On the numerical range of operators on locally and on H-locally convex spaces

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Abstract. The spatial numerical range for a class of operators on locally convex space was studied by Giles, Joseph, Koehler and Sims in [3]. The purpose of this paper is to consider some additional properties of the numerical range on locally convex and especially on H-locally convex spaces.

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1. Introduction.

Let X be a locally convex Hausdorff space over the real or complex field K. Each system of seminorms $P = \{p_\alpha, \alpha \in\triangle\}$ inducing its topology will be called a *calibration*. Such a space is said to be H-*locally convex* with respect to a calibration P if P consists of Hilbertian seminorms, i.e. for each $p_\alpha \in P$ there is a semi-inner product $(,)_\alpha$ (it is only nonnegative definite) such that $p^2_\alpha(x) = (x, x)_\alpha$, $x \in X$. Such spaces have been studied e.g. in [6], [7] and [8].

For a given calibration P we denote by $Q_P(X)$ the algebra of *quotient bounded operators* on X, i.e. the set of all linear operators T on X for which

$$p_\alpha(Tx) \leq C_\alpha p_\alpha(x), \quad x \in X, \quad \alpha \in\triangle$$

and by $B_P(X)$ the algebra of *universally bounded operators* on X, i.e. the set of all $T \in Q_P(X)$ for which $C = C_\alpha$ is independent of $\alpha \in\triangle$ ([3]). The family $Q_P(X)$ is a unital l.m.c. algebra with respect to seminorms $\widehat{P} = \{q_\alpha, \alpha \in\triangle\}$ where

$$q_\alpha(T) = \sup\{p_\alpha(Tx) : p_\alpha(x) \leq 1,\ x \in X\}, \quad \alpha \in\triangle, \quad T \in Q_P(X)$$

and $B_P(X)$ is a unital normed algebra with respect to the norm

$$\|T\|_P = \sup\{q_\alpha(T) : \alpha \in\triangle\}.$$

For each $\alpha \in\triangle$ let J_α denote the null space of p_α and X_α the quotient space X/J_α. This is a normed space with the norm $\|x_\alpha\|_\alpha := p_\alpha(x)$, $x_\alpha = x + J_\alpha$, and $\widetilde{X}_\alpha$ is the completion of X_α. For a given $T \in Q_P(X)$ we define T_α on X_α by $T_\alpha x_\alpha := (Tx)_\alpha$, and denote by $\widetilde{T}_\alpha$ its continuous linear extension on $\widetilde{X}_\alpha$ ([3]).

Let (X, P) be an H-locally convex space. Then an operator $T \in Q_P(X)$ has an adjoint operator T^0 if and only if $(Tx, y)_\alpha = (x, T^0 y)_\alpha$ for each $\alpha \in\triangle$ and $x, y \in X$. In this case $(\widetilde{T^0}) = (\widetilde{T}_\alpha)^*$ for all $\alpha \in\triangle$ ([5]) where $(\widetilde{T}_\alpha)^*$ is the adjoint operator of $\widetilde{T}_\alpha$ in the Hilbert space $\widetilde{X}_\alpha$.

2. The spatial numerical range.

The spatial numerical range for a given operator $T \in Q_P(X)$ in a locally convex space (X, P) is defined by

$$V(X, P, T) = \bigcup V\{(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha) : \alpha \in \triangle\}$$

where on the right hand side there are numerical ranges on normed spaces $\widetilde{X}_\alpha$. The above numerical range has the usual properties ([3])

$$V(X, P, \lambda T + \mu I) = \lambda V(X, P, T) + \mu, \quad T \in Q_P(X), \quad \lambda, \mu \in K$$

and

$$V(X, P, T + S) \subseteq V(X, P, T) + V(X, P, S), \quad T, S \in Q_P(X).$$

We shall consider some additional properties of the numerical range in locally convex and especially in H-locally convex spaces.

Let (X, P) be an H-locally convex space. Then $\widetilde{X}_\alpha$ are Hilbert spaces and $V(\widetilde{X}_\alpha, \|\cdot\|, \widetilde{T}_\alpha)$ are convex sets. Unfortunately, their union i.e. $V(X, P, T)$ is in general not convex. In [3] there was defined the algebra numerical range of an element a for a unital l.m.c. algebra $(A, \widehat{P})$ as

$$V(A, \widehat{P}, a) = \bigcup\{V(A_\alpha, \|\cdot\|_\alpha, a_\alpha),\ \alpha \in \triangle\}$$

where A_α are quotient algebras with respect to the null spaces N_α of $q_\alpha \in \widehat{P}$ and $a_\alpha = a + N_\alpha$, $\|a_\alpha\|_\alpha = q_\alpha(a)$. In particular, for the l.m.c. algebra $Q_P(X)$ the following relation holds

$$V(Q_P(X), \widehat{P}, T) = \bigcup\{V(B(\widetilde{X}_\alpha), \|\cdot\|_\alpha, \widetilde{T}_\alpha),\ \alpha \in \triangle\} \tag{2.1}$$

where on the right hand side there are algebra numerical ranges on Banach algebras $B(\widetilde{X}_\alpha)$ ([3]).

For a locally convex space (X, P) the following inclusions were proved in [3]: $V(X, P, T) \subset V(Q_P(X), \widehat{P}, T) \subset \overline{co}\, V(X, P, T)$ where $\overline{co}\, M$ denotes closed convex hull of a set M. For an H-locally convex space we have

Theorem 2.1. *Let (X, P) be an* H*-locally convex space and $T \in Q_P(X)$. Then*

(i) $V(X, P, T) \subset V(Q_P(X), \widehat{P}, T) \subset \overline{V(X, P, T)}$,

(ii) $\overline{V(Q_P(X), \widehat{P}, T)} = \overline{V(X, P, T)}$.

Proof: We have to prove the second inclusion in (i). Let us take into account the connection between the spatial and the algebra numerical range in Hilbert spaces $\widetilde{X}_\alpha$

$$V(Q_P(X), \widehat{P}, T) = \bigcup\{V(B(\widetilde{X}_\alpha), \|\cdot\|_\alpha, \widetilde{T}_\alpha),\ \alpha \in \triangle\} = \tag{2.2}$$

$$= \bigcup\{\overline{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)},\ \alpha \in \triangle\} \subset \overline{\bigcup\{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha),\ \alpha \in \triangle\}} = \overline{V(X, P, T)}$$

Thus (i) holds and taking the closure implies (ii). $\square$

Remark. The relation (ii) can also be found in [3] for the special case when X is a product of Hilbert spaces.

When $\widehat{P}$ is a directed family, $V(Q_P(X), \widehat{P}, T)$ is a convex set ([3]) and we have

Corollary 2.2. *Let (X, P) be an* H*-locally convex space and P a calibration such that $\widehat{P}$ is directed. Then for $T \in Q_P(X)$ the set $\overline{V(X, P, T)}$ is convex.*

3. The numerical range and the spectrum.

Let $T \in Q_P(X)$. Then the number $\lambda \in K$ is in the resolvent set ($\lambda \in \varrho(Q, T)$) if and only if there exists $(T - \lambda I)^{-1} \in Q_P(X)$. The spectrum of T is the set $\sigma(Q, T) := \varrho(Q, T)^c$ ([6]). Let $\sigma_\alpha(\widetilde{T}_\alpha)$ denote the spectrum of $\widetilde{T}_\alpha$ in $\widetilde{X}_\alpha$. Then ([3])

Proposition 3.1. *If (X, P) is a complete locally convex space and $T \in Q_P(X)$, then*

$$\sigma(Q, T) = \bigcup \{\sigma_\alpha(\widetilde{T}_\alpha),\ \alpha \in \Delta\}.$$

As in a Banach space we can define the following four main subsets of the spectrum: $\sigma_p(Q, T)$, $\sigma_c(Q, T)$, $\sigma_r(Q, T)$ and $\sigma_a(Q, T)$ — the point, the continuous, the residual and the approximate spectrum respectively.

Definition 3.2. *For $T \in Q_P(X)$ and $\lambda \in K$ in a locally convex space (X, P) we have*

(i) $\lambda \in \sigma_p(Q, T)$ if and only if $\ker(T - \lambda I) \neq \{0\}$,
(ii) $\lambda \in \sigma_c(Q, T)$ if and only if there exists $(T - \lambda I)^{-1}$ on the set $\operatorname{im}(T - \lambda I)$ which is dense in X and $(T - \lambda I)^{-1} \notin Q_P(X)$,
(iii) $\lambda \in \sigma_r(Q, T)$ if and only if $(T - \lambda I)^{-1}$ exists on the set $\operatorname{im}(T - \lambda I)$ which is not dense in X,
(iv) $\lambda \notin \sigma_a(Q, T)$ if and only if for each $\alpha \in \Delta$ there exists $C_\alpha > 0$ such that $p_\alpha((T - \lambda I)x) \geq C_\alpha p_\alpha(x)$, $x \in X$.

Let us write down the following connection.

Proposition 3.3. *For $T \in Q_P(X)$ in a locally convex space (X, P) the following holds*

$$\sigma_a(Q, T) \cup \sigma_r(Q, T) = \sigma(Q, T).$$

Proof: Let $\lambda \in \sigma_a(Q, T)^c \cap \sigma_r(Q, T)^c$ and $y \in X$. Since $\operatorname{im}(T - \lambda I)$ is dense, there exists a net $\{x_\delta\}$ such that $y_\delta := Tx_\delta - \lambda x_\delta \to y$. Since $\lambda \notin \sigma_a(Q, T)$ by the above definition there exists on $\operatorname{im}(T - \lambda I)$ the inverse operator which is continuous in the sense $p_\alpha((T - \lambda I)^{-1} z) \leq D_\alpha p_\alpha(z)$, $\alpha \in \Delta$, $z \in \operatorname{im}(T - \lambda I)$. Hence the sequence $x_\delta = (T - \lambda I)^{-1} y_\delta$ is also convergent, $x_\delta \to x$ and by continuity of $T - \lambda I$ it follows $(T - \lambda I)x = y$. Thus, $\operatorname{im}(T - \lambda I) = X$ and by the above inequality $(T - \lambda I)^{-1} \in Q_P(X)$, which means $\lambda \in \sigma(Q, T)^c$. The reverse inclusion $\sigma_a(Q, T) \cup \sigma_r(Q, T) \subset \sigma(Q, T)$ is obvious. $\square$

Some connections between parts of the spectrum on X and on the quotient spaces $\widetilde{X}_\alpha$ are

Proposition 3.4. *For $T \in Q_P(X)$ on a separated locally convex space (X, P) the following two relations hold:*

(i) $\sigma_p(Q, T) \subset \cup\{\sigma_p(\widetilde{T}_\alpha), \alpha \in \Delta\}$,

(ii) $\sigma_a(Q, T) = \cup\{\sigma_a(\widetilde{T}_\alpha), \alpha \in \Delta\}$.

Proof: (i) We may choose $\lambda = 0 \in \sigma_p(Q, T)$. Then there is some $x \neq 0$ such that $Tx = 0$. Since X is separated there exists some $\beta \in \Delta$ such that $p_\beta(x) \neq 0$, hence x_β is a nonzero vector in $\ker(\widetilde{T}_\beta)$. Thus, $0 \in \sigma_p(\widetilde{T}_\beta) \subset \cup\{\sigma_p(\widetilde{T}_\alpha), \alpha \in \Delta\}$.

(ii) Again we may choose $\lambda = 0 \notin \sigma_a(Q, T)$. Then for each $\alpha \in \Delta$ there exists $C_\alpha > 0$ such that $p_\alpha(Tx) \geq C_\alpha p_\alpha(x)$, $x \in X$ and consequently $\|T_\alpha x_\alpha\|_\alpha \geq C_\alpha \|x_\alpha\|_\alpha$, $x_\alpha \in X_\alpha$. The same estimate then holds on the space $\widetilde{X}_\alpha$. This means $0 \notin \sigma_a(\widetilde{T}_\alpha)$ for all $\alpha \in \Delta$. Conversely, suppose $0 \notin \sigma_a(\widetilde{T}_\alpha)$ for all $\alpha \in \Delta$, then for each $\alpha \in \Delta$ there is some $C_\alpha \geq 0$ such that $\|\widetilde{T}_\alpha x_\alpha\| \geq C_\alpha \|x_\alpha\|$, $x_\alpha \in \widetilde{X}_\alpha$, in particular we have the same estimate for T_α and it follows

$$p_\alpha(Tx) \geq C_\alpha p_\alpha(x), \quad x \in X,\ \alpha \in \Delta,$$

which means $0 \notin \sigma_a(Q, T)$. □

Corollary 3.5. *For $T \in Q_P(X)$ in a separated locally convex space (X, P), $\lambda \in \sigma_a(Q, T)$ if and only if there exists an $\alpha \in \Delta$ and a sequence $\{x_n\} \subset X$, $\{x_n\} \subset J_\alpha^c$ such that $p_\alpha((T - \lambda I)x_n) \to 0$.*

We can prove also a result concerning the boundary points of the spectrum. There it must be supposed an additional assumption since the spectrum in general is not closed.

Theorem 3.6. *Let (X, P) be a complete separated locally convex space and $T \in Q_P(X)$. Then*

$$\sigma(Q, T) \cap \partial\sigma(Q, T) \subset \sigma_a(Q, T).$$

Proof: Let $\lambda \in \sigma(Q, T) \cap \partial\sigma(Q, T)$. Then there exists an $\alpha \in \Delta$ such that $\lambda \in \sigma(\widetilde{T}_\alpha)$. If λ were an inner point of $\sigma(\widetilde{T}_\alpha)$, there would exist an open neighborhood S with the property $\lambda \in S \subset \sigma(\widetilde{T}_\alpha)$. Then S would be contained also in $\sigma(Q, T)$ and λ would not be a boundary point of the spectrum. Thus, $\lambda \in \partial\sigma(\widetilde{T}_\alpha)$. By such a theorem for normed spaces ([1]), $\lambda \in \sigma_a(\widetilde{T}_\alpha)$ and by Proposition 3.4 we have $\lambda \in \sigma_a(Q, T)$. □

In the following we shall consider the connections between the spectrum and the numerical range of an operator. The following result is basic to this subject ([3]).

Theorem 3.7. *Let (X, P) be a complete separated locally convex space and $T \in Q_P(X)$. Then*

$$\sigma(Q, T) \subset \overline{V(X, P, T)}.$$

Let us take $\lambda \in \sigma_p(Q, T)$, then there is some $\alpha \in \Delta$ such that $\lambda \in \sigma_p(\widetilde{T}_\alpha) \subset V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)$, consequently the following holds

Proposition 3.8. *Given a locally convex space* (X, P) *and* $T \in Q_P(X)$*, then*

$$\sigma_p(Q, T) \subset V(X, P, T).$$

Let, now, (X, P) be an H-locally convex space.

Proposition 3.9. *Let* (X, P) *be an* H-*locally convex space, let* $T \in B_P(X)$ *and* $\lambda \in V(X, P, T)$ *with the property* $|\lambda| = \|T\|_P$. *Then* $\lambda \in \sigma_a(Q, T)$.

PROOF: Let $\lambda \in V(X, P, T)$. Then λ is in some $V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)$ and by assumption $|\lambda| \leq \|\widetilde{T}_\alpha\| \leq \|T\|_P = |\lambda|$, hence $|\lambda| = \|\widetilde{T}_\alpha\|$. By a similar theorem for Hilbert spaces ([4]), and by Proposition 3.4 it follows $\lambda \in \sigma_a(\widetilde{T}_\alpha) \subset \sigma_a(Q, T)$. □

In the Hilbert space the convex hull of the spectrum of a normal operator is equal to closedness of the numerical range. A generalization of this result is

Theorem 3.10. *Let* (X, P) *be a complete* H-*locally convex space, let* $T \in Q_P(X)$ *be an operator for which* T^0 *exists and let* T *be normal operator. Then*

$$\overline{co}\, \sigma(Q, T) = \overline{co}\, V(X, P, T).$$

PROOF: First, by Theorem 3.7, $\overline{co}\, \sigma(Q, T) \subset \overline{co}\, V(X, P, T)$. Conversely, since T is normal, $T^0 T = T T^0$, all operators $\widetilde{T}_\alpha$ are normal, too. Thus, in Hilbert spaces $\widetilde{X}_\alpha$ we have

$$co\, \sigma(\widetilde{T}_\alpha) = \overline{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)} = V(B(\widetilde{X}_\alpha), \|\cdot\|_\alpha, \widetilde{T}_\alpha), \quad \alpha \in \triangle\, .$$

Let us take the union for all $\alpha \in \triangle$, then (2.1) implies

$$V(Q_P(X), \widehat{P}, T) = \bigcup \{V(B(\widetilde{X}_\alpha), \|\cdot\|_\alpha, \widetilde{T}_\alpha), \alpha \in \triangle\} = \bigcup \{co\, \sigma(\widetilde{T}_\alpha), \alpha \in \triangle\} \subset$$

$$\subset co \bigcup \{\sigma(\widetilde{T}_\alpha), \alpha \in \triangle\} = co\, \sigma(Q, T).$$

By Theorem 2.1

$$\overline{V(X, P, T)} = \overline{V(Q_P(X), \widehat{P}, T)} \subset \overline{co}\, \sigma(Q, T).$$

□

Corollary 3.11. *Let* (X, P) *be a complete* H-*locally convex space and* $T \in Q_P(X)$ *an operator such that* T^0 *exists and let* T *be normal. When* P *is a calibration such that* $\widehat{P}$ *is directed then*

$$\overline{co}\, \sigma(Q, T) = \overline{V(X, P, T)}.$$

Let us denote by $d(\lambda, M)$ the distance between λ and the set M in the complex plane. Then

Theorem 3.12. *Let* (X,P) *be a complete* H-*locally convex space, let* $T \in Q_P(X)$ *and* $\lambda \notin \overline{V(X,P,T)}$. *Then* $(T-\lambda I)^{-1} \in B_P(X)$ *and*

$$\|(T-\lambda I)^{-1}\|_P \le (d(\lambda, \overline{V(X,P,T)}))^{-1}\,. \tag{3.1}$$

Proof: One may suppose $\lambda = 0$. Let $0 \notin \overline{V(X,P,T)}$, then by Theorem 3.7, $0 \in \rho(Q,T)$ and by Proposition 3.1, $0 \in \rho(\widetilde{T}_\alpha)$ for each $\alpha \in \triangle$. Thus

$$\|\widetilde{T}_\alpha^{-1} x_\alpha\|_\alpha \le \|\widetilde{T}_\alpha^{-1}\|_\alpha \|x_\alpha\|_\alpha, \quad x_\alpha \in \widetilde{X}_\alpha$$

for each $\alpha \in \triangle$ and then it is easy to see that $p_\alpha(T^{-1}x) \le \|\widetilde{T}_\alpha^{-1}\|_\alpha p_\alpha(x)$, for all $x \in X$ and $\alpha \in \triangle$. Hence

$$q_\alpha(T^{-1}) \le \|\widetilde{T}_\alpha^{-1}\|_\alpha, \quad \alpha \in \triangle\ . \tag{3.2}$$

For each $\alpha \in \triangle$ the inclusion in (2.2) implies $0 \notin \overline{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)}$. By an analogous inequality as is (3.1) for Hilbert space ([4]) and again by the inclusion in (2.2) we obtain

$$\|\widetilde{T}_\alpha^{-1}\|_\alpha \le (d(0, \overline{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)}))^{-1} \le (d(0, \bigcup\{\overline{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha)}, \alpha \in \triangle\}))^{-1}$$

$$\le (d(0, \overline{\bigcup\{V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha), \alpha \in \triangle\}}))^{-1} = (d(0, \overline{V(X,P,T)}))^{-1}\,.$$

By (3.2) we obtain $q_\alpha(T^{-1}) \le (d(0, \overline{V(X,P,T)}))^{-1}$ for each $\alpha \in \triangle$. Thus, $T^{-1} \in B_P(X)$ and $\|T^{-1}\|_P \le (d(0, \overline{V(X,P,T)}))^{-1}$. □

In a separated complex locally convex space (X,P), an operator $T \in Q_P(X)$ is *hermitian* if $V(X,P,T) \subset \mathcal{R}$ ([3]). This definition is consistent with the notion of a hermitian operator in an H-locally convex space ([6]), namely

Proposition 3.13. *In a complex* H-*locally convex space for an operator* $T \in Q_P(X)$ *the following two relations are equivalent:*

(i) $V(X,P,T) \subset \mathcal{R}$,
(ii) $(Tx,y)_\alpha = (x,Ty)_\alpha$, $\alpha \in \triangle$, $x, y \in X$.

Proof: If $V(X,P,T) \subset \mathcal{R}$, then $V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha) \subset \mathcal{R}$ for all $\alpha \in \triangle$, consequently $\widetilde{T}_\alpha^* = \widetilde{T}_\alpha$. Thus, $(Tx,y)_\alpha = (x,Ty)_\alpha$, $\alpha \in \triangle$, $x, y \in X$. Conversely, when the last equalities are valid, they hold for all $\widetilde{T}_\alpha$, too, hence $V(\widetilde{X}_\alpha, \|\cdot\|_\alpha, \widetilde{T}_\alpha) \subset \mathcal{R}$ for all $\alpha \in \triangle$, thus, $V(X,P,T) \subset \mathcal{R}$. □

Definition 3.14. *Let (X, P) be a locally convex space and $T \in Q_P(X)$.*

(i) *When $\sigma(Q, T)$ is a bounded set, we define* **the spectral radius** *of T by the relation*

$$r(Q, T) = \sup\{|\lambda| : \lambda \in \sigma(Q, T)\}.$$

(ii) *When $V(X, P, T)$ is bounded, we define* **the numerical radius** *of T by the relation*

$$v(Q, T) = \sup\{|\lambda| : \lambda \in V(X, P, T)\}.$$

By $r(\widetilde{T}_\alpha)$ and $v(\widetilde{T}_\alpha)$ we denote the spectral radius and the numerical radius of $\widetilde{T}_\alpha$ in $\widetilde{X}_\alpha$, respectively. By the above definition the following equality follows

$$v(Q, T) = \sup\{v(\widetilde{T}_\alpha), \alpha \in \triangle\}. \tag{3.3}$$

It was proved in [3] that for $T \in Q_P(X)$ the numerical range is bounded if and only if $T \in B_P(X)$.

Proposition 3.15. *For $T \in B_P(X)$ in a locally convex space (X, P) the following holds:*

$$r(Q, T) \leq v(Q, T) \leq \|T\|_P\,.$$

Proof: The first inequality follows by Theorem 3.7. Let us prove the second one. Clearly, $v(\widetilde{T}_\alpha) \leq \|\widetilde{T}_\alpha\|_\alpha = q_a(T) \leq \|T\|_P$ for each $\alpha \in \triangle$, hence taking the supremum we obtain $v(Q, T) \leq \|T\|_P$. □

In [3] it was also proved that when a hermitian operator $T \in Q_P(X)$ has a bounded spectrum, then $T \in B_P(X)$. For an H-locally convex space one can somewhat generalize this result.

Theorem 3.16. *Let (X, P) be a complete* H*-locally convex space and $T \in Q_P(X)$ an operator for which T^0 exists, let T be normal and let $r(Q, T) < \infty$. Then the following two assertions hold:*

(i) $T \in B_P(X)$,

(ii) $r(Q, T) = v(Q, T) = \|T\|_P$.

Proof: Using the equality $(\widetilde{T}_\alpha)^* = (\widetilde{T^0})_\alpha$ ([5]), normality of T implies the normality of all $\widetilde{T}_\alpha$, $\alpha \in \triangle$. Consequently

$$q_\alpha(T) = \|T_\alpha\|_\alpha = \|\widetilde{T}_\alpha\|_\alpha = r(\widetilde{T}_\alpha) \leq r(Q, T), \quad \alpha \in \triangle\,.$$

Thus, $\sup q_\alpha(T) < \infty$, which implies $T \in B_P(X)$ and the inequality $\|T\|_P \leq r(Q, T)$. The reverse inequality follows by Proposition 3.15. □

Corollary 3.17. *Let (X,P) be as above and let $S,T\in B_P(X)$ be such that their adjoint exist and they are normal, then the following inequality holds*

$$v(Q,ST)\le v(Q,S)v(Q,T).$$

The numerical radius in locally convex spaces has the same properties as the one in normed spaces.

Proposition 3.18. *Let (X,P) be a locally convex space. Then the numerical radius is a norm on $B_P(X)$, equivalent to $\|\cdot\|_P$. Precisely, the following inequalities hold:*

$$e^{-1}\cdot\|T\|_P\le v(Q,T)\le\|T\|_P,\quad T\in B_P(X).$$

Proof: Clearly, by the definition $v(Q,T)\ge 0$ and $v(Q,\lambda T)=|\lambda|v(Q,T)$. If $v(Q,T)=0$, by (3.3), $v(\widetilde{T}_\alpha)=0$ and hence $\widetilde{T}_\alpha=0$, for all $\alpha\in\triangle$, so $T=0$. For $S,T\in Q_P(X)$ and all $\alpha\in\triangle$ the following inequality holds:

$$v(\widetilde{S_\alpha}+\widetilde{T}_\alpha)\le v(\widetilde{S_\alpha})+v(\widetilde{T}_\alpha).$$

Then by (3.3) also $v(Q,S+T)\le v(Q,S)+v(Q,T)$. For any $\alpha\in\triangle$ we have the inequality $e^{-1}\cdot\|\widetilde{T}_\alpha\|\le v(\widetilde{T}_\alpha)$ ([1]). Then such an inequality holds also for the supremum, thus, the left inequality in the above proposition is proved. □

For the case of an H-locally convex space we can generalize more inequalities from the Hilbert space.

Proposition 3.19. *Let (X,P) be an* H*-locally convex space and $S,T\in B_P(X)$. Then the following inequalities hold:*

(i) $\frac{1}{2}\|T\|_P\le v(Q,T)\le\|T\|_P$,
(ii) $v(Q,ST)\le 4v(Q,S)v(Q,T)$,
(iii) $v(Q,T^n)\le v(Q,T)^n$, $n\in N$.

Proof: (i) Since $\widetilde{X}_\alpha$ are Hilbert spaces, we have $\|\widetilde{T}_\alpha\|_\alpha\le 2v(\widetilde{T}_\alpha)$, for all $\alpha\in\triangle$. Taking the supremum we obtain $\|T\|_P\le 2v(Q,T)$. The second inequality is known by the previous proposition. The estimate (ii) follows by (i). For each $\alpha\in\triangle$ the Berger inequality $v(\widetilde{T}^n_\alpha)\le v(\widetilde{T}_\alpha)^n$, $n\in N$, holds and taking the supremum we obtain (iii). □

Finally, we give a result concerning Q-equivalent calibrations. Two calibrations P and P' on a locally convex space X are Q-*equivalent* (denoted by $P\simeq P'$) if each seminorm $p\in P$ is equivalent to some $p'\in P'$ and vice versa (see [5]). It is easy to see that $P\simeq P'$ implies $Q_P(X)=Q_{P'}(X)$.

Theorem 3.20. *Let (X,P) be a complex complete locally convex space and $T\in Q_P(X)$ such that $\sigma(Q,T)$ is bounded. Then*

$$\overline{co}\,\sigma(Q,T)=\bigcap\{\overline{co}\,V(X,P',T):P'\simeq P\}.$$

PROOF: Since $\sigma(Q,T)$ is independent of calibrations, by Theorem 3.7, $\overline{co}\,\sigma(Q,T) \subset \overline{co}\,V(X,P',T)$, for all $P' \simeq P$, hence $\overline{co}\,\sigma(Q,T) \subset \cap\{\overline{co}\,V(X,P',T) : P' \simeq P\}$. Let us prove the opposite inclusion. Since $\overline{co}\,\sigma(Q,T)$ is compact and convex it is an intersection of the open circular discs containing $\overline{\sigma(Q,T)}$. Take any such an open disc $S = \{\lambda : |\lambda - \lambda_0| < r'\}$. Clearly $r(Q,T-\lambda_0 I) < r'$. Let us choose a number ε such that $0 < \varepsilon < r' - r(Q,T-\lambda_0 I)$. Then by [3] there exists a calibration $P' = \{p'_\alpha, \alpha \in \triangle\}$ on X which has the same indexing as P such that for each $\alpha \in \triangle$ the corresponding norm $\|\cdot\|'_\alpha$ on $\widetilde{X}_\alpha$ is equivalent to $\|\cdot\|_\alpha$, such that $T - \lambda_0 I \in B_{P'}(X)$ and such that

$$r(Q,T-\lambda_0 I) \leq \|T-\lambda_0 I\|_{P'} \leq r(Q,T-\lambda_0 I) + \varepsilon.$$

It is obvious that P' and P are Q-equivalent. Suppose that $\lambda \in \overline{V(X,P',T)}$ then $\lambda - \lambda_0 \in \overline{V(X,P',T-\lambda_0 I)}$ and by Proposition 3.15 we have

$$|\lambda - \lambda_0| \leq \|T - \lambda_0 I\|_{P'} < r',$$

which means that S contains $\overline{V(X,P',T)}$ and then also $\overline{co}\,V(X,P',T)$. Thus, the set $\cap\{\overline{co}\,V(X,P',T) : P' \simeq P\}$ is contained in every circular disc that contains $\overline{\sigma(Q,T)}$ and the opposite inclusion is proved. □

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