A strengthening of the Katětov-Tong insertion theorem

Tomasz Kubiak

Abstract. Normal spaces are characterized in terms of an insertion type theorem, which implies the Katětov-Tong theorem. The proof actually provides a simple necessary and sufficient condition for the insertion of an ordered pair of lower and upper semicontinuous functions between two comparable real-valued functions. As a consequence of the latter, we obtain a characterization of completely normal spaces by real-valued functions.

Keywords: normal space, semicontinuous functions, insertion, limit functions, completely normal space

Classification: 54D15, 54C30

Urysohn's lemma states that if X is a normal space and K is its closed subset included in an open subset U, then there is a continuous function f on X such that $\chi_K \leq f \leq \chi_U$. The well-known Katětov-Tong insertion theorem strengthens this characterization by replacing χ_K and χ_U by arbitrary upper and lower semicontinuous functions. Another characterization due to Urysohn asserts that each two separated F_σ-sets in X have disjoint open neighbourhoods. In terms of the upper and lower limit functions, this property can be stated as follows: given an F_σ-set A and a G_δ-set B with $(\chi_A)^* \leq \chi_B$ and $\chi_A \leq (\chi_B)_*$, there is a lower semicontinuous function g on X such that $\chi_A \leq g \leq g^* \leq \chi_B$.

We shall show that this is possible in the more general case in which the characteristic functions are replaced respectively by a countable supremum of upper semicontinuous functions and a countable infimum of lower semicontinuous functions. This result can indeed be claimed to be a strengthening of the Katětov-Tong theorem, because a pair of lower and upper semicontinuous functions between an upper and a lower semicontinuous function is all we need to arrive at the Katětov-Tong result, in which one has a continuous function in-between. Namely, applying the insertion procedure repeatedly (as in the Urysohn's lemma construction) will produce the desired function. Since we will not refer to the theorem of Katětov and Tong, the present proof provides a technique of proving it, which is different from those of Katětov [8, Theorem 1], Tong [18, Theorem 2], Priestley [15, Theorem 2], Jameson [7, 12.1], and Engelking [6, 2.7.2 (c)]. (See also Michael [13, Theorem 3.1'], Blatter and Seever [5, Interposition Theorem], Lane [12, Theorem 2.1], Preiss and Vilímovský [14, 3.4], Blair [3, 3.5], Blair and Swardson [4, 1.6], Kubiak [10, 3.7],

During the preparation of this paper, the author was visiting Rhodes University as a Research Fellow supported by the Foundation for Research Development, Pretoria. A Rhodes University research grant is also acknowledged

and Kotzé and Kubiak [9, 3.6] for generalizations of the Katětov-Tong theorem.) In fact, the proof does slightly more than this, and gives a simple condition which is necessary and sufficient for the insertion of an ordered pair of lower and upper semicontinuous functions between two comparable real-valued functions. (Note that such a condition for the insertion of a continuous function is given in the above mentioned theorems of [5] (implicitly), [12], [14], and [3].) As a consequence of the latter, we obtain a characterization of completely normal spaces in terms of real-valued functions. By using a different method, this was proved directly in [11, Theorem 1].

1. Preliminaries.

Let us now introduce some notation and terminology. For X a topological space, $F(X)$ is the set of all real-valued functions on X with the usual pointwise ordering. All the 'sup's ($\bigvee$) and 'inf's ($\bigwedge$) in $F(X)$ are understood in this pointwise order. Given $f \in F(X)$ and $t \in \mathbf{R}$, $[f < t] = \{x \in X : f(x) < t\}$ and $[f > t] = \{x \in X : f(x) > t\}$; a similar convention applies to $[f \leq t]$ and $[f \geq t]$. The characteristic function of $A \subset X$ is denoted by χ_A. The constant map in $F(X)$ whose constant value is $t \in \mathbf{R}$ will be denoted by $\mathbf{t}$.We denote by $LSC\,(X)$, $USC\,(X)$, and $C(X)$ the collections of all lower semicontinuous, upper semicontinuous, and continuous members of $F(X)$. As always, $\mathbf{Q}$ is the set of all rationals and $I = [0, 1]$.

In formulating our results, we shall use the upper and lower limit functions a^* and a_* of a given $a \in F(X)$, which are defined as follows: for all $x \in X$

$$a^*(x) = \bigwedge\{\bigvee a(U) : U \text{ is an open nbhd of } x\},$$

and, dually,

$$a_*(x) = \bigvee\{\bigwedge a(U) : U \text{ is an open nbhd of } x\}.$$

In general, a^* and a_* are extended real-valued functions. In this paper, however, we shall deal only with the case that $a^* \leq b$ and $a \leq b_*$, with $a, b \in F(X)$. Then $a^*, b_* \in F(X)$, since $a \leq a^*$ and $b_* \leq b$. Also, $a^* \in USC\,(X)$ and $a_* \in LSC\,(X)$, and $a^* = a$ iff $a \in USC\,(X)$, while $a_* = a$ iff $a \in LSC\,(X)$. Clearly, $a \leq b$ implies $a^* \leq b^*$ and $a_* \leq b_*$. Given a family $\mathcal{A} \subset F(X)$, we write $\mathcal{A}^* = \{a^* : a \in \mathcal{A}\}$ and $\mathcal{A}_* = \{a_* : a \in \mathcal{A}\}$. (See [2, 5.4] or [17].)

We note that these concepts are not indispensable for this paper and we use them just for convenience. Otherwise, instead of $a^* \leq b$, for example, one has to write $a \leq h \leq b$ with $h \in USC\,(X)$. Moreover, the use of them emphasizes the parallelism between the operators of interior and closure on the power set of X and the operators $(\cdot)_*$ and $(\cdot)^*$ on $F(X)$. In particular, one has $(\chi_A)_* = \chi_{\mathrm{Int}\,A}$ $(\chi_A)^* = \chi_{\overline{A}}$.

2. Insertion theorems.

We begin with the following lemma which extends the so-called normalization lemma (see Aleksandrov and Pasynkov [1, Chap. I, § 5, Lemma 2]) from sets to functions. The proof remains the same (as that of "regularity plus Lindelöfness implies normality").

Lemma 2.1 (NORMALIZATION LEMMA)**.** *Let X be a topological space and $a \le b$ in $F(X)$. Let $\mathcal{G} \subset LSC\,(X)$ and $\mathcal{H} \subset USC\,(X)$ be countable families such that $a \le \bigvee \mathcal{G} \le \bigvee \mathcal{G}^* \le b$ and $a \le \bigwedge \mathcal{H}_* \le \bigwedge \mathcal{H} \le b$. Then there exists an $f \in LSC\,(X)$ such that $a \le f \le f^* \le b$.*

PROOF: Let $\mathcal{G} = \{g_n : n \in \mathbf{N}\}$ and $\mathcal{H} = \{h_n : n \in \mathbf{N}\}$. Define $f_1 = g_1$ and $f_n = g_n \wedge \bigwedge\{(h_i)_* : i < n\}$ for all $n \ge 2$. Then $f = \bigvee\{f_n : n \in \mathbf{N}\} \in LSC\,(X)$ has the required properties. Indeed, we have

$$a \le \bigvee \mathcal{G} \wedge \bigwedge \mathcal{H}_* \le g_1 \vee \bigvee_{n \ge 2} (g_n \wedge \bigwedge_{i<n} (h_i)_*) = f.$$

Since $f_m \le \bigvee\{g_i^* : i \le n\}$ if $m \le n$, and $f_m \le h_n$ if $m > n$, hence $f_m \le h_n \vee \bigvee\{g_i^* : i \le n\} \in USC\,(X)$ for all m and n. Therefore

$$f^* \le \bigwedge_{n \in \mathbf{N}} (h_n \vee \bigvee_{i \le n} g_i^*) \le \bigwedge \mathcal{H} \vee \bigvee \mathcal{G}^* \le b.$$

□

Remark 2.2. Let us note the following obvious facts. Given $a, b \in F(X)$, $\mathcal{A} \subset F(X)$, and $t \in \mathbf{R}$ one has:

(1) $[\bigvee \mathcal{A} > t] = \bigcup\{[a \ge t + 1/n] : a \in \mathcal{A} \text{ and } n \in \mathbf{N}\}$.
(2) $[\bigwedge \mathcal{A} < t] = \bigcup\{[a \le t - 1/n] : a \in \mathcal{A} \text{ and } n \in \mathbf{N}\}$.
(3) If $a^* \le b$ and $a \le b_*$, then $[b < t] \subset [a^* < t] \subset [a \le t]$ and $[b < t] \subset [b_* \le t] \subset [a \le t]$, i.e. $[a > t]$ and $[b < t]$ are separated sets.

It follows from 2.1 that each two separated F_σ-sets in a normal space have disjoint open neigbourhoods (cf. [1, Chap. I, § 5, Lemma 3]), a fact which we need in proving 2.3 which follows. Note that 2.3 (2) can be obtained by combining 2.1 and the Katětov-Tong theorem. However, rather than refer to this theorem, we give an independent argument, and then deduce the Katětov-Tong theorem from the condition (2) of 2.3.

Let us denote by $USC_\sigma(X)$ and $LSC_\delta(X)$, respectively, the collections of 'sup's and 'inf's of all the countable families of $USC\,(X)$ and $LSC\,(X)$. (Note that if $a \in USC_\sigma(X)$ and $b \in LSC_\delta(X)$, then $a \le b$ implies that both a and b are in $F(X)$. In actual fact, all those warnings we make about the finiteness of the involved functions do not really matter, because all the results of this paper are valid for extended functions.)

Theorem 2.3. *For X a topological space, the following statements are equivalent:*

(1) *X is normal.*
(2) *If $a \in USC_\sigma(X)$, $b \in LSC_\delta(X)$, $a^* \le b$ and $a \le b_*$, then there exists an $f \in LSC\,(X)$ such that $a \le f \le f^* \le b$.*
(3) (KATĚTOV-TONG THEOREM) *If $a \in USC\,(X)$, $b \in LSC\,(X)$, and $a \le b$, then there exists an $f \in C(X)$ such that $a \le f \le b$.*

Proof: (1) $\Rightarrow$ (2): By 2.2, $[a > r]$ and $[b < r]$ are separated F_σ-sets for each $r \in \mathbf{Q}$. By normality, for each $r \in \mathbf{Q}$ there exists an open set U_r such that

$$(*) \qquad [a > r] \subset U_r \subset \overline{U}_r \subset [b \geq r].$$

We first consider the case in which a and b are I-valued. By $(*)$ with $\mathcal{G} = \{\mathbf{r} \wedge \chi_{U_r} : r \in \mathbf{Q} \cap I\} \subset LSC\,(X)$ and $\mathcal{H} = \{\mathbf{r} \vee \chi_{\overline{U}_r} : r \in \mathbf{Q} \cap I\} \subset USC\,(X)$ one has

$$a = \bigvee_{r \in \mathbf{Q} \cap I} \mathbf{r} \wedge \chi_{[a>r]} \leq \bigvee \mathcal{G} \leq \bigvee \mathcal{G}^* \leq \bigvee_{r \in \mathbf{Q} \cap I} \mathbf{r} \wedge \chi_{[b \geq r]} = b$$

and

$$a = \bigwedge_{r \in \mathbf{Q} \cap I} \mathbf{r} \vee \chi_{[a>r]} \leq \bigwedge \mathcal{H}_* \leq \bigwedge \mathcal{H} \leq \bigwedge_{r \in \mathbf{Q} \cap I} \mathbf{r} \vee \chi_{[b \geq r]} = b.$$

By 2.1, there exists an f in $LSC\,(X)$ with $a \leq f \leq f^* \leq b$.

Suppose now that a and b are $\mathbf{R}$-valued. Let $h : \mathbf{R} \to (0,1)$ be an increasing homomorphism. Since h is inf- and sup-preserving, we have $(h \circ a)^* = h \circ a^*$ and $(h \circ b)_* = h \circ b_*$ with $h \circ a \in USC_\sigma(X)$ and $h \circ b \in LSC_\delta(X)$ being $(0,1)$-valued. By what was just proved, there is an f in $LSC\,(X)$ such that $h \circ a \leq f \leq f^* \leq h \circ b$ and, thus, with values in $(0,1)$. Then $h^{-1} \circ f$ is the required function.

(2) $\Rightarrow$ (3): As above, we may assume that a and b take values in I. By (2) there exists $g_0 \in LSC\,(X)$ such that $a \leq g_0 \leq g_0^* \leq b = g_1$. A procedure perfectly analogous to that in the Urysohn's lemma construction (e.g. [6, 1.5.10]) can now be used to define a family $\{g_r : r \in \mathbf{Q} \cap I\} \subset LSC\,(X)$ such that

$$g_0 \leq g_r \leq g_1 \quad \text{and} \quad g_r^* \leq g_s \quad \text{whenever } r < s.$$

Next, define

$$\begin{aligned} U_r &= [g_r > 1 - r] && \text{if } 0 \leq r \leq 1, \\ &= \emptyset && \text{if } r < 0, \\ &= X && \text{if } r > 1. \end{aligned}$$

Then each U_r is open and $\overline{U}_r \subset U_s$ whenever $r < s$ in $\mathbf{Q}$. For if $r < s$ in $\mathbf{Q} \cap I$, then $U_r \subset [g_r^* \geq 1 - r] \subset U_s$. Further, it is well-known that g defined by $g(x) = \inf\{r \in \mathbf{Q} : x \in U_r\}$, $x \in X$, is continuous, and clearly takes values in I. Since $a \leq g_r \leq b$, hence $[a > 1 - r] \subset U_r \subset [b > 1 - r]$ for all r, and we conclude that $1 - b \leq g \leq 1 - a$. Therefore $f = 1 - g$ is the required function.

(3) $\Rightarrow$ (1): Obvious. □

We note that the proof of the implication (1) $\Rightarrow$ (2) of 2.3 merely depends upon the condition that $[a > r]$ and $[b < r]$ have disjoint open neighbourhoods. Since this condition is trivially a necessary one for the insertion of f and f^* between a and b, we thus have the following

Theorem 2.4. *Let X be a topological space. For $a \leq b$ in $F(X)$, the following are equivalent:*

(1) *There exists an $f \in LSC\,(X)$ such that $a \leq f \leq f^* \leq b$.*
(2) *For every r in $\mathbf{Q}$, $[a > r]$ and $[b < r]$ have disjoint open neighbourhoods.*

Note in passing that the condition in the normalization lemma is another one which is necessary and sufficient.

One may invent further special cases of 2.4 apart from 2.3 (2). The following one is of some interest, since it characterizes complete normality in terms of real functions and seems to be new in the metric context as well. A self-contained proof based on a more traditional method is given in [11, Theorem 1].

Corollary 2.5 ([11])**.** *Let X be a topological space. The following conditions are equivalent:*

(1) *X is completely normal.*
(2) *If $a, b \in F(X)$, $a^* \leq b$ and $a \leq b_*$, then there exists an $f \in LSC\,(X)$ such that $a \leq f \leq f^* \leq b$.*

Proof: (1) $\Rightarrow$ (2): By 2.2 (3), $[a > r]$ and $[b < r]$ are separated for each $r \in \mathbf{Q}$, hence have disjoint open neighbourhoods by complete normality.

(2) $\Rightarrow$ (1): As in [11], if $A, B \subset X$ are such that $\overline{A} \subset B$ and $A \subset \operatorname{Int} B$, then $(\chi_A)^* \leq \chi_B$ and $\chi_A \leq (\chi_B)_*$. Given $f \in LSC\,(X)$ with $\chi_A \leq f \leq f^* \leq \chi_B$, one gets $A \subset U \subset \overline{U} \subset B$ with open $U = [f > \frac{1}{2}]$. □

Remark 2.6. Both 2.3 and 2.5 may have a fuzzy topological interpretation. With X a topological space, $(X, LSC\,(X, I))$ is a fuzzy topological space, and 2.3 states that X is normal iff $(X, LSC\,(X, I))$ is fuzzy normal, while 2.5 states that X is completely normal iff $(X, LSC\,(X, I))$ is fuzzy completely normal. Whether it is so in the latter case, was an open question by the present author in [16].

References

[1] Aleksandrov P.S., Pasynkov B.A., *Introduction to Dimension Theory* (in Russian), Nauka, Moscow, 1973.
[2] Aumann G., *Reelle Funktionen*, Springer-Verlag, Berlin, 1954.
[3] Blair R.L., *Extension of Lebesgue sets and real-valued functions*, Czechoslovak Math. J. **31** (1981), 63–74.
[4] Blair R.L., Swardson M.A., *Insertion, approximation, and extension of real-valued functions*, Proc. Amer. Math. Soc. **93** (1985), 169–175.
[5] Blatter J., Seever G.L., *Interposition and lattice cones of functions*, Trans. Amer. Math. Soc. **222** (1976), 65–96.
[6] Engelking R., *General Topology*, P.W.N., Warszawa, 1977.
[7] Jameson G.J.O., *Topology and Normed Spaces*, Chapman and Hall, London, 1974.
[8] Katětov M., *On real-valued functions in topological spaces*, Fund. Math. **38** (1951), 85–91; Correction **40** (1953), 203–205.
[9] Kotzé W., Kubiak T., *Insertion of a measurable function*, J. Austral. Math. Soc., to appear.
[10] Kubiak T., *L-fuzzy normal spaces and Tietze extension theorem*, J. Math. Anal. Appl. **125** (1987), 141–153.
[11] ———, *Completely normal spaces and insertion of semicontinuous functions*, to appear.
[12] Lane E.P., *Insertion of a continuous function*, Topology Proc. **4** (1979), 463–478.

[13] Michael E., *Continuous selections I*, Annals of Math. **63** (1956), 361–382.
[14] Preiss D., Vilímovský J., *In-between theorems in uniform spaces*, Trans. Amer. Math. Soc. **261** (1980), 483–501.
[15] Priestley H.A., *Separation theorems for semicontinuous functions on normally ordered topological spaces*, J. London Math. Soc. (2) **3** (1971), 371–377.
[16] Rodabaugh S.E., Höhle U., Klement E.P. (Eds.), *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publ., Dordrecht, 1992, p. 348.
[17] Schmid J., *Rational extension of $C(X)$ and semicontinuous functions*, Dissert. Math. **270** (1988), 1–27.
[18] Tong H., *Some characterizations of normal and perfectly normal spaces*, Duke Math. J. **19** (1952), 289–292.

Department of Mathematics (Pure & Applied), Rhodes University, P.O.Box 94,
6140 Grahamstown, South Africa
and (permanent address)
Instytut Matematyki, Uniwersytet im. Adama Mickiewicza, ul. Matejki 48/49,
60–769 Poznań, Poland

(*Received* September 25, 1991)