \omega_1$ such that ${}^{\mu}({}^{\nu}p) \simeq_{\mathrm{RK}} \Sigma_q {}^{\mu(n)}q \leq_{\mathrm{RK}} {}^{\nu}p \otimes {}^{\lambda}p \leq_{\mathrm{RK}} {}^{\theta}p$.

(7) We are going to prove the first inequality because the second one is shown in an analogous fashion. Assume that for each $0 < \nu < \mu < \omega_1$ there is $\theta < \omega_1$ such that ${}^{\nu}p \leq_{\mathrm{RK}} p^{\theta}$. If $\mu = \lambda + 1$, then there is $\delta < \omega_1$ such that ${}^{\mu}p = {}^{\lambda+1}p \simeq_{\mathrm{RK}} p \otimes {}^{\lambda}p \leq_{\mathrm{RK}} p \otimes p^{\delta}$. By 1.6 (4), we can find $\theta < \omega_1$ satisfying ${}^{\mu}p \leq_{\mathrm{RK}}$

$p\otimes p^\delta \leq_{\mathrm{RK}} p^\theta$. Let us suppose now that ${}^\mu p \simeq_{\mathrm{RK}} \Sigma_p{}^{\mu(n)}p$. By induction hypothesis, for each $n<\omega$, there is $\delta_n<\omega_1$ such that ${}^{\mu(n)}p \leq_{\mathrm{RK}} p^{\delta_n}$. If $\delta=\sup\{\delta_n : n<\omega\}$, then ${}^{\mu(n)}p \leq_{\mathrm{RK}} p^\delta$ for all $n<\omega$. Thus, using 1.5 (1) and 1.6 (4), we obtain ${}^\mu p \simeq_{\mathrm{RK}} \Sigma_p{}^{\mu(n)}p \leq_{\mathrm{RK}} \Sigma_p p^\delta \simeq_{\mathrm{RK}} p\otimes p^\delta \leq_{\mathrm{RK}} p^\theta$ for some $\theta<\omega_1$. □

Observe that we do not have a statement in 1.7 analogous to that in 1.6 (1). In fact, because of 1.5(1) we obtain the following inequality: $\Sigma_p{}^{(\omega+1)(n)}p \simeq_{\mathrm{RK}} \Sigma_p{}^{\omega(n)+1}p \simeq_{\mathrm{RK}} \Sigma_p{}^{n+1}p <_{\mathrm{RK}} \Sigma_p{}^{\omega}p \simeq_{\mathrm{RK}} p\otimes{}^\omega p \simeq_{\mathrm{RK}} {}^{\omega+1}p$.

Notation 1.8. For $p\in\omega^*$ we put $L(p)=\{{}^\nu p : \nu<\omega_1\}$ and $R(p)=\{p^\nu : \nu<\omega_1\}$.

2. SFU(M)-spaces and WFU(M)-spaces.

The Fréchet-Urysohn spaces and sequential spaces can be generalized using p-limits as follows:

Definition 2.1. Let $p\in\omega^*$ and X be a space. Then

(1) (Comfort-Savchenko) X is a FU(p)-space if for each $A\subseteq X$ and $x\in \mathrm{Cl}(A)$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x=p\text{-}\lim x_n$;

(2) (Kombarov [Ko]) X is p-sequential if for every non-closed subset A of X there is $x\in\mathrm{Cl}(A)\setminus A$ and a sequence $(x_n)_{n<\omega}$ in A such that $x=p\text{-}\lim x_n$.

The p-limits and subsets of ω^* can be used to produce the following classes of spaces, which are closely related to the FU(p)-property.

Definition 2.2 (Kočinac [Koč]). Let $\emptyset\neq M\subseteq\omega^*$ and let X be a space. Then

(1) X is a WFU(M)-space if for $A\subseteq X$ and $x\in A^-$ there are $p\in M$ and a sequence $(x_n)_{n<\omega}$ in A such that $x=p\text{-}\lim x_n$;

(2) X is a SFU(M)-space if for $A\subseteq X$ and $x\in A^-$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x=p\text{-}\lim x_n$ for all $p\in M$.

Notice that the concept of SFU(ω^*)-space (resp. WFU(ω^*)-space) coincides with the concept of Fréchet-Urysohn space (resp. countable tightness). If $p\in\omega^*$, then SFU($\{p\}$)-space = WFU($\{p\}$)-space = FU(p)-space. The fundamental properties of the notions given in 2.2 are stated in the next theorem.

Theorem 2.3. *Let $\emptyset\neq M\subseteq\omega^*$. Then*

(1) *if $p\in M$,* SFU(M)*-space* $\Rightarrow$ FU(p)*-space* $\Rightarrow$ WFU(M)*-space;*
(2) SFU(M)*-space* $\Leftrightarrow$ SFU($\mathrm{Cl}_{\beta(\omega)}(M)$)*-space;*
(3) FU(p)*-space* $\Leftrightarrow$ WFU($T(p)$)*-space, for $p\in\omega^*$;*
(4) WFU(M)*-space* $\Rightarrow$ WFU($\mathrm{Cl}_{\beta(\omega)}(M)$)*-space.*

For a nonempty closed subset M of ω^*, we define $\xi(M)=\omega\cup\{M\}$, where ω has the discrete topology and the neighborhood system of M is $\{\{M\}\cup A : A\subseteq\omega$ and $M\subseteq A^*\}$. Then $\xi(M)$ is a WFU(M)-space for each $\emptyset\neq M\subseteq\omega^*$. Observe that, for $A\subset\omega$, $M\in\mathrm{Cl}_{\xi(M)}(A)$ iff there is $p\in M$ such that $A\in p$, and if M is closed,

$$M=\bigcap\{B^* : B\subseteq\omega \text{ and } M\subseteq B^*\}.$$

This kind of spaces will supply some important examples. We are also going to analyze when $\xi(M)$ is a SFU(M)-space and when it is a Fréchet-Urysohn space.

Lemma 2.4. *Let $M \subset \omega^*$ be closed. Then $\xi(M)$ is a* SFU(M)*-space iff for each $A \subset \omega$ satisfying $A^* \cap M \neq \emptyset$, there exists $f : \omega \to A$ such that $\overline{f}[M] \subset M \cap A^*$.*

PROOF: Necessity. Let $A \subset \omega$ such that $A^* \cap M \neq \emptyset$. Thus, $M \in \mathrm{Cl}_{\xi(M)}(A)$. Hence, there is a sequence $(a_n)_{n<\omega}$ in A such that $M = p\text{-}\lim a_n$ for every $p \in M$. Let $f : \omega \to A$ defined by $f(n) = a_n$. It is not difficult to see that $\overline{f}(M) \subset M \cap A^*$.

Sufficiency. $M \in \mathrm{Cl}_{\xi(M)}(A)$ implies that $M \cap A^* \neq \emptyset$. By hypothesis, there exists $f : \omega \to A$ for which $\overline{f}[M] \subset M \cap A^*$. The sequence $(f(n))_{n<\omega}$ q-converges to M for every $q \in M$. □

Next, we give some equivalent conditions which guarantee that $\xi(M)$ is Fréchet-Urysohn. The statement (1) $\Leftrightarrow$ (2) below is due to Malyhin [M, Theorem 1].

Theorem 2.5. *Let M be a closed subset of ω^*. Then the following statements are equivalent*

(1) *M is a regular closed subset of ω^*;*
(2) *$\xi(M)$ is a Fréchet-Urysohn space;*
(3) *$\xi(M)$ is a* SFU(M)*-space and $\mathrm{In}_{\omega^*}(M) \neq \emptyset$.*

PROOF: (1) $\Rightarrow$ (2). Assume that $M = \mathrm{Cl}_{\omega^*}(\mathrm{In}_{\omega^*}(M))$ and $M \in \mathrm{Cl}_{\xi(M)}(A)$. Then there is $p \in M$ such that $A \in p$. We claim that $A^* \cap \mathrm{In}_{\omega^*}(M) \neq \emptyset$. If not, then $A^* \cap M = \emptyset$ which would be a contradiction. Let $D \subseteq \omega$ such that $D^* \subseteq A^* \cap \mathrm{In}_{\omega^*}(M)$. We may suppose that $D \subseteq A$. Enumerate faithfully D by $\{d_n : n < \omega\}$. We shall verify that $d_n \to M$. Let $B \subseteq \omega$ be such that $M \subseteq B^*$. If $|D \setminus B| = \omega$, then there is $q \in (D \setminus B)^* \subseteq D^* \subseteq M \subseteq B^*$, but this is impossible. Thus, $|D \setminus B| < \omega$ and so there is $m < \omega$ such that $d_n \in B$ for all $m \leq n < \omega$. This shows that $d_n \to M$.

(2) $\Rightarrow$ (3). We only need to show that $\mathrm{In}_{\omega^*}(M) \neq \emptyset$. By assumption there is a sequence $(n_k)_{k<\omega}$ of positive integers such that $n_k \to M$. Set $A = \{n_k : k < \omega\}$. We claim that $A^* \subset M$. Indeed, let $p \in A^*$ and suppose that $p \notin M$. Then we can find $B \subset A$ such that $B \in p$ and $B^* \cap M = \emptyset$. Since $M \subset (\omega \setminus B)^*$, there is $m < \omega$ such that $n_k \in A \setminus B$ whenever $m \leq k < \omega$, but this is impossible because B is an infinite subset of A.

(3) $\Rightarrow$ (1). We shall verify that $\mathrm{In}_{\omega^*}(M)$ is dense in M. Fix $p \in M$ and $A \in p$. Then $M \in \mathrm{Cl}_{\xi(M)}(A)$ and so there is a sequence $(x_n)_{n<\omega}$ in A such that $M = q\text{-}\lim x_n$ for all $q \in M$. By hypothesis, there is $B \subset \omega$ satisfying $B^* \subset \mathrm{In}_{\omega^*}(M)$. If $q \in B^*$, then $\{n < \omega : x_n \in B^*\} \in q$. Hence, $|A \cap B| = \omega$ and so $\emptyset \neq A^* \cap B^* \subset A^* \cap \mathrm{In}_{\omega^*}(M)$. □

Examples 2.6. (1) If $p \in \omega^*$, then $\xi(p)$ is a FU(p)-space and not a SFU($T(p)$)-space.

(2) Let $p, q \in \omega^*$ be RK-incomparable (see [CN, 10.4]). Then $\xi(p)$ is a WFU($\mathrm{Cl}_{\beta(\omega)} T(q)$)-space and not a WFU($T(q)$)-space since $\xi(p)$ cannot be

a FU(q)-space (by [G-F$_1$, 2.2]). Also, $\xi(p)$ is not q-sequential and is a WFU($\{p,q\}$)-space; this shows that WFU(M)-space does not imply r-sequential for $r \in M$.

(3) If $p, q \in \omega^*$, and p is not $\simeq_{\mathrm{RK}}$-equivalent to q, then $\xi(\{p,q\})$ is not a SFU($\{p,q\}$)-space.

(4) Let $p \in \omega^*$ and $\{p_n : n < \omega\}$ be a discrete subset of $T(p)$. If $M = \mathrm{Cl}_{\omega^*}(\{p_n : n < \omega\})$, then $\xi(M)$ is a SFU(M)-space and is not Fréchet-Urysohn. In fact, since $\mathrm{In}_{\omega^*}(M) = \emptyset$, $\xi(M)$ cannot be Fréchet-Urysohn (2.5). Since $\{p_n : n < \pi\}$ is discrete, we can find a partition $\{A_n : n < \omega\}$ of ω such that $A_n \in p_n$ for each $n < \omega$. Let $A \subset \omega$ be such that $A^* \cap M \neq \emptyset$. Choose $r \in A^* \cap M$. Without loss of generality, we may assume that $r \neq p_n$ for all $n < \omega$. Then there is $m < \omega$ such that $p_m \in A^*$. Since $p_n \simeq_{\mathrm{RK}} p_m$ and $p_n \in A^* \cap A_n^*$, for each $m \neq n < \omega$, there is a bijection $\sigma_n : A_n \to A$ such that $\overline{\sigma}_n(p_n) = p_m$. Define $\sigma = \bigcup_{m \neq n < \omega} \sigma_n : \omega \to A$. Then we have that $\overline{\sigma}[M] = \{p_m\} \in A^* \cap M$ and the conclusion follows from 2.4.

In the next theorem, we will show that the WFU($L({}^{\nu}p)$)-property agrees with the WFU($R(p^{\mu})$)-property for each $0 < \nu, \mu < \omega_1$. First, we prove a lemma.

Lemma 2.7. *Let $N, M \subseteq \omega^*$ such that $N \neq \emptyset \neq M$ and $\forall p \in M \exists q \in N$ $(p \leq_{\mathrm{RK}} q)$. Then every* WFU(M)*-space is a* WFU(N)*-space.*

Proof: Let X be a WFU(M)-space and $A \subseteq X$. Fix $x \in \mathrm{Cl}(A)$. Then, there is a sequence $(x_n)_{n<\omega}$ in A and $p \in M$ such that $x = p\text{-}\lim x_n$. By assumption, there is $q \in N$ such that $p \leq_{\mathrm{RK}} q$. Let $f : \omega \to \omega$ be a surjection such that $\overline{f}(q) = p$. By 1.2, we have that $x = q\text{-}\lim x_{f(n)}$. Thus, x is a WFU(N)-space. □

Theorem 2.8. *If $p \in \omega^*$ and $0 < \nu, \mu < \omega_1$, then a space X is* WFU($L({}^{\nu}p)$)*-space iff it is a* WFU($R(p^{\mu})$)*-space.*

Proof: By 1.7 (6′), 1.7 (7), 1.5 (3) and 1.8 (6) for each $\nu, \mu, \theta < \omega_1$ there are $\gamma, \tau < \omega_1$ such that ${}^{\theta}({}^{\nu}p) \leq_{\mathrm{RK}} (p^{\mu})^{\gamma} \leq_{\mathrm{RK}} {}^{\tau}({}^{\nu}p)$. Then the conclusion is a consequence of 2.7. □

3. p-sequential p-compact spaces.

We saw in 2.6 (2) that a WFU(M)-space is not necessarily r-sequential whenever $r \in M$. There are also r-sequential spaces with $r \in M \subset \omega^*$, which are not WFU(M)-spaces; for instance, every p-sequential which is not a FU(p)-space, for $p \in \omega^*$ (see [G-F$_1$]). The situation is quite different in the class of p-compact spaces when $M = L(p)$, as we shall prove in this section (3.8). First some preliminary lemmas and definitions.

Definition 3.1. Let p be a free ultrafilter on $\omega \times \omega$ and $(x_{n,m})_{n,m<\omega}$ a bisequence in a space X. Then we say $x = p\text{-}\lim x_{n,m}$ if for every $V \in \mathcal{N}(x)$ we have that $\{(n,m) \in \omega \times \omega : x_{n,m} \in V\} \in p$.

Lemma 3.2. *Let $p, q_n \in \omega^*$, for $n < \omega$, and let $(x_{n,m})_{n,m<\omega}$ be a bisequence in a space X. If $q_n\text{-}\lim_{m\to\infty} x_{n,m}$ exists for all $n < \omega$, then $x = (\Sigma_p q_n)\text{-}\lim x_{n,m}$ iff*

$$x = p\text{-}\lim_{n\to\infty}(q_n\text{-}\lim_{m\to\infty} x_{n,m}).$$

Proof: Necessity. Assume that $x \neq p\text{-}\lim_{n\to\infty}(q_n\text{-}\lim_{m\to\infty} x_{n,m})$. Then there is $V \in \mathcal{N}(x)$ such that $\{n < \omega : q_n\text{-}\lim_{m\to\infty} x_{n,m} \notin \mathrm{Cl}(V)\} \in p$. By assumption, $A = \{(n,m) \in \omega\times\omega : x_{n,m} \in V\} \in \Sigma_p q_n$; that is, $\{n < \omega : \{m < \omega : x_{n,m} \in V\} \in q_n\} \in p$. Thus, there is $N < \omega$ such that $q_N\text{-}\lim_{m\to\infty} x_{N,m} \notin \mathrm{Cl}(V)$ and $\{m < \omega : x_{N,m} \in V\} \in q_N$, but this is a contradiction.

Sufficiency. If $V \in \mathcal{N}(x)$, then $\{n < \omega : q_n\text{-}\lim_{m\to\infty} x_{n,m} \in V\} \in p$ and hence $\{n < \omega : \{m < \omega : x_{n,m} \in V\} \in q_n\} \in p$. Thus, $\{(n,m) \in \omega\times\omega : x_{n,m} \in V\} \in \Sigma_p q_n$. Therefore, $x = (\Sigma_p q_n)\text{-}\lim x_{n,m}$. □

We remark that the conclusion of 3.2 does not hold if we drop the condition $q_n\text{-}\lim_{m\to\infty} x_{n,m}$ exists for each $n < \omega$. For instance, in the space $\xi(p\otimes p) = \omega\times\omega\cup\{p\otimes p\}$ we have that $p\otimes p = p\otimes p\text{-}\lim(n,m)$, but $p\text{-}\lim_{n\to\infty}(n,m)$ does not exist for each $n < \omega$.

Definition 3.3. Let X be a space, $A \subset X$ and $p \in \omega^*$. We put $A_{p,0} = A$, and, if $A_{p,\lambda}$ is already defined for every $\lambda < \mu \le \omega_1$, then $A_{p,\mu} = \{x \in X : x = p\text{-}\lim x_n$ for some sequence $(x_n)_{n<\omega}$ in $\bigcup_{\lambda<\mu} A_{p,\lambda}\}$. When it is clear what p we are talking about, we write A_λ instead of $A_{p,\lambda}$. We also define $L(q,A) = \{x \in X : x = q\text{-}\lim x_n$ for some $(x_n)_{n<\omega} \subset A\}$. Because of 1.2, if $p \le_{\mathrm{RK}} q$, then $L(p,A) \subset L(q,A)$.

We omit the proof of the next easy lemma.

Lemma 3.4. *Let $p \in M \subseteq \omega^*$, and let X be a space. Then*

(1) *X is p-sequential iff for every $A \subset X$, $\mathrm{Cl}_X(A) = \bigcup_{\lambda<\omega_1} A_{p,\lambda}$;*
(2) *X is a $\mathrm{WFU}(M)$-space iff for every $A \subset X$, $\mathrm{Cl}_X(A) = \bigcup_{p\in M} L(p,A)$;*
(3) *X is a $\mathrm{FU}(p)$-space iff for every $A \subset X$, $\mathrm{Cl}_X(A) = L_{p,A}$.*

Definition 3.5. Let $p \in \omega^*$. A p-sequential space X has a <u>degree of p-sequentiality</u> equal to $\mu \le \omega_1$ if μ is the least ordinal such that for every $A \subset X$, $\mathrm{Cl}_X(A) = A_\mu$ (see the notation in 3.3).

Theorem 3.6. *For $p \in \omega^*$, every p-sequential space is a $\mathrm{WFU}(L(p))$-space. Moreover, if X has a degree of p-sequentiality equal to $\mu < \omega_1$ (resp. $0 < \mu < \omega$) then X is a $\mathrm{FU}({}^{\mu+1}p)$-space (resp. $\mathrm{FU}({}^{\mu}p)$-space).*

Proof: Let $p \in \omega^*$, X a p-sequential space and $A \subseteq X$. In order to prove all the statements in the theorem, it is enough to show that $A_\lambda \subset L({}^{\lambda+1}p, A)$ for every $0 < \lambda < \omega_1$, and $A_\lambda \subset L({}^{\lambda}p, A)$ if $0 < \lambda < \omega$ (see 3.4). We proceed by induction. Evidently, $A_1 \subset L(p,A)$. Suppose that for every $\lambda < \mu < \omega_1$, $A_\lambda \subset L({}^{\lambda+1}p, A)$ (resp. for every $0 < \lambda < \mu < \omega$, $A_\lambda \subset L({}^{\lambda}p, A)$). Let $x \in A_\mu$, so $x = p\text{-}\lim_{n\to\infty} x_n$, where $x_n \in \bigcup_{\lambda<\mu} A_\lambda$ for all $n < \omega$. For each $n < \omega$ there is $\lambda_n < \mu$ such that $x_n \in A_{\lambda_n}$. Let $\nu = \sup\{\lambda_n : n < \omega\}$. By hypothesis $x_n \in L({}^{\nu}p, A)$ for every $n < \omega$. Then, for each $n < \omega$ there exists a sequence $(x_{n,m})_{m<\omega} \subset A$ such that $x_n = {}^{\nu}p\text{-}\lim_{m\to\infty} x_{n,m}$. Then, because of 3.2, $x = p\text{-}\lim_{n\to\infty}({}^{\nu}p\text{-}\lim_{m\to\infty} x_{n,m}) = {}^{\nu+1}p\text{-}\lim x_{n,m}$; that is, $x \in L({}^{\nu+1}p, A) \subset L({}^{\mu}p, A)$ if $0 < \mu < \omega$, and $x \in L({}^{\nu+1}p, A) \subset L({}^{\mu+1}p, A)$ if $\omega \le \mu < \omega_1$. □

The following lemma is a direct consequence of [G-F$_2$, 2.7 (3)], 1.2 and 1.7 (7).

Lemma 3.7. *For $p \in \omega^*$ and $0 < \nu < \omega_1$, p-compactness, ${}^{\nu}p$-compactness and p^{ν}-compactness are equivalent.*

We are ready now to prove that the converse of Theorem 3.6 holds in the class of p-compact spaces.

Theorem 3.8. *Let $p \in \omega^*$. If X is a p-compact, $\mathrm{WFU}(L(p))$-space, then X is p-sequential. In addition, if X is a $\mathrm{FU}({}^{\mu}p)$-space for some $0 < \mu < \omega_1$, then X has a degree of p-sequentiality $\leq \mu$.*

PROOF: Let $A \subset X$. We will prove by induction that for every $0 < \lambda < \omega_1$, $L({}^{\lambda}p, A) \subset A_{\lambda}$ (see the definition in 3.4). It is clear that $A_1 = L(p, A)$. Assume that, for every $0 < \lambda < \mu < \omega_1$, we have $L({}^{\lambda}p, A) \subset A_{\lambda}$. Let $x \in L({}^{\mu}p, A)$, then $x = {}^{\mu}p\text{-}\lim_{n\to\infty} x_n$ for some sequence $(x_n)_{n<\omega}$ in A. First, suppose that $\mu = \lambda + 1$, so $x = p \otimes {}^{\lambda}p\text{-}\lim x_{n,m}$ where $x_{n,m} \in A$ for all $n, m < \omega$ (this is possible because of 1.2). Since X is p-compact, by 3.7, X is ${}^{\lambda}p$-compact and so ${}^{\lambda}p\text{-}\lim_{m\to\infty} x_{n,m}$ exists for each $n < \omega$. In virtue of 3.2, $x = p\text{-}\lim_{n\to\infty} ({}^{\lambda}p\text{-}\lim_{m\to\infty} x_{n,m})$. By induction hypothesis, we have that, for each $n < \omega$, $y_n = {}^{\lambda}p\text{-}\lim_{m\to\infty} x_{n,m} \in A_{\lambda}$. Therefore, $x = p\text{-}\lim_{n\to\infty} y_n \in A_{\lambda+1} = A_{\mu}$.

Now assume that μ is a limit ordinal. So ${}^{\mu}p \simeq_{\mathrm{RK}} \Sigma_p {}^{\mu(n)}p$ and hence, by 1.2, $x = \Sigma_p {}^{\mu(n)}p\text{-}\lim x_{n,m}$ where $x_{n,m} \in A$ for all $n, m < \omega$. According to 3.7, X is ${}^{\mu(n)}p$-compact for all $n < \omega$. Then, ${}^{\mu(n)}p\text{-}\lim_{m\to\infty} x_{n,m}$ exists for each $n < \omega$. By 3.2, $x = p\text{-}\lim_{n\to\infty} ({}^{\mu(n)}p\text{-}\lim_{m\to\infty} x_{n,m})$. By assumption, for each $n < \omega$, $y_n = {}^{\mu(n)}p\text{-}\lim_{m\to\infty} x_{n,m} \in A_{\mu(n)}$. Therefore, $x \in A_{\mu}$, and so $L({}^{\mu}p, A) \subset A_{\mu}$. □

As a direct consequence of 2.8, 3.6 and 3.8 we have:

Corollary 3.9. *Let $p \in \omega^*$, $0 < \nu < \omega_1$ and X be a p-compact space. Then the following are equivalent*

(a) *X is p-sequential;*
(b) *X is ${}^{\nu}p$-sequential;*
(c) *X is p^{ν}-sequential.*

Observe that if $p \in \omega^*$, then $\xi(p^2)$ is p^2-sequential, but it is not p-sequential, by [G-F$_1$, 2.2].

If we assume CH, then the situation for p-compact $\mathrm{FU}(p)$-spaces is quite different to that described in 3.9. In fact, Boldjiev and Malyhin [BM] have shown that, under CH, for every P-point p of ω^* there is a compact Franklin space X_p (this space is constructed from a suitable almost disjoint family on ω) which is a compact $\mathrm{FU}(p^2)$-space and is not a $\mathrm{FU}(p)$-space. The answer to the following question remains unknown.

Question 3.10. Does ZFC imply that there is a p-compact, $\mathrm{FU}(p^2)$-space which is not a $\mathrm{FU}(p)$-space, for each $p \in \omega^*$?

References

[Be] Bernstein A.R., *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185–193.

[Bl] Blass A.R., *Kleene degrees of ultrafilters*, in: Recursion Theory Weak (OberWolfach, 1984), 29–48, Lecture Notes in Math. **1141**, Springer, Berlin-New York, 1985.

[Bo] Booth D.D., *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970), 1–24.

[BM] Boldjiev B., Malyhin V., *The sequentiality is equivalent to the $\mathcal{F}$-Fréchet-Urysohn property*, Comment. Math. Univ. Carolinae **31** (1990), 23–25.

[CN] Comfort W.W., Negrepontis S., *The Theory of Ultrafilters*, Grundlehren der Mathematichen Wissenschaften, vol. 211, Springer-Verlag, 1974.

[F] Frolík Z., *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91.

[G-F$_1$] Garcia-Ferreira S., *On* FU(p)*-spaces and p-sequential spaces*, Comment. Math. Univ. Carolinae **32** (1991), 161–171.

[G-F$_2$] ———, *Three orderings on $\beta(\omega) \setminus \omega$*, Top. Appl., to appear.

[K] Katětov M., *Products of filters*, Comment. Math. Univ. Carolinae **9** (1968), 173–189.

[Koč] Kočinac L.D., *A generalization of chain net spaces*, Publ. Inst. Math. (Beograd) **44** (58) (1988), 109–114.

[Ko] Kombarov A.P., *On a theorem of A.H. Stone*, Soviet Math. Dokl. **27** (1983), 544–547.

[M] Malyhin V.I., *On countable space having no bicompactification of countable tightness*, Soviet Math. Dokl. **13** (1972), 1407–1411.

[V] Vopěnka P., *The construction of models of set-theory by the method of ultraproducts*, Z. Math. Logik Grundlagen Math. **8** (1962), 293–306.

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(*Received* September 21, 1992)