A note on simple medial quasigroups

K.K. Ščukin

Abstract. A solvable primitive group with finitely generated abelian stabilizers is finite.

Keywords: permutation group, primitive

Classification: 20B15

In [1], J. Ježek and T. Kepka described simple medial quasigroups. Among others, these quasigroups turned out to be finite of prime power order. Now, using multiplication groups of the quasigroups (see [2]), this result can be translated into the language of permutation groups. In the present short note we give a direct proof of the permutation group analogue. In fact, we are going to prove the following more general result:

Theorem. *Let G be a solvable primitive permutation group on a non-empty set Q such that the stabilizers are finitely generated abelian groups. Then G is finite, Q is finite of a prime power order and the stabilizers are cyclic groups.*

Proof: By [3, Theorem 7, p. 37], G is the semidirect product $G = M \rightthreetimes N$, where $M = M(Q,+)$ is the regular representation of an abelian group $(Q,+)$ defined on Q and N is the stabilizer of the zero element 0. Moreover, since N is maximal in G, no non-trivial proper subgroup of M is normal in G. Further, the subring R generated by N in the endomorphism ring of $(Q,+)$ is a finitely generated commutative ring. Now, if $q \in Q$ and $f \in R$ are non-zero, then $\mathrm{Ker}\,(f)$ is a proper subgroup of $(Q,+)$ and $\mathrm{Ker}\,(f)$ is invariant under N, which means that $M(\mathrm{Ker}\,(f))$ is normal in G and consequently $\mathrm{Ker}\,(f) = 0$ and $f(q) \neq 0$. This implies that $R(q)$ is a non-zero subgroup of $(Q,+)$ and, since it is also invariant under N, we have $R(q) = Q$. If $0 \neq p \in Q$, then $p = g(q)$ and $q = hf(q)$ for suitable $g, h \in R$ and $hf(p) = hfg(q) = ghf(q) = g(q) = p$. Thus $hf = 1$ and we have shown that R is a field. However, it is a well known fact that every field, finitely generated as a ring, is finite. In particular, R is a finite field and $\mathrm{card}\,(R) = \mathrm{card}\,(Q)$ is a power of a prime number. Finally, N is a subgroup of the cyclic group R^*, and therefore N is also cyclic. □

References

[1] Ježek J., Kepka T., *Varieties of abelian quasigroups*, Czech. Math. J. **27** (1977), 473–503.

[2] Kepka T., *Multiplication groups of some quasigroups*, Colloquia Math. Soc. J. Bolyai **29** (1977), 459–465.

[3] Suprunĕnko D.A., *Gruppy matric*, Nauka, Moskva, 1972.
[4] Ščukin K.K., *On simple medial quasigroups*, Abstracts of talks of the international conference "Universal algebra, quasigroups and related systems", Jadwisin, Poland, 1989, p. 32.

Catedra de algebra, Universitateu de Stat diu Rep. Moldova, Strada Mateevici 60, Chişinău 14, Moldova

(*Received* November 20, 1992)