Contact manifolds, harmonic curvature tensor and (k,μ)-nullity distribution

Basil J. Papantoniou

Abstract. In this paper we give first a classification of contact Riemannian manifolds with harmonic curvature tensor under the condition that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. Next it is shown that the dimension of the (k,μ)-nullity distribution is equal to one and therefore is spanned by the characteristic vector field ξ.

Keywords: contact Riemannian manifold, harmonic curvature, D-homothetic deformation

Classification: 53C05, 53C20, 53C21

It is well known that there exist contact Riemannian manifolds $[M^{2n+1}, (\varphi,\xi,\eta,g)]$ for which the curvature tensor R in the direction of the characteristic vector field ξ satisfies $R_{XY}\xi = 0$, for any tangent vector fields X, Y of M^{2n+1}. The tangent sphere bundle of a flat Riemannian manifold, for example, admits such a structure [2]. Applying a D-homothetic deformation [7] on M^{2n+1} with $R_{XY}\xi = 0$, we find a new class of contact metric manifolds satisfying the relation

$$(1.1)\qquad R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad (k,\mu)\in\mathbb{R}^2$$

where $2h$ is the Lie derivative of φ with respect to ξ. An interesting property of this class is that the form of (1.1) is invariant under a D-homothetic deformation.

The purpose of this paper is, on the one hand, the classification of the contact Riemannian manifolds having a harmonic curvature tensor under the condition that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution, i.e. satisfies the condition (1.1), and on the other hand, to prove that the (k,μ)-nullity distribution, which we will denote by $N(k,\mu)$ for $k<1$, $k\neq 0$, is a 1-dimensional subspace of T_pM for every $p\in M$ and is spanned by the characteristic vector field ξ.

2. Preliminaries and known results.

Manifolds and tensor fields are supposed to be of the class C^∞.

Let $M = M^{2n+1}$ be a connected differentiable manifold with contact form η, i.e. a tensor field of type $(0,1)$ satisfying $\eta\wedge(d\eta)^n \neq 0$. It is well known that such a manifold admits a vector field ξ, called the *characteristic vector field* such that $\eta(\xi)=1$ and $d\eta(\xi,X)=0$, for every $X\in\chi(M)$ ($\chi(M)$ being the Lie algebra of the

*This work was done while the author was a visiting scholar at Michigan State University.

The author would like to express his sincere thanks to Prof. D.E. Blair for contributing valuable information, making this study possible.

vector fields of M). Moreover, M admits a Riemannian metric g and a tensor field φ of type (1.1) such that

(2.1) (i) $\varphi^2 = -I + \eta \otimes \xi$, (ii) $g(X,\xi) = \eta(X)$, (iii)$g(X,\varphi Y) = d\eta(X,Y)$.

We then say that (φ, ξ, η, g) is a *contact metric structure*. As a consequence of these relations, one has

(2.2) (i) $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, (ii) $\varphi\xi = 0$, (iii) $\eta\varphi = 0$.

Denoting by $\mathcal{L}$ and R the Lie differentiation and the curvature tensor respectively, we define the operators ℓ and h by

$$\text{(2.3)} \qquad \text{(i) } \ell X = R(X,\xi)\xi, \ \text{(ii) } hX = \frac{1}{2}(\mathcal{L}_\xi \varphi)X.$$

The $(1,1)$ tensors ℓ and h are self-adjoint and satisfy

(2.4) (i) $h\xi = 0$, (ii) $\ell\xi = 0$, (iii) $tr\ h = tr\ h\varphi = 0$, (iv) $h\varphi = -\varphi h$.

Since h anticommutes with φ, if X is an eigenvector of h corresponding to the eigenvalue λ, then φX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. If $\triangledown$ is the Riemannian connection of g, then

(2.5) (i) $\triangledown_X \xi = -\varphi X - \varphi h X$, (ii)$\triangledown_X \varphi = 0$, (iii) $\varphi\ell\varphi - \ell = 2(h^2 + \varphi^2)$.

A contact metric manifold for which ξ is a Killing vector field is called a K-*contact* manifold. It is well known that a contact manifold is K-contact if and only if $h = 0$. Moreover, on a K-contact manifold it is valid $R(X,\xi)\xi = X - \eta(X)\xi$. A contact metric manifold is said to be a *Sasakian* manifold if

$$\text{(2.6)} \qquad (\triangledown_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$$

in which case

(2.7) (i) $\triangledown_X \xi = -\varphi X$, (i) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$.

Note that a Sasakian manifold is K-contact, but the converse holds if and only if $\dim M = 3$.

A contact manifold is said to be η-*Einstein* if

$$\text{(2.8)} \qquad Q = a\,I\,d + b\eta \otimes \xi,$$

where Q is the Ricci operator and a, b are smooth functions on M. The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a ξ-*sectional curvature*, while the sectional curvature $K(X,\varphi X)$ is called a φ-*sectional curvature*. The (k,μ)-*nullity* distribution of a contact metric manifold for the pair $(k,\mu) \in \mathbb{R}^2$, is a distribution

$$N(k,\mu) : p \to N_p(k,\mu) = \{Z \in T_pM \mid R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ + \mu[g(Y,Z)hX - g(X,Z)hY]\}.$$

So, if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution we have

$$\text{(2.9)} \qquad R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

Now the following lemma is well known [4], but for completness, we also give the proof.

Lemma 2.1. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then*

(2.10)

1. $\ell X = k(X-\eta(X)\xi)+\mu hX,\ \forall\, X\in\chi(M)$
2. $R(\xi,X)Y = k(g(X,Y)\xi-\eta(Y)X)+\mu(g(hX,Y)\xi-\eta(Y)hX)$
3. $h^2=(k-1)\varphi^2,\ k\le 1$
4. $QX=[2(n-1)-n\mu]X+[2(n-1)+\mu]hX+[2(1-n)+n(2k+\mu)]\eta(X)\xi,\quad n\ge 1$
5. $\varphi Q = Q\varphi - 2[2(n-1)+\mu]h\varphi.$

Proof: 1. Using the relations (2.3 (i)) and (2.9) we have

$$\begin{aligned}\ell X &= R(X,\xi)\xi = k(\eta(\xi)X-\eta(X)\xi)+\mu(\eta(\xi)hX-\eta(X)h\xi)\\ &= k(X-\eta(X)\xi)+\mu hX.\end{aligned}\tag{2.11}$$

2. Using the relation (2.9) and $g(hX,Y)=g(X,hY)$ we have

$$\begin{aligned}g(R(\xi,X)Y,Z) &= g(R(Y,Z)\xi,X) = g(k(\eta(Z)Y-\eta(Y)Z),X)+g(\mu(\eta(Z)hY\\ &\quad -\eta(Y)hZ),X) = k[g(X,Y)\eta(Z)-g(X,Z)\eta(Y)]+\mu[g(X,hY)\eta(Z)\\ &\quad -g(X,hZ)\eta(Y)] = k[g(X,Y)g(\xi,Z)-\eta(Y)g(X,Z)]\\ &\quad +\mu[g(hX,Y)g(\xi,Z)-\eta(Y)g(hX,Z)]\end{aligned}$$

and since this equation is valid for any $Z\in\chi(M)$, we get the required result.

3. Using (2.5 (iii)), (2.10 (i)), and (2.4 (iv)) we have

$$\begin{aligned}(-\ell+\varphi\ell\varphi)X &= -\ell X+\varphi\ell\varphi X\\ &= -k(X-\eta(X)\xi)-\mu hX+\varphi(k\varphi X+\mu h\varphi X)\\ &= 2k\varphi^2X-\mu h(X+\varphi^2X)=2k\varphi^2X\end{aligned}$$

but on the other hand, $-\ell+\varphi\ell\varphi=2(h^2+\varphi^2)$, so we easily get the result. Now using the definition of the Ricci operator Q and the orthonormal basis $\{e_i\}$ one easily computes that

$$Q\xi=\sum_{i=1}^{2n+1}R(\xi,e_i)e_i=(2n+1)k\xi-k\xi+\mu(tr\ h)\xi=2nk\xi.$$

But on any contact manifold $Q(\xi,\xi)=2n-\|h\|^2$, hence we have $\|h\|^2=2n(1-k)$ ≥ 0, from which $k\le 1$.

4.–5. Similarly, one can easily prove these cases as well. □

For more details concerning contact metric manifolds we refer the reader to [2].

We close this section with a brief discussion of the harmonicity of the curvature tensor of a Riemannian manifold. It is well known that, if the divergence of the curvature tensor of a Riemannian manifold is equal to zero, then this curvature tensor is called harmonic. So, a Riemannian manifold has harmonic curvature tensor if and only if the Ricci operator Q, which is given by $g(QX,Y) = S(X,Y)$ where S is the Ricci tensor, satisfies the following relation:

$$(\nabla_X Q)Y - (\nabla_Y Q)X = 0. \tag{2.12}$$

3. Contact manifolds with harmonic curvature tensor and ξ belonging to the (k,μ)-nullity distribution.

Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution, i.e.

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad (k,\mu) \in \mathbb{R}^2. \tag{3.1}$$

Let Q be the Ricci operator of M, then the manifold has the harmonic curvature tensor if, as mentioned above,

$$(\nabla_X Q)Y - (\nabla_Y Q)X = 0 \tag{3.2}$$

for any vector fields X, Y of M.

We first prove the following lemma.

Lemma 3.1. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact Riemannian manifold with ξ belonging to the (k,μ)-nullity distribution. Then*

$$\begin{aligned} g((\nabla_X Q)Y - (\nabla_Y Q)X,\xi) = {} & 2[2(n+k-1) - \mu(k-1)]g(X,\varphi Y) \\ & + 2g(Y,Q\varphi X) - 2[2(n-1)+\mu]g(Y,h\varphi X) \\ & + g(Y,(Q\varphi h + hQ\varphi)X) \end{aligned} \tag{3.3}$$

for any $X, Y \in \chi(M)$.

Proof: Using the symmetry of the operator $\nabla_X Q$ and $(2.10,4)$ we have

$$g((\nabla_X Q)Y,\xi) = g(Y,(\nabla_X Q)\xi) = -2nkg(Y,\varphi X + \varphi hX) + g(Y,Q(\varphi X + \varphi hX)).$$

Similarly,

$$g((\nabla_Y Q)X,\xi) = -2nkg(X,\varphi Y + \varphi hY) + g(X,Q(\varphi Y + \varphi hY)).$$

Hence

$$\begin{aligned} g((\nabla_X Q)Y - (\nabla_Y Q)X,\xi) = {} & 4nkg(X,\varphi Y) \\ & + g(Y,Q\varphi X) + g(Y,Q\varphi hX) \\ & + g(Y,\varphi QX) + g(Y,h\varphi QX). \end{aligned} \tag{3.4}$$

Now using $(2.10,5)$ and $(2.10,3)$ we have

$$\begin{aligned}
g((\nabla_X Q)Y-(\nabla_Y Q)X,\xi) &= 4nkg(X,\varphi Y)+g(Y,Q\varphi X)+g(Y,Q\varphi hX)\\
&\quad+g(Y,Q\varphi X-2[2(n-1)+\mu]h\varphi X)\\
&\quad+g(Y,hQ\varphi X-2[2(n-1)+\mu](k-1)\varphi^3 X)\\
&=2[2(k+n-1)-\mu(k-1)]g(X,\varphi Y)+2g(Y,Q\varphi X)\\
&\quad-2[2(n-1)+\mu]g(Y,h\varphi X)+g(Y,(Q\varphi h+hQ\varphi)X)
\end{aligned}$$

and the proof is complete. □

We now state the main result.

Theorem 3.1. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with harmonic curvature tensor and ξ belonging to the (k,μ)-nullity distribution. Then M is either*

(i) *an Einstein Sasakian manifold, or*
(ii) *an η-Einstein manifold, or*
(iii) *locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant curvature equal to 4, including a flat contact metric structure for $n=1$.*

The proof of this theorem depends largely on the following results.

Lemma 3.2 [4]**.** *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then $k\le 1$. If $k<1$, then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$, $D(-\lambda)$ defined by the eigenspaces of h, where $\lambda=\sqrt{1-k}>0$.*

Theorem 3.2 [2]**.** *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with $R_{XY}\xi=0$ for all vector fields X,Y of M. Then M is locally the product of a flat $(n+1)$-dimensional manifold of positive constant curvature equal to 4, including a flat contact metric structure for $n=1$.*

Theorem 3.3 [4]**.** *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. If $k<1$ then for any X orthogonal to ξ*

(1) *The ξ-sectional curvature $K(X,\xi)$ is given by*

$$K(X,\xi)=\begin{cases} k+\lambda\mu, & \text{if } X\in D(\lambda)\\ k-\lambda\mu, & \text{if } X\in D(-\lambda),\end{cases}$$

(2) *the sectional curvature of a plane section $\{X,Y\}$ normal to ξ is given by*

$$K(X,Y)=\begin{cases} 2(1+\lambda)-\mu, \ \text{if } X,Y\in D(\lambda),\\ -(k+\mu)(g(X,\varphi Y))^2, \ \text{for any unit vectors } X\in D(\lambda),\ Y\in D(-\lambda)\\ 2(1-\lambda)-\mu, \ \text{if } X,Y\in D(-\lambda),\ n>1.\end{cases}$$

Next we prove the following lemma.

Lemma 3.3. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then*

$$\text{(i) If } X \in D(\lambda),\ h(\nabla_\xi X) = \lambda(\nabla_\xi X + \mu\varphi X) \tag{3.5}$$

$$\text{(ii) If } X \in D(-\lambda),\ h(\nabla_\xi X) = -\lambda(\nabla_\xi X + \mu\varphi X). \tag{3.6}$$

Proof: (i) Since $X \in D(\lambda)$, applying (3.1) we easily get

$$R(\xi,X)\xi = -(k+\lambda\mu)X. \tag{1}$$

On the other hand, using the definition of the curvature tensor we have

$$\begin{aligned} R(\xi,X)\xi &= \nabla_\xi\nabla_X\xi - \nabla_{[\xi,X]}\xi = -\nabla_\xi(\varphi X + \varphi hX) \\ &\quad + \varphi[\xi,X] + \varphi h[\xi,X] = -\lambda\varphi\nabla_\xi X + \varphi h\nabla_\xi X + \varphi(\varphi X + \varphi hX) \\ &\quad + \varphi h(\varphi X + \varphi hX) = -\lambda\varphi\nabla_\xi X + \varphi h\nabla_\xi X - (1-\lambda^2)X \end{aligned}$$

and since $k = 1-\lambda^2$, we have

$$R(\xi,X)\xi = -\lambda\varphi\nabla_\xi X + \varphi h\nabla_\xi X - kX. \tag{2}$$

Now comparing (1) with (2) we get

$$-\lambda\varphi\nabla_\xi X + \varphi h\nabla_\xi X = -\lambda\mu X, \tag{3.7}$$

or applying with φ and using $h\xi = 0$ and $g(\nabla_\xi X,\xi) = 0$ we get the required result (3.5).

(ii) For $X \in D(-\lambda)$, again applying (3.1) we have

$$R(\xi,X)\xi = -(k-\lambda\mu)X. \tag{3}$$

On the other hand, using the definition of the curvature tensor we easily have

$$R(\xi,X)\xi = \lambda\varphi\nabla_\xi X + \varphi h\nabla_\xi X - kX. \tag{4}$$

So, comparing (3) and (4) we have

$$\varphi h\nabla_\xi X = \lambda(-\varphi\nabla_\xi X + \mu X)$$

and acting with φ we get

$$h(\nabla_\xi X) = -\lambda(\nabla_\xi X + \mu\varphi X)$$

and the proof is complete. □

We are now going to give the proof of the main Theorem 3.1.

Proof of Theorem 3.1: The case of $k=1$, $\mu\in\mathbb{R}$ gives $\lambda=\sqrt{1-k}=0$, or equivalently $h=0$. So, $R(X,Y)\xi=\eta(Y)X-\eta(X)Y$ and the manifold is a Sasakian. Now using Lemma 3.1 we easily get that this manifold with harmonic curvature tensor is an Einstein manifold. Let $k<1$ and $\mu\in\mathbb{R}$, and suppose $X\in D(\lambda)$, $Y\in D(-\lambda)$. Then one easily proves that $g(Y,Q\varphi hX+hQ\varphi X)=0$ and using the harmonicity of the curvature tensor, applying Lemma 3.1, we get

$$g(Q\varphi X,Y)=\{\lambda[2(n-1)+\mu]-\lambda^2\mu-2(n-\lambda^2)\}g(X,\varphi Y). \tag{1}$$

Replacing Y by φZ $(Z\in D(\lambda))$ and using (2.2 (i)) and (2.10, 5) we deduce

$$g(QX,Z)=c_1g(X,Z),\quad \forall\, X,Z\in D(\lambda), \tag{3.8}$$

where

$$c_1=\lambda[2(n-1)+\mu]+\lambda^2\mu+2(n-\lambda^2)=\text{ const.} \tag{3.9}$$

Next, replacing X by φW $(W\in D(-\lambda))$ in (1) and using (2.2 (i)) we get

$$g(QW,Y)=c_2g(W,Y),\quad \forall\, Y,W\in D(-\lambda), \tag{3.10}$$

where

$$c_2=-\lambda[2(n-1)+\mu]+\lambda^2\mu+2(n-\lambda^2). \tag{3.11}$$

Now differentiating (2.10, 4) with respect to ξ and again using (3.8) we get

$$\begin{aligned} g((\nabla_\xi Q)X+Q(-\varphi X-\varphi hX),Z)+g(QX,-\varphi Z-\varphi hZ)&\\ =c_1[-g(\varphi X+\varphi hX,Z)-g(X,\varphi Z+\varphi hZ)]&\end{aligned}$$

or

$$\begin{aligned} g((\nabla_\xi Q)X,Z)-g(Q(\varphi X+\varphi hX),Z)-g(QX,\varphi Z+\varphi hZ)&\\ =c_1[g(\varphi X+\varphi hX,Z)+g(X,\varphi Z+\varphi hZ)].&\end{aligned} \tag{3}$$

But one easily can prove that

$$g(\varphi X+\varphi hX,Z)=(1+\lambda)g(\varphi X,Z),\ g(X,\varphi Z+\varphi hZ)=-(1+\lambda)g(Z,\varphi X) \tag{4}$$

and

$$\begin{aligned} g(Q\varphi X+Q\varphi hX,Z)&=(1+\lambda)g(Q\varphi X,Z),\\ g(QX,\varphi Z+\varphi hZ)&=-(1+\lambda)g(\varphi QX,Z).\end{aligned} \tag{5}$$

So, the equation (3) is reduced to

$$g((\nabla_\xi Q)X,Z)=0,\quad \forall\, X,Z\in D(\lambda). \tag{3.12}$$

Now, since the curvature tensor is harmonic, using (4) and (5) and $g(\varphi X, Z) = 0$, we have

$$0 = g((\nabla_\xi Q)X, Z) = g((\nabla_X Q)\xi, Z) = -2nkg(\varphi X + \varphi hX, Z) \\ + g[Q(\varphi X + \varphi hX), Z] = (1+\lambda)g(Q\varphi X, Z).$$

Hence, $g(\varphi X, QZ) = 0$ and also since $g(QZ, \xi) = 0$, we conclude from (3.8) and Lemma 3.2 that

$$QX = c_1 X, \quad \forall X \in D(\lambda). \tag{3.13}$$

Similarly, one can obtain

$$QX = c_2 X, \quad \forall X \in D(-\lambda). \tag{3.14}$$

Now differentiating (3.13) with respect to ξ we have

$$(\nabla_\xi Q)X + Q\nabla_\xi X = c_1 \nabla_\xi X, \quad \forall X \in D(\lambda). \tag{3.15}$$

Now suppose that

$$\nabla_\xi X = (\nabla_\xi X)_\lambda + (\nabla_\xi X)_{-\lambda}\,. \tag{6}$$

Using (3.15) and this equation, we have

$$(\nabla_X Q)\xi = (\nabla_\xi Q)X = -Q\nabla_\xi X + c_1 \nabla_\xi X \\ = -Q[(\nabla_\xi X)_\lambda + (\nabla_\xi X)_{-\lambda}] + c_1(\nabla_\xi X)_\lambda + c_1(\nabla_\xi X)_{-\lambda}\,.$$

But from (3.13) and (3.14) we have

$$Q(\nabla_\xi X)_\lambda = c_1(\nabla_\xi X)_\lambda,\ Q(\nabla_\xi X)_{-\lambda} = c_2(\nabla_\xi X)_{-\lambda}\,.$$

So,

$$(\nabla_X Q)\xi = (c_1 - c_2)(\nabla_\xi X)_{-\lambda}\,, \tag{3.16}$$

where

$$c_1 - c_2 = 2\lambda[2(n-1) + \mu]. \tag{3.17}$$

On the other hand,

$$(\nabla_X Q)\xi = 2nk\nabla_X \xi + Q(\varphi X + \varphi hX) = -2nk(\varphi X + \varphi hX) + (1+\lambda)Q\varphi X$$

and using (3.14), we have

$$(\nabla_X Q)\xi = (1+\lambda)(c_2 - 2nk)\varphi X. \tag{3.18}$$

Comparing (3.16), (3.17) and (3.18) we get

$$2\lambda[2(n-1)+\mu](\nabla_\xi X)_{-\lambda} = (1+\lambda)(c_2 - 2nk)\varphi X. \tag{3.19}$$

Now, if we substitute the equation (6) into equation (3.5) of Lemma 3.3, we easily deduce that

$$(\nabla_\xi X)_{-\lambda} = -\frac{\mu}{2}\varphi X.$$

Substituting this equation into equation (3.19) and using (3.11) we conclude either

$$\text{(i) } \mu + 2(n-1) = 0, \text{ or } \text{ (ii) } k = \mu. \tag{3.20}$$

If the first (i) equality holds, then applying Lemma 2.1, we conclude that the Ricci operator Q is given by

$$QX = 2(n^2-1)X + 2(1+nk-n^2)\eta(X)\xi \tag{3.21}$$

which is of the form (2.8) and therefore, the manifold M^{2n+1} is η-Einstein.

If the second (ii) equality holds, then from Theorem 3.3 we get for the ξ-sectional curvatures

$$K(X,\xi) = (1+\lambda)k, \ \forall X \in D(\lambda), \quad K(X,\xi) = (1-\lambda)k, \ \forall X \in D(-\lambda) \tag{3.22}$$

and for the sectional curvatures

$$\begin{aligned} &\text{(i) } K(X,Y) = 2(1+\lambda) - k = (1+\lambda)^2, \quad \forall X,Y \in D(\lambda),\\ &\text{(ii) } K(X,Y) = 2(1-\lambda) - k = (1-\lambda)^2, \quad \forall X,Y \in D(-\lambda),\\ &\text{(iii) } K(X,Y) = 2(\lambda^2-1)(g(X,\varphi Y))^2, \quad \forall X \in D(\lambda), \quad \forall Y \in D(-\lambda). \end{aligned} \tag{3.23}$$

On the other hand, another implication of $k=\mu$ may be taken from Lemma 2.1, and therefore, we get

$$QX = [2(n-1) - nk]X + \lambda[2(n-1)+k]X, \quad \forall X \in D(\lambda). \tag{3.24}$$

But, as we proved $QX = c_1 X$ for every X, so we will have

$$2n-2-nk+2(n-1)\lambda+\lambda(1-\lambda^2) = 2(n-1)\lambda+\lambda(1-\lambda^2)+\lambda^2(1-\lambda^2)+2n-2\lambda^2,$$

from which we get

$$\lambda^4 + (1+n)\lambda^2 - (2+n) = 0. \tag{3.25}$$

The only positive root of this equation is $\lambda = 1$ and since $k = 1-\lambda^2$ (Lemma 3.2), we conclude that $k = \mu = 0$. Hence $R_{XY}\xi = 0$ for all vector fields X, Y. Now, the equation (3.23) gives (i) K(X,Y)=4, $\forall X, Y \in D(\lambda)$, or (ii) K(X,Y)=0, either $X, Y \in D(-\lambda)$ or $X \in D(\lambda)$, $Y \in D(-\lambda)$. Therefore, we conclude that the manifold is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4 and the proof of the theorem is complete. □

4. The dimension of the (k,μ)-nullity distribution.

In the previous paragraph we considered the (k,μ)-nullity distribution $N(k,\mu)$ of the contact metric manifold $[M^{2n+1},(\varphi,\xi,\eta,g)]$. Hence it is natural to ask how large $N(k,\mu)$ can be. If $k=\mu=0$ then $R_{XY}\xi=0$ for any X,Y and so the manifold locally is isometric to the product $E^{n+1}(0)\times S^n(4)$, with ξ belonging to the Euclidean factor [3]. Thus $\dim N(0,0)=n+1$.

Recently, the following theorem has been proved [4]:

Theorem 4.1. *Let M^{2n+1} be a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution. Then $k\le 1$, and if $k=1$ holds, then M is a Sasakian. If $k<1$ then M admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h, where $\lambda=\sqrt{1-k}$. Moreover,*

$$(4.1)\quad \begin{aligned}
&1.\ R(X_\lambda,Y_\lambda)Z_{-\lambda}=(k-\mu)[g(\varphi X_\lambda,Z_{-\lambda})\varphi X_\lambda-g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda]\\
&2.\ R(X_{-\lambda},Y_{-\lambda})Z_\lambda=(k-\mu)[g(\varphi Y_{-\lambda},Z_\lambda)\varphi X_{-\lambda}-g(\varphi X_{-\lambda},Z_\lambda)\varphi Y_{-\lambda}]\\
&3.\ R(X_\lambda,Y_{-\lambda})Z_{-\lambda}=kg(\varphi X_\lambda,Z_{-\lambda})\varphi Y_{-\lambda}+\mu g(\varphi X_\lambda,Y_{-\lambda})\varphi Z_{-\lambda}\\
&4.\ R(X_\lambda,Y_{-\lambda})Z_\lambda=-kg(\varphi Y_{-\lambda},Z_\lambda)\varphi X_\lambda-\mu g(\varphi Y_{-\lambda},X_\lambda)\varphi Z_\lambda\\
&5.\ R(X_\lambda,Y_\lambda)Z_\lambda=[2(1+\lambda)-\mu][g(Y_\lambda,Z_\lambda)X_\lambda-g(X_\lambda,Z_\lambda)Y_\lambda]\\
&6.\ R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda}=[2(1-\lambda)-\mu][g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda}-g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}]
\end{aligned}$$

where $X_\lambda,Y_\lambda,Z_\lambda\in D(\lambda)$ and $X_{-\lambda},Y_{-\lambda},Z_{-\lambda}\in D(-\lambda)$.

We now state and prove the main result of this section.

Theorem 4.2. *Let $[M^{2n+1},(\varphi,\xi,\eta,g)]$ be a contact metric manifold of dimension $2n+1\ge 5$ such that ξ belongs to the (k,μ)-nullity distribution $N(k,\mu)$. If $k<1$ and $k\ne 0$ then $\dim N(k,\mu)=1$ and $N(k,\mu)$ is just the span of ξ.*

Proof: If $P\in M$ then by definition

$$(4.2)\quad \begin{aligned} N_P(k,\mu)=\{Z\in T_PM\mid R(X,Y)Z&=k(g(Y,Z)X-g(X,Z)Y)\\ &+\mu(g(Y,Z)hX-g(X,Z)hY)\}.\end{aligned}$$

Suppose that there exist a unit vector $Z\in N(k,\mu)$ orthogonal to ξ. Then $Z=aZ_\lambda+bZ_{-\lambda}$ where $Z_\lambda,Z_{-\lambda}$ are unit vectors and $a,b\ge 0$.

Suppose that $X,Y\in D(\lambda)$, then using Theorem 4.1 we get

$$(4.3)\quad \begin{aligned} R(X,Y)Z=a[2(1+\lambda)-\mu]&[g(Y,Z_\lambda)X-g(X,Z_\lambda)Y]\\ &+b(k-\mu)[g(\varphi Y,Z_{-\lambda})\varphi X-g(\varphi X,Z_{-\lambda})\varphi Y].\end{aligned}$$

On the other hand, from (4.2) we have

$$(4.4)\quad R(X,Y)Z=a(k+\lambda\mu)[g(Y,Z_\lambda)X-g(X,Z_\lambda)Y].$$

Now comparing these two equations, we get

$$(4.5)\quad \begin{aligned} a(1+\lambda)(1+\lambda-\mu)&[g(Y,Z_\lambda)X-g(X,Z_\lambda)Y]\\ &+b(k-\mu)[g(\varphi Y,Z_{-\lambda})\varphi X-g(\varphi X,Z_{-\lambda})\varphi Y]=0\end{aligned}$$

for all $X, Y \in D(\lambda)$.

Suppose that $g(X,Y) = 0$ and choose $\varphi Y = Z_{-\lambda}$. Then this equation is reduced to

$$a(1+\lambda)(1+\lambda-\mu)[g(Y,Z_\lambda)X - g(X,Z_\lambda)Y] = b(k-\mu)\cdot\varphi X = 0,$$

from which, by taking inner products with φX we deduce

$$b(k-\mu) = 0 \tag{4.6}$$

and

$$a(1+\lambda)(1+\lambda-\mu) = 0. \tag{4.7}$$

Now suppose that $X, Y \in D(-\lambda)$, then working similarly we get

$$\begin{aligned} b(\lambda-1)(\lambda+\mu-1)&[g(Y,Z_{-\lambda})X - g(X,Z_{-\lambda})Y] \\ &+ a(k-\mu)[g(\varphi Y,Z_\lambda)\varphi X - g(\varphi X,Z_\lambda)\varphi Y] = 0. \end{aligned} \tag{4.8}$$

If we choose X, Y to be such that $g(X,Y) = 0$ and $\varphi Y = Z_\lambda$ then the equation (4.8) is reduced to

$$b(\lambda-1)(\lambda+\mu-1)[g(Y,Z_{-\lambda})X - g(X,Z_{-\lambda})Y] + a(k-\mu)\varphi X = 0, \tag{4.9}$$

from which, taking the inner products with φX, we conclude that

$$a(k-\mu) = 0 \tag{4.10}$$

and

$$b(\lambda-1)(\lambda+\mu-1) = 0. \tag{4.11}$$

Now if $k \neq \mu$, (4.6) and (4.10) imply $a = b = 0$ and the proof is complete, since we have $Z = 0$. So suppose $k = \mu$. Then since $k = 1-\lambda^2$, (4.7) and (4.11) become

$$a\lambda(1+\lambda^2) = 0 \tag{4.12}$$

and

$$b\lambda(\lambda-1)^2 = 0. \tag{4.13}$$

But $\lambda \neq 0$ $(k < 1)$ and $\lambda \neq \pm 1$ $(k \neq 0)$ so we also conclude that $a = b = 0$. Therefore, there does not exist a vector Z perpendicular to ξ belonging to the (k,μ)-nullity distribution, $N(k,\mu)$ is spanned by ξ and hence $\dim N(k,\mu) = 1$. $\square$

References

[1] Baikoussis C., Koufogiorgos T., *On a type of contact manifolds*, to appear in Journal of Geometry.
[2] Blair D.E., *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics **509**, Springer-Verlag, Berlin, 1979.
[3] ———, *Two remarks on contact metric structures*, Tôhoku Math. J. **29** (1977), 319–324.
[4] Blair D.E., Koufogiorgos T., Papantoniou B.J., *Contact metric manifolds with characteristic vector field satisfying* $R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$, submitted.
[5] Deng S.R., *Variational problems on contact manifolds*, Thesis, Michigan State University, 1991.
[6] Koufogiorgos T., *Contact metric manifolds*, to appear in Annals of Global Analysis and Geometry.
[7] Tanno S., *Ricci curvatures of contact Riemannian manifolds*, Tôhoku Math J. **40** (1988), 441–448.

University of Patras, Department of Mathematics, 26110 Patras, Greece

(*Received* July 23, 1992)