Global in time solvability of the initial boundary value problem for some nonlinear dissipative evolution equations

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Abstract. The global in time solvability of the one-dimensional nonlinear equations of thermoelasticity, equations of viscoelasticity and nonlinear wave equations in several space dimensions with some boundary dissipation is discussed. The blow up of the solutions which might be possible even for small data is excluded by allowing for a certain dissipative mechanism.

Keywords: nonlinear thermoelasticity, viscoelasticity, nonlinear wave equation, global solutions

Classification: 35L20, 35L70, 73B30, 73F99

This lecture note is based on the lecture in the summer school held at Prague, June 29–July 3, EVEQ–92. In this note, I will discuss the global in time solvability of the following three equations:

(1) Neumann problem for one-dimensional nonlinear thermoelastic equations;
(2) Dirichlet problem for some nonlinear viscoelastic equations;
(3) Nonlinear wave equations with some boundary dissipation.

As is well-known, one of the methods of solving nonlinear evolution equations is based on the decay structure of the corresponding linearized equations. If you consider the nonlinear wave equation in a bounded domain $\Omega \subset \mathbf{R}^n$ with Dirichlet or Neumann boundary condition:

$$(*)\qquad \begin{cases} u_{tt} - \operatorname{div}\,(\nabla u/\sqrt{1+|\nabla u|^2}) = 0 & \text{in } \Omega \quad \text{for } t>0, \\ u=0 \;\text{ or }\; \nu\cdot\nabla u = 0 & \text{on } \partial\Omega \;\text{ for } t>0, \\ u(0,x)=u_0(x), \;\; u_t(0,x)=u_1(x) & \text{in } \Omega, \end{cases}$$

where t denotes a time, $x=(x_1,\dots,x_n)\in\Omega$, $\partial\Omega$ denotes the boundary of Ω which is a C^∞ and compact hypersurface, $\nabla u = (\partial u/\partial x_1,\dots,\partial u/\partial x_n)$ and $\nu=(\nu_1,\dots,\nu_n)$ denotes the exterior unit normal to $\partial\Omega$, then in general smooth solutions to the problem (*) blow up in finite time no matter how small and smooth initial data

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are. This was shown by MacCamy-Mizel [42] and Klainerman-Majda [39] in the one-dimensional case. And also, P. Godin [26] showed a blow up theorem in the case of the zero Neumann boundary condition in the multi-dimensional space and R. Racke [56, p. 220] showed a blow up theorem in some special related cases of nonlinear wave equations in a bounded domain of three space dimension with zero Dirichlet condition. The reason why this equation is not solvable globally in time is, I think, that the corresponding linear problem:

$$(**)\qquad \begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \quad \text{for } t > 0, \\ u = 0 \ \text{ or } \ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega \ \text{ for } t > 0, \end{cases}$$

has no decay structure, where $\Delta u = \partial^2 u/\partial x_1^2 + \cdots + \partial^2 u/\partial x_n^2$. In fact, let $k \in \mathbf{R}$ be a real number such that there exists a non-trivial function $\phi(x)$ satisfying the homogeneous equation:

$$\begin{cases} (\Delta + k^2)\phi = 0 & \text{in } \Omega, \\ \phi = 0 \ \text{ or } \ \nu \cdot \nabla \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

such k existing infinitely many. Put $u(t,x) = e^{ikt}\phi(x)$, and then u solves (**) and u does not decay as t goes to infinity.

One approach of searching for solutions to (*) is to disregard smoothness and to consider weak solutions. In fact, in the one dimensional case, the existence of weak solutions is rather well-known. But, in the multi-dimensional case, I do not know any existence theorems of weak solutions to (*) and for me to show the existence of weak solutions seems to be a quite hard problem.

Another approach is to allow for the possibility of certain dissipative mechanism. My subjects here are in this direction.

1. Neumann problem of one-dimensional nonlinear thermoelastic equations.

First, let me formulate the problem. Let $\Omega = (0,1)$ be the unit interval of $\mathbf{R}$ identified with reference body with natural temperature $\tau_0 > 0$. The deformation of the reference body Ω after time t past is described by the deformation map:

$$X : x \in \Omega \longmapsto X(t,x) \in \mathbf{R}.$$

Let $T(t,x)$ be the absolute temperature of the point $X(t,x)$. The equation of motion and the balance of energy are described by the following equations:

$$(1.1)\qquad \rho(x)X_{tt} - S_x = f \qquad \text{for } x \in \Omega \text{ and } t > 0,$$

$$(1.2)\qquad \begin{aligned} &(e + \frac{\rho(x)}{2}X_t^2)_t - (SX_t)_x \\ &\qquad = q_x + X_t f + g \qquad \text{for } x \in \Omega \text{ and } t > 0. \end{aligned}$$

Here, the subscripts stand for partial derivatives, S is the stress function, e is the internal energy function, q is the heat flux, $\rho(x)$ is the mass density, f is an external force and g is an external heat supply. For simplicity, I assume that

$$\rho(x) = 1, \ \ g = 0 \ \text{ and } \ q = \kappa T_x,$$

κ being a positive constant. In this lecture note, as a boundary condition, I consider the traction free and thermally insulated condition which is described by the following formula:

$$S = q = 0 \qquad \text{for } x \in \partial\Omega. \tag{N.N}$$

Note that $\partial\Omega$ consists of only two points 0 and 1. The initial condition is given by the formula:

$$X(0,x) = X_0(x), \;\; X_t(0,x) = X_1(x), \;\; T(0,x) = T_0(x). \tag{1.3}$$

Now, let me explain the assumptions. The first one is that

$$f = f(x), \;\; \text{that is } f \text{ depends only on } x. \tag{A.1}$$

The next one is concerned with the constitutive relation. Let F be a variable corresponding to X_x. Let ψ be a Helmholtz free energy function and η be an entropy function. The second assumption is that

(A.2) S, e, ψ and η are functions in (F, T) only, that is
$S = S(F,T)$, $e = e(F,T)$, $\psi = \psi(F,T)$, $\eta = \eta(F,T)$,
and they are in $C^\infty(G(\delta))$,

where

$$G(\delta) = \{(F,T) \in \mathbf{R}^2 \mid |(F,T) - (1,\tau_0)| \le \delta\}.$$

In this lecture, all the functions are real-valued unless it is mentioned. The 2nd Law of Thermodynamics tells us that the following two relations are equivalent:

$$de = SdF + Td\eta \iff d\psi = SdF - \eta dT,$$

from which the following constitutive relations follow:

$$S = \frac{\partial \psi}{\partial F}, \;\; \eta = -\frac{\partial \psi}{\partial T}, \;\; e = \psi - T\frac{\partial \psi}{\partial T}. \tag{A.3}$$

The next assumption is that

$$\frac{\partial^2 \psi}{\partial F^2} > 0, \;\; \frac{\partial^2 \psi}{\partial F \partial T} \neq 0, \;\; \frac{\partial^2 \psi}{\partial T^2} < 0 \quad \text{in } G(\delta). \tag{A.4}$$

Under the constitutive relation (A.3), (1.2) is equivalent to the following equation:

$$T\eta_t = q_x \qquad \text{for } x \in \Omega \qquad \text{and } t > 0. \tag{1.4}$$

In fact,

$$\begin{aligned} e_t &= \frac{\partial \psi}{\partial T}T_t + \frac{\partial \psi}{\partial F}X_{xt} - T_t\frac{\partial \psi}{\partial T} - T\left(\frac{\partial \psi}{\partial T}\right)_t = T\eta_t + SX_{xt}, \\ \left(\frac{1}{2}X_t^2\right)_t &= X_tX_{tt} = S_xX_t + fX_t. \end{aligned}$$

Combining these two equations implies (1.4). Except for finding a conservative quantity, we usually solve (1.1) and (1.4) instead of (1.1) and (1.2).

Example 1.1. As a Helmholtz free energy function, let us choose

$$\psi(F,T)=\sqrt{1+F^2}-T^2-\gamma TF,\ \ \gamma\neq 0.$$

Then,

$$S=F/\sqrt{1+F^2}-\gamma T,\ \ e=\sqrt{1+F^2}+T^2,\ \ \eta=\gamma F+2T.$$

The corresponding equations are that

$$\begin{cases} X_{tt}-\left(X_x/\sqrt{1+X_x^2}-\gamma T\right)_x=f,\\ T(2T+\gamma X_x)_t=\kappa T_{xx},\end{cases}$$

which is one of thermo-damping equations corresponding to (*) in one-dimensional case.

As a class of solutions, let me consider the following space: for $t_0>0$ I put

$$Z(t_0)=\{(X(t,x),T(x,t))\ |$$

$$\text{(1.5)}\quad X\in\bigcap_{j=0}^{3}C^j([0,t_0);H^{3-j}),$$

$$\text{(1.6)}\quad T\in C^2([0,t_0);L^2)\cap\bigcap_{j=0}^{1}C^j([0,t_0);H^{3-j}),\ \ \partial_t^2T\in L^2((0,t_0);H^1),$$

$$\text{(1.7)}\quad (X_x(t,x),T(t,x))\in G(\delta)\ \text{ and }\ T(t,x)>0\ \text{ for any }(t,x)\in[0,t_0)\times\bar\Omega\}.$$

I will look for solutions $(X,T)\in Z(\infty)$. Here and hereafter, L^2 denotes the set of all square integrable functions on Ω. H^k denotes the set of all L^2 functions whose distributional derivatives of order up to k also belong to L^2. $C^k(I;B)$ denotes the set of all B-valued k-times continuously differentiable functions on I. $L^2(I;B)$ denotes the set of all B-valued square integrable functions on I.

Now, let me explain the conditions on initial data X_0, X_1, T_0 and a right member f. To do this, I assume for a moment that solutions $(X,T)\in Z(t_0)$ exist. Put

$$X_j(x)=\partial_t^jX(0,x)\ \text{ and }\ T_j(x)=\partial_t^jT(0,x)$$

which are successively determined through the equations (1.1) and (1.4) in terms of X_0, X_1, T_0, f and their derivatives. For example,

$$X_2(x)=S(X_0'(x),T_0(x))+f(x),$$

$$T_1(x)=(T_0(x)\frac{\partial\eta}{\partial T}(X_0'(x),T_0(x))^{-1}\{\kappa T_0''(x)-T_0(x)\frac{\partial\eta}{\partial F}(X_0'(x),T_0(x))X_1'(x)\},$$

and so on. According to (1.5) and (1.6), I assume that

(A.5)

$$X_j(x)\in H^{3-j},\ 0\le j\le 3;\quad T_j(x)\in H^{3-j},\ j=0,1;\quad T_2(x)\in L^2;\quad f(x)\in H^1.$$

Since

$$T_x(t,x),\quad S(X_x(t,x),T(t,x)) \in \bigcap_{j=0}^{1} C^j([0,t_0);H^{2-j}),$$

in view of the trace theorem to the boundary, the boundary condition (N.N) requires the following conditions:

$$\text{(A.6)}\qquad \begin{aligned} &S(X_0'(x),T_0(x))=0,\\ &\frac{\partial S}{\partial F}(X_0'(x),T_0(x))X_1'(x)+\frac{\partial S}{\partial T}(X_0'(x),T_0(x))T_1(x)=0,\\ &T_0'(x)=T_1'(x)=0 \end{aligned}$$

for $x\in\Omega$. In fact, these conditions come from the facts that $S=S_t=T_x=T_{xt}=0$ on $\partial\Omega$. (A.6) is called the compatibility condition. In addition, I assume that

$$\text{(A.7)}\qquad \int_0^1 f(x)dx=\int_0^1 X_0(x)dx=\int_0^1 X_1(x)dx=0.$$

But, (A.7) does not give us any restrictions. In fact, let me consider the compensating function $r(t)$ defined by the formula:

$$r(t)=\int_0^1 X_0(x)dx+t\int_0^1 X_1(x)dx+\frac{t^2}{2}\int_0^1 f(x)dx.$$

Put $\tilde{X}(t,x)=X(t,x)-r(t)$. Then,

$$\begin{aligned} &\int_0^1 \tilde{X}(0,x)dx=\int_0^1 \tilde{X}_t(0,x)dx=0,\\ &\tilde{X}_{tt}-S(\tilde{X}_x,T)_x=X_{tt}-S(X_x,T)_x-r''(t)=f(x)-\int_0^1 f(x)dx,\\ &\int_0^1\left(f(x)-\int_0^1 f(x)dx\right)dx=0. \end{aligned}$$

From these observations, you see that (A.7) actually does not give us any restrictions on initial data and right members.

To find the possible asymptotic behaviour as $t\to\infty$, for unknown functions X_∞ and T_∞, let me consider the stationary problem corresponding to (1.1), (1.4) and (N.N):

$$\begin{aligned} &S(X_\infty'(x),T_\infty(x))'=-f(x)\ \text{ and }\ T_\infty''(x)=0 &&\text{in } \Omega,\\ &S(X_\infty'(x),T_\infty(x))=T_\infty'(x)=0 &&\text{on } \partial\Omega. \end{aligned}$$

It follows from the boundary condition that T_∞ is a constant and that

$$\text{(1.8)}\qquad S(X_\infty'(x),T_\infty)=-F(x),$$

where

$$F(x) = \int_0^x f(y)dy.$$

Note that it follows from (A.7) that $F(0) = F(1) = 0$. A pair $(X_\infty(x), T_\infty)$ satisfying (1.8) is not unique. Another requirement on $(X_\infty(x), T_\infty)$ comes from the following energy conservation law. Integrating (1.2) over $(0,t) \times \Omega$ and noting the relation:

$$X_t f = (X_t F)_x - \frac{\partial}{\partial t}(X_x F),$$

you see that

$$\int_0^1 \left\{ e(X_x(t,x), T(t,x)) + \frac{1}{2}X_t(t,x)^2 + X_x(t,x)F(x) \right\} dx = e_0, \tag{1.9}$$

where

$$e_0 = \int_0^1 \left\{ e(X_0'(x), T_0(x)) + \frac{1}{2}X_1(x)^2 + X_0'(x)F(x) \right\} dx.$$

Since the motion is expected to stop at $t = \infty$, that is $X_t \to 0$ as $t \to \infty$, and since a pair (X_x, T) is expected to converge to (X_∞', T_∞) as t tends to infinity, (1.9) implies that

$$\int_0^1 \{ e(X_\infty'(x), T_\infty) + X_\infty'(x)F(x) \} dx = e_0. \tag{1.10}$$

Put

$$M_g(F,T) = \frac{\partial S}{\partial F}(F,T)\frac{\partial g}{\partial T}(F,T) - \frac{\partial S}{\partial T}(F,T)\frac{\partial g}{\partial F}(F,T) \text{ for } g = e \text{ and } \eta.$$

Assume that

$$S(1, \tau_0) = 0, \tag{A.8}$$

which means that $(1, \tau_0)$ is an equilibrium state with $f = g = 0$. And then, (A.3), (A.4) and (A.8) imply that

$$\begin{aligned} M_e(1,\tau_0) &= M_\eta(1,\tau_0) = \\ &= \tau_0 \left\{ -\frac{\partial^2\psi}{\partial F^2}(1,\tau_0)\frac{\partial^2\psi}{\partial T^2}(1,\tau_0) + \frac{\partial^2\psi}{\partial F}\partial T(1,\tau_0)^2 \right\} > 0. \end{aligned} \tag{1.11}$$

In view of (1.11), by the implicit function theorem you can prove the following lemma concerning the unique existence of a pair $(X_\infty(x), T_\infty)$ satisfying (1.8) and (1.10).

Lemma 1.2. *Suppose that* (A.3), (A.4), (A.5), (A.7) *and* (A.8) *hold. Then, for any* $\sigma > 0$ *there exists a* $\kappa > 0$ *such that if*

$$\|(X_0', T_0) - (1, \tau_0)\|_\infty + \|X_1\| + \|f\|_1 < \kappa$$

then there exist a $X_\infty(x) \in H^3$ *and a constant* $T_\infty > 0$ *satisfying* (1.8) *and* (1.10) *and the following conditions:*

$$\|X_\infty' - 1\|_2 + |T_\infty - \tau_0| < \sigma \ \text{ and } \ (X_\infty'(x), T_\infty) \in G(\delta/2) \quad \text{for all } x \in \bar{\Omega}.$$

In particular, $S(X_\infty'(x), T_\infty) = 0$ *for* $x \in \partial\Omega$.

Here and hereafter, $\|\cdot\|$ denotes the usual L^2-norm on Ω and put

$$\|v\|_k = \left\{ \sum_{j=0}^{k} \|\frac{d^j v}{dx^j}\|^2 \right\}^{1/2} \quad \text{and} \quad \|v\|_\infty = \sup_{x\in\Omega} |v(x)|.$$

To state a main result exactly, let me introduce an additional notation. Put

$$\begin{aligned}
&u(t,x) = X(t,x) - X_\infty(x), \ \ \theta(t,x) = T(t,x) - T_\infty, \\
&N(t) = \sup_{0<s<t} \|\bar{D}^2(u_x, u_t, \theta)(s,\cdot)\|, \\
&N_\alpha(t) = \sup_{0<s<t} e^{\alpha s} \left\{ \|\bar{D}^2(u_x, u_t, \theta)(s,\cdot)\| + \|(\theta_{xxt}, \theta_{xxx})(s,\cdot)\| \right\}, \\
&M_\alpha(t) = \left\{ \int_0^t e^{2\alpha s} \|(D^2u, D^3u, D^1\theta, D^2\theta, \theta_{xtt}, \theta_{xxt}, \theta_{xxx})(s,\cdot)\|^2 ds \right\}^{1/2}, \\
&E_0 = \|X_0' - X_\infty'\|_2 + \|T_0 - T_\infty\|_3 + \sum_{j=1}^{3} \|X_j\|_{3-j} + \|T_1\|_2 + \|T_2\|.
\end{aligned}$$

Here and hereafter, I use the following symbols:

$$D^k u = (\frac{\partial^k u}{\partial t^j \partial x^{j-k}}, j = 0, 1, \cdots, k) \ \text{ and } \ \bar{D}^k u = (u, D^1 u, \ldots, D^k u).$$

Under these preparations, I can state the main result of this section which was proved by Kawashima and Shibata [37] in the following way.

Theorem 1.3. *Suppose that* (A.1)–(A.8) *hold and that* $T_0(x) > 0$ *for* $x \in \bar{\Omega}$. *Then, there exists an* $\epsilon > 0$ *such that if* $\|(X_0', T_0) - (1, \tau_0)\|_\infty + E_0 + \|f\|_1 < \epsilon$, *then the problem* (1.1), (1.2), (N.N) *and* (1.3) *admits a unique solution* $(X(t,x), T(t,x)) \in Z(\infty)$ *satisfying the estimate:*

$$N_\alpha(t)^2 + M_\alpha(t)^2 \leq C E_0^2 \tag{1.12}$$

for suitable positive constants α and C.

Notes:

(1) **One-dimensional problem:** Concerning a global in time existence theorem for small and smooth data, let me give you a rough survey. The first result in one dimension was given for bounded domains and the following boundary conditions:

$$\text{(D.N)} \qquad X = x \ \text{ and } \ T_x = 0 \qquad \text{for } x \in \partial\Omega \qquad \text{and } t > 0,$$

$$\text{(N.D)} \qquad S = 0 \ \text{ and } \ T = \tau_0 \qquad \text{for } x \in \partial\Omega \qquad \text{and } t > 0,$$

by Slemrod [64] in 1981, similar results were obtained by Zheng [68]. Then, the Cauchy problem was treated by Kawashima and Okada [35] and [36], by Zheng and Shen [69] and by Hrusa and Tarabek [28]. Jiang [29] investigated the half-line for (D.N) and (N.D) boundary conditions. The result for bounded domains and the following boundary condition:

$$\text{(D.D)} \qquad X = x \ \text{ and } \ T = \tau_0 \qquad \text{for } x \in \partial\Omega \qquad \text{and } t > 0,$$

was proved by Racke, Shibata and Zheng [57] and [58]. The (D.D) problem for the half-line was discussed by Jiang [32] and the (N.N) problem for a bounded domain by Kawashima and Shibata [38] and [60] and Jiang [33] who also discussed the half-line case. Periodic solutions were studied by Feireisl [21] for (D.N) and (N.D) and by Racke, Shibata and Zheng [58] for (D.D). The existence of periodic solutions for (N.N) and the case that external forces depend on time for (N.N) are open problems.

The development of singularities for large data was shown by Dafermos and Hsiao [15] and Hrusa and Messaoudi [27]. I think that except for two open problems mentioned above, unique existence theorems and studies of the asymptotic behaviour of solutions as time goes to infinity have been settled.

(2) **Multi-dimensional problem:** Compared with one-dimensional problem, the multi-dimensional case has been less studied. One of the reasons is that we can not expect a good decay structure coming from the thermo-damping. In fact, let me consider three dimensional linear thermoelastic equations:

$$\text{(L.T)} \qquad \begin{cases} u_{tt} - (\lambda+\mu)\nabla\mathrm{div}\ u - \mu\Delta u - \gamma\nabla\theta = 0, \\ \theta_t - \Delta\theta - \gamma\mathrm{div}\ u_t = 0, \end{cases}$$

where λ and μ are Lamé constants such that $\mu > 0$ and $\lambda + \mu > 0$, γ is a coupling constant $\neq 0$, θ is a temperature and $u = {}^t(u_1, u_2, u_3)$ is a displacement vector (${}^tM$ means the transposed M). If you take the divergence to the first equation of (L.T), then

$$\text{(D)} \qquad \begin{cases} (\mathrm{div}\ u)_{tt} - (\lambda+2\mu)\Delta(\mathrm{div}\ u) - \gamma\Delta\theta = 0, \\ \theta_t - \Delta\theta - \gamma(\mathrm{div}\ u)_t = 0. \end{cases}$$

Therefore, div u and θ have a good decay structure like the one-dimensional case. But, if you take the rotation to the first equation of (L.T), then

$$(\text{rot } u)_{tt} - \mu\Delta(\text{rot } u) = 0. \tag{R}$$

Namely, rot u does not get any influence from the temperature θ and the behaviour of rot u is governed by the wave equation. This is one of the reasons why multi-dimensional thermoelasticity does not give us a good decay structure caused by thermo-damping. The asymptotic behaviour of solutions to (L.T) was proved by Dassios and Grillakis [17], Racke and Ponce [48] and [53] and Dafermos [13].

The asymptotic behaviour of solutions as $|x| \to \infty$ was investigated by Jiang [30] and [31] for the nonlinear Cauchy problem both in one and in three dimensions. Local in time existence theorems were proved by Jiang and Racke [34], Chrzęszczyk [10] and Dan [16]. Global in time existence theorems for the Cauchy problem were proved by Racke and Ponce [48] and [53]. A blow up theorem for the Cauchy problem was also proved by Racke [54]. The desired proofs of global existence theorems for smooth solutions cannot imitate those for the Cauchy problem. Explicit representation formulae are missing and the decomposition into divergence-free and rotation-free components mentioned in (D) and (R) above is in general not compatible with the boundary conditions. Not only nonlinear but also linear equations both in bounded and unbounded domains require future research.

Finally, I would like to mention a lecture note [55] and a book [56] by R. Racke as excellent books of the mathematical theory of thermoelastodynamics.

2. Dirichlet problem for some nonlinear viscoelastic equations.

In this section, I consider the initial boundary value problem for nonlinear wave equations with linear viscosity of the following type:

$$\mathbf{u}_{tt} - \text{div } \mathbf{a}(\nabla\mathbf{u}) - \mathcal{B}\mathbf{u}_t = 0 \quad \text{in } \Omega \quad \text{for } t > 0, \tag{2.1}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \quad \text{for } t > 0, \tag{2.2}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad \mathbf{u}(0,x) = \mathbf{u}_1(x) \quad \text{in } \Omega, \tag{2.3}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$ being a C^∞ hypersurface, $\mathbf{u} = {}^t(u_1, \dots, u_d)$ is a d-vector of unknown functions (${}^tM$ means the transposed M), $\nabla\mathbf{u} = (\partial_i u_j, i = 1, \dots, n, j = 1, \dots, d)$, $\partial_i = \partial/\partial x_i$, $\mathbf{a}(\nabla\mathbf{u}) = \big((a_{ij}(\nabla\mathbf{u})\big)$ is a $d \times n$ matrix of smooth functions $a_{ij}(U)$, $i = 1, \dots, d$, $j = 1, \dots, n$, ($U = (u_{k\ell}$, $k = 1, \dots, d$, $\ell = 1, \dots, n$) and $u_{k\ell}$ are independent variables corresponding to $\partial_\ell u_k$),

$$\text{div } \mathbf{a}(\nabla\mathbf{u}) = {}^t(\partial_j a_{1j}(\nabla\mathbf{u}), \dots, \partial_j a_{dj}(\nabla\mathbf{u})) \text{ and } \mathcal{B} = \mathbf{b}_{ij}\partial_i\partial_j,$$

where the summation convention is understood with subscripts i, j, $k, \cdots$ ranging over 1, $2, \cdots, n$ and $\mathbf{b}_{ij}$ are $d \times d$ real constant matrices satisfying the following assumptions:

$${}^t\mathbf{b}_{ij} = \mathbf{b}_{ji}. \tag{A.9}$$

(A.10) There exists a $\delta > 0$ such that for any $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$,

$$\mathbf{b}_{ij}\xi_i\xi_j \geq \delta|\xi|^2\mathbf{I}_d$$

where $\mathbf{I}_d$ is the $d \times d$ identity matrix.

Example 2.1. A nonlinear acoustic equation with linear viscosity is given by the following equation:

$$u_{tt} - \operatorname{div}\,(\nabla u/\sqrt{1+|\nabla u|^2}) - \lambda\Delta u_t = 0 \tag{2.4}$$

with λ being a positive constant. If you choose d, $\mathbf{a}(\nabla\mathbf{u})$ and $\mathcal{B}$ in such a way that $d=1$, $\mathbf{u}=u_1=u$ (scalar function),

$$\begin{aligned} \mathbf{a}(\nabla\mathbf{u}) &= a_1(\nabla u) = \nabla u/\sqrt{1+|\nabla u|^2}, \\ \mathcal{B} &= \lambda\Delta u \qquad (\mathbf{b}_{ij} = \delta_{ij}\ :\ \text{Kronecker's delta symbol}), \end{aligned}$$

then (2.4) can be described in the form of (2.1).

The purpose of considering the linear viscosity is to obtain smooth solutions globally in time for arbitrary smooth data without any smallness assumptions. As it will be seen later, under some growth condition on the derivatives of $\mathbf{a}$, it is true. A method is to use the L^p-theory of the linear parabolic operator: $\mathbf{v}_t - \mathcal{B}\mathbf{v}$. First of all, I introduce the space of solutions. Put

$$X^{p,N}(J) = \bigcap_{j=0}^{N} C^{j+1}(J; H^{p,N+2-j}) \cap C^{N+2}(J; L^p),$$

where L^p is a usual L^p-space on Ω with norm $\|\cdot\|_p$,

$$\begin{aligned} &H^{p,N} = \left\{u \in L^p \mid \|u\|_{p,N} = \|\bar{\partial}_x^N u\| < \infty\right\},\ \ L^p = H^{p,0}, \\ &\|\cdot\|_{p,0} = \|\cdot\|_p, \quad \|\cdot\|_2 = \|\cdot\|, \quad \|\cdot\|_{2,N} = \|\cdot\|_N, \quad \bar{\partial}_x^N u = (u, \partial_x^1 u, \dots, \partial_x^N u), \\ &\partial_x^k u = (\partial_x^\alpha u = \frac{\partial^k u}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, |\alpha| = |(\alpha_1,\dots,\alpha_n)| = \alpha_1+\cdots+\alpha_n = k). \end{aligned}$$

To explain conditions on initial data, I assume for a moment that a solution $\mathbf{u} \in X^{p,N}([0,t_0])$ to the problem (2.1)–(2.3) exists. Put

$$\mathbf{u}_k(x) = \partial_t^k \mathbf{u}(0,x),$$

and then, through the equation (2.1), $\mathbf{u}_k(x)$ $(k \ge 2)$ are determined successively in terms of $\mathbf{u}_0$, $\mathbf{u}_1$ and their derivatives. Whatever $\mathbf{u} \in X^{p,N}([0,t_0])$ implies the following condition on initial data:

$$\mathbf{u}_0 \in H^{p,N+2},\ \ \mathbf{u}_{j+1} \in H^{p,N+2-j},\ \ 0 \le j \le N,\ \ \mathbf{u}_{N+2} \in L^p. \tag{A.11}$$

Since $\partial_t^j \mathbf{u} = 0$, $0 \le j \le N+1$, on $\partial\Omega$ for $t \in [0,t_0]$ as follows from (2.2) and the trace theorem to the boundary, you arrive at the following condition:

$$\mathbf{u}_j = 0 \qquad \text{on } \partial\Omega \qquad \text{for } 0 \le j \le N+1, \tag{A.12}$$

which is called the compatibility condition of order $N+1$.

My local and global in time existence theorems for the problem (2.1)–(2.3) are stated in the following way.

Theorem 2.2. *Let N be an integer ≥ 1 and $p \in (n, \infty)$. Suppose that* (A.9)–(A.12) *hold. Let K be a number such that $\|(\mathbf{u}_0, \mathbf{u}_1)\|_p \leq K$. Then, there exists a time $T > 0$ depending essentially on K only such that the problem* (2.1)–(2.3) *admits a unique solution $\mathbf{u} \in X^{p,N}([0,T])$.*

Theorem 2.3. *In addition to all the assumptions in Theorem 2.2, suppose that*
(A.13) *the first and second derivatives of $a_{ij}(U)$ are bounded, i.e.*

$$\left|\frac{\partial a_{ij}}{\partial u_{k\ell}}(U)\right| \leq B, \quad \left|\frac{\partial^2 a_{ij}}{\partial u_{k\ell}\partial u_{pq}}(U)\right| \leq B$$

for all $U \in \mathbf{R}^{nd}$ with some positive constant B. Then, the problem (2.1)–(2.3) *admits a unique solution $\mathbf{u} \in X^{p,N}([0,\infty))$.*

Remark 2.4. (1) Applying Theorem 2.3, we know that the problem (2.4), (2.2) and (2.3) admits a unique solution $u \in X^{p,N}([0,\infty))$. In fact, since

$$a_{1j}(U) = u_j/\sqrt{1 + u_1^2 + \cdots + u_n^2},$$

you see easily that

$$\begin{aligned}\frac{\partial a_{1j}}{\partial u_k}(U) &= \delta_{jk}(1 + u_1^2 + \cdots + u_n^2)^{-1/2} - u_j u_k (1 + u_1^2 + \cdots + u_n^2)^{-3/2},\\ \frac{\partial^2 a_{1j}}{\partial u_k \partial u_\ell}(U) &= -(\delta_{jk}u_\ell + \delta_{j\ell}u_k + \delta_{k\ell}u_j)(1 + u_1^2 + \cdots + u_n^2)^{-3/2}\\ &\quad + \frac{3}{2}u_j u_k u_\ell (1 + u_1^2 + \cdots + u_n^2)^{-5/2}\end{aligned}$$

and then the first and second derivatives of a_{1j} are all bounded for all $U = (u_1, \ldots, u_n) \in \mathbf{R}^n$.

(2) If $\mathbf{u}_0$ and $\mathbf{u}_1$ are in $C_0^\infty(\Omega) = \{\mathbf{u} \in C^\infty(\Omega) \mid \text{supp}\ \mathbf{u} \subset \Omega\}$ (the support of $\mathbf{u}$ does not meet the boundary $\partial\Omega$), then all the $\mathbf{u}_j$ also belong to $C_0^\infty(\Omega)$. Therefore, you get a unique global in time existence of solutions in $C^\infty([0,\infty); H^{p,\infty})$ to the problem (2.1)–(2.3) provided that the initial data belong to $C_0^\infty(\Omega)$. In particular, the solutions are in $C^\infty([0,\infty) \times \bar{\Omega})$.

(3) As a challenging problem, there is a vanishing viscosity problem. Namely, to get a weak solution to the problem (2.4), (2.2) and (2.3), you consider the limiting process of λ tending to 0.

(4) Theorems 2.2 and 2.3 were obtained by Kobayashi, Pecher and Shibata [40]. The proof in [40] is based on classical results concerning L^p-estimates of solutions to linear elliptic equations and linear parabolic equations which the readers find in [1], [2] and [65].

As I mentioned before, one of the main purposes of considering the viscosity is to get a global in time existence theorem without any smallness assumptions on initial data. But, up to now, to get it in general we need an additional assumption on the derivatives of $a_{ij}(U)$. Unfortunately, I think that not so many physical examples satisfy special growth order conditions on the derivatives of $a_{ij}(U)$. Therefore, I come to a question if a global existence of smooth solutions holds in the general setting of the problem under at least some smallness assumptions on initial data. My answer will be stated in the following framework. Let me consider the following second order quasilinear hyperbolic system with nonlinear viscosity:

$$
\begin{aligned}
&(2.5)\quad A_0(\mathbf{U})\partial_t^2\mathbf{u} + A_j(\mathbf{U})\partial_j\partial_t\mathbf{u} - A_{ij}(\mathbf{U})\partial_i\partial_j\mathbf{u}\\
&\qquad\qquad - B_{ij}(\mathbf{U})\partial_i\partial_j\partial_t\mathbf{u} = 0 \quad \text{in } \Omega \quad \text{for } t>0,\\
&(2.6)\quad \mathbf{u} = 0 \quad \text{on } \partial\Omega \quad \text{for } t>0,\\
&(2.7)\quad \mathbf{u}(0,x) = \mathbf{u}_0(x),\ \ \mathbf{u}_t(0,x) = \mathbf{u}_1(x) \quad \text{in } \Omega,
\end{aligned}
$$

where $\mathbf{U} = (\mathbf{u}_t, \nabla\mathbf{u}, \nabla\mathbf{u}_t)$, A_0, A_j, A_{ij} and B_{ij} are $d\times d$ matrices of smooth functions in $\mathcal{U}\in G(\delta)$,

$$G(\delta) = \{\mathcal{U}\in\mathbf{R}^{(2n+1)d} \mid |\mathcal{U}|\le\delta\},$$

and $\mathcal{U}$ is an independent variable corresponding to $\mathbf{U}$. I introduce the following assumptions:

(A.14) $\quad {}^tA_0 = A_0,\ \ {}^tA_j = A_j,\ \ {}^tA_{ij} = A_{ji},\ \ {}^tB_{ij} = B_{ji};$

(A.15) there exists an $\alpha_0>0$ such that

$$A_0(0)\ge\alpha_0\mathbf{I}_d, \tag{2.8}$$

$$A_{ij}(0)\xi_i\xi_j\ge\alpha_0|\xi|^2\mathbf{I}_d \ \text{ and } \ B_{ij}(0)\xi_i\xi_j\ge\alpha_0|\xi|^2\mathbf{I}_d \tag{2.9}$$

for all $\xi=(\xi_1,\dots,\xi_n)\in\mathbf{R}^n$.
The condition (2.9) guarantees that the inequalities:

$$(A_{ij}(\mathbf{U})\partial_j\mathbf{u},\partial_i\mathbf{u})\ge\beta\|\nabla\mathbf{u}\|^2 \ \text{ and } \ (B_{ij}(\mathbf{U})\partial_j\mathbf{u},\partial_i\mathbf{u})\ge\beta\|\nabla\mathbf{u}\|^2, \tag{2.10}$$

hold with a suitable constant $\beta>0$ for any $\mathbf{u}(x)\in H_0^1=\{\mathbf{u}\in H^{2,1} \mid \mathbf{u}=0 \text{ on } \partial\Omega\}$ and any vector $\mathbf{U}(x)$ of functions satisfying the condition: $|\mathbf{U}(x)|\le\delta$ for any $x\in\Omega$. Here and hereafter, $(\cdot,\cdot)$ denotes the usual L^2-innerproduct on Ω. Then, the following theorem holds, which was obtained by Kawashima-Shibata [37].

Theorem 2.5. *Let N be an integer $\ge[n/2]+1$ and suppose that* (A.14), (A.15), (A.11) *with* $p=2$ *and* (A.12) *hold, where $\mathbf{u}_j$, $j\ge2$, are defined in the same manner as in the discussion for Theorems 2.2 and 2.3 and $[r]$ denotes the largest integer $\le r$. Then, there exists an $\varepsilon>0$ such that if*

$$\|\mathbf{u}_0\|_{[n/2]+3} + \sum_{j=0}^{[n/2]+1}\|\mathbf{u}_{j+1}\|_{[n/2]+3-j}\le\varepsilon, \tag{2.11}$$

then the problem (2.5)–(2.7) *admits a unique solution* $\mathbf{u} \in X^{2,N}([0,\infty))$. *Moreover,*

$$\|\mathbf{u}(t,\cdot)\|_{N+2} + \sum_{j=0}^{N} \|\partial_t^{j+1}\mathbf{u}(t,\cdot)\|_{N+2-j} \le C_N e^{-\beta t} \tag{2.12}$$

for suitable positive constants C_N *and* β *where* C_N *depends not only on* N *but also* $\|\mathbf{u}_0\|_{N+2}$, $\|\mathbf{u}_1\|_{N+2}$, $\dots$, $\|\mathbf{u}_{N+1}\|_2$, *but* β *is independent of them.*

Remark 2.6. Exploiting the exponential stability of (2.12), by using the technique due to Rabinowitz [52] and Matsumura [43], you can establish the existence of periodic solutions for small and smooth periodic forces.

Notes:

Concerning global existence theorems of smooth solutions, the problem to nonlinear wave equations with viscosity was first studied in one space dimension and scalar case, and classical solutions for arbitrary smooth data, their asymptotic properties and so on were found in [3], [4], [6], [14], [23], [25], [49] and [67]. Compared with the one-dimensional case, the many dimensional case has been less studied. Pecher [46] proved the global existence of classical solutions for arbitrary data in two dimensional Cauchy problem case under some growth order condition on the derivatives of $\mathbf{a}$ which was completely different from (A.13). Global weak solutions were studied by Clemént [11] and Rybka [58]. The semilinear case: $u_{tt} - \Delta u - \Delta u_t + f(u, \nabla u, u_t, \nabla u_t) = 0$, was studied in [5], [6], [46] and [66]. Global strong solutions (not so smooth, e.g. ∇u, ∇u_t are Hölder continuous) were studied by [20] and [22] under some growth condition on the derivatives of $\mathbf{a}$ which was also completely different from (A.13). In the treatment of the above studies in the higher dimensional case, it plays an essential role the estimation of L^p-norm of solutions to the linear parabolic part: $\partial_t - \mathcal{B}$, and the nonlinear term: $\mathbf{a}(\nabla \mathbf{u})$ was restricted strongly, because the estimation of derivatives of solutions of higher order was not enough.

In order to study more general equations containing important physical models, one approach is to abandon arbitrariness of initial data and to look for small solutions. In this direction, the L^2 framework is better, because the equation has a hyperbolic feature with respect to derivatives (cf. the definition of $X^{p,N}(J)$). In the case of the bounded domain with zero Dirichlet condition, Ebihara [18], [19] studied small and smooth solutions to the strongly damped scalar wave equation of the form: $u_{tt} - \Delta u_t = f(u, \nabla u, \partial_x^2 u, u_t, \nabla u_t)$, by using the so called $\overset{o}{H}{}^k$ - Galerkin method. Mizohata and Ukai [44] studied also small and smooth solutions in the L^2 framework to an acoustic wave equation in a viscous conducting fluid described by the following equation: $u_{tt} - a\Delta u - b\Delta u_t = c\left(|\nabla u|^2 + u_t^2\right)_t$ with some constants $a > 0$, $b > 0$ and $c \neq 0$. For the Cauchy problem, Ponce [47] proved the existence of solutions globally in time to the equation (1) for the scalar operators, that is, $d = 1$, under the condition (A.2) with small and smooth initial data.

3. Nonlinear wave equations with some boundary dissipation.

To get a good decay structure of wave equations in bounded domains, I would like to consider the boundary dissipation in this section. Namely, let me consider the following equations:

$$
(3.1) \qquad \begin{cases} u_{tt} - \operatorname{div} \mathbf{a}(\nabla u) = 0 & \text{in } \Omega \quad \text{for } t > 0, \\ \nu \cdot \mathbf{a}(\nabla u) + u_t = 0 & \text{on } \partial\Omega \quad \text{for } t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \Omega, \end{cases}
$$

where $\mathbf{a}(\nabla u) = (a_1(\nabla u), \dots, a_n(\nabla u))$ which describes a stress tensor. In this section, I use the same notation as in §2. I assume that each $a_j(U)$ ($U = (u_1, \dots, u_n)$ and u_j, $j = 1, \dots, n$, are corresponding variables to $\partial_j u$) is in $C^\infty(\mathbf{R}^n)$ and that

$$
(\mathrm{A.16}) \qquad \frac{\partial a_i}{\partial u_j} = \frac{\partial a_j}{\partial u_i}, \qquad a(0) = 0,
$$

$$
(\mathrm{A.17}) \qquad \frac{\partial a_i}{\partial u_j}(U)\xi_i\xi_j \geq \delta|\xi|^2
$$

for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$ and $|U| \leq U_0$ with some positive constant δ depending only on U_0.

Roughly speaking, Shibata and Zheng [63] proved the following result.

Theorem 3.1. *If initial data u_0 and u_1 are small and smooth enough, then the problem* (3.1) *admits a unique solution $u \in C^2([0,\infty) \times \bar{\Omega})$.*

To prove Theorem 3.1, the main step of our approach is to show the decay property of solutions to the linear equations:

$$
(3.2) \qquad \begin{cases} u_{tt} - a_{ij}\partial_i\partial_j u = 0 & \text{in } \Omega \times (0,\infty), \\ \nu_i a_{ij}\partial_j u + u_t = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) & \text{in } \Omega, \end{cases}
$$

where $a_{ij} = (\partial a_i/\partial u_j)(0)$. In [63], we proved the following theorem which is concerned with the decay rate of solutions to (3.2).

Theorem 3.2. *Let u be a smooth solution to* (3.2)*. Then, for any integer $K \geq 1$, we have*

$$
\begin{aligned} &\|u_t(t,\cdot)\|^2 + (a_{ij}\partial_j u(t,\cdot), \partial_i u(t,\cdot)) \\ &\qquad \leq C(K)(1+t)^{-2K}\left\{\|\bar{\partial}_t^{K+4}u(0,\cdot)\|^2 + \|\bar{\partial}_t^{K+3}u(0,\cdot)\|_1^2\right\}. \end{aligned}
$$

Combining the decay results of solutions to the linearized problem of (3.1) and their derivatives which are obtained by using Theorem 3.2 and a local existence theorem for (3.1) which was proved by Shibata and Nakamura [61] and Shibata and Kikuchi [62] in the more general framework, you can continue local in time solutions to any time interval, and then Theorem 3.1 is established. Our proof of Theorem 3.2 is to use a spectral analysis of the reduced problem with spectral parameter k. Sometimes, the spectral analysis is a very strong tool of showing a decay structure of linear equations, so that I shall explain a rough idea of our proof of Theorem 3.2 after Notes.

Notes:

Quinn and Russell [51] got a polynomial decay rate of solutions to (3.2). Later on, Chen [9] got the exponential decay rate of solutions to (3.2) and Lagnese [41] got the exponential decay rate of solutions to the three-dimensional elastic wave equations with damping boundary condition. But, in these works, the results were obtained under some additional assumptions concerning the shape of the boundary and so on.

Shibata and Zheng [63] got a polynomial decay rate of solutions to (3.2) without any assumptions on the shape of the boundary. And also, in Bardos, Lebeau and Rauch [7] and [8], the exponential decay result for (3.2) was obtained without any assumptions on the shape of the boundary. The exponential decay result for some nonlinear wave equations with damping boundary condition was found in Zuazua [70].

Concerning a global in time solvability of (3.1) with small and smooth data, the first work was done by Greenberg and Li [24] in the case of the nonlinear conservation system in one space dimension with damping boundary condition. And, Nagasawa [45] also considered the equation of the one-dimensional motion of the polytropic ideal gas which is not fixed on the boundary. In higher space dimension, the work due to Qin [50] was first, but he used the result due to Chen [9] so that the boundary was divided into two non-empty parts such as the boundary where the damping boundary condition was posed, was star-shaped and on the other part of the boundary Dirichlet condition was imposed. Shibata and Zheng [63] proved Theorem 3.1 without any assumptions on the shape of the boundary.

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