Comment.Math.Univ.Carolin. 34,2 (1993)341–346


# Bernoulli sequences and Borel measurability in $(0,1)$

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*Abstract.* The necessary and sufficient condition for a function $f:(0,1)\to[0,1]$ to be Borel measurable (given by Theorem stated below) provides a technique to prove (in Corollary 2) the existence of a Borel measurable map $H:\{0,1\}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$ such that $\mathcal{L}(H(\mathbf{X}^p))=\mathcal{L}(\mathbf{X}^{1/2})$ holds for each $p\in(0,1)$, where $\mathbf{X}^p=(X_1^p,X_2^p,\dots)$ denotes Bernoulli sequence of random variables with $P[X_i^p=1]=p$.

*Keywords:* Borel measurable function, Bernoulli sequence of random variables, Strong law of large numbers

*Classification:* 60A10, 28A20

## 1. The main result and notation.

Consider a sequence $X_n$, $n\in\mathbb{N}$, of mutually independent random variables assuming the values 1 and 0 with probabilities $p$ and $1-p$, where $p\in(0,1)$. Denote the distribution of the random variable

$$Y=\sum_{n=1}^{\infty}2^{-n}X_n$$

by $\lambda_p$. Identifying Borel spaces $(0,1)$ and $\{0,1\}^{\mathbb{N}}$ by the irrational dyadic expansion map we can also define these measures by

$$\lambda_p\left(\{x\in\{0,1\}^{\mathbb{N}}\mid x_1=a_1,\dots,x_n=a_n\}\right)=\prod_{i=1}^{n}p^{a_i}(1-p)^{1-a_i},\ n\in\mathbb{N},\ a\in\{0,1\}^n$$

or equivalently by

$$\lambda_p=\bigotimes_1^{\infty}(1-p)\varepsilon_0+p\varepsilon_1\,,$$

where $\varepsilon_x$ denotes the atomic measure supported by $\{x\}$.

Our main result is

I am very grateful to Professor J. Štěpán for his assistance. The Corollaries 1, 2 and 3 belong to him (see [2])

**Theorem.** *For each function $f\colon (0,1) \to [0,1]$, the following assertions are equivalent:*

(a) *$f$ is a Borel measurable;*
(b) *there exists a Borel set $B \subseteq (0,1)$ such that $f(p) = \lambda_p(B)$ for all $p \in (0,1)$.*

Corollaries to this result related to Bernoulli sequences of random variables are stated and proved in the part 3 of the present paper.

The following terminology and notation will be used in the sequel: Let $x \in (0,1)$. By the dyadic expansion of $x$ we mean the sequence $(x_1, x_2, \dots) \in \{0,1\}^{\mathbb{N}}$ with infinitely many $x_i$'s zeros such that $x = \sum_{i=1}^{\infty} x_i 2^{-i}$. In this case we write $x = (x_1, x_2, \dots)$. Put

$$\mathcal{I}(n,a) = \{x \in (0,1) \mid x_1 = a_1, \dots, x_n = a_n\} \text{ for } n \in \mathbb{N},\ a = (a_1, \dots, a_n) \in \{0,1\}^n$$

and denote by $\mathcal{K}$ the algebra generated by the sets $\mathcal{I}(n,a)$. Note that the algebra $\mathcal{K}$ consists exactly of finite (possibly empty) unions of the sets $\mathcal{I}(n,a)$ and generates Borel $\sigma$-algebra $\mathcal{B}(0,1)$. Putting

$$\Lambda(B) = \{\, x \in (0,1) \mid \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i \in B \,\}, \qquad B \subseteq (0,1),$$

it follows easily by Strong law of large numbers that

$$\text{(1)} \quad \Lambda(B) \in \mathcal{B}(0,1) \text{ and } \lambda_p(\Lambda(B)) = I_B(p) \text{ for each } B \in \mathcal{B}(0,1) \text{ and } p \in (0,1).$$

Finally, let us agree that if $\mathcal{T}_1$, $\mathcal{T}_2$ are two decompositions of a set $\mathcal{S}$ and if for all $T_1 \in \mathcal{T}_1$, $T_2 \in \mathcal{T}_2$ either $T_1 \cap T_2 = \emptyset$ or $T_1 \subseteq T_2$, then we shall write $\mathcal{T}_1 \preccurlyeq \mathcal{T}_2$.

## 2. Proof of Theorem.

**Lemma 1.** *Let $p \in (0,1)$ and $K \in \mathcal{K}$. Then $\{\, \lambda_p(D)\,;\, K \supseteq D \in \mathcal{K} \,\}$ is a dense set in the interval $[0, \lambda_p(K)]$.*

The assertion follows easily by the inequality

$$\lambda_p(\mathcal{I}(m,a)) \le \max\{\, p^m, (1-p)^m \,\}, \qquad m \in \mathbb{N},\ a \in \{0,1\}^m,$$

using the fact that for almost all $m \in \mathbb{N}$ there exists a set $A_m \subseteq \{0,1\}^m$ such that $\{\, \mathcal{I}(m,a)\,;\, a \in A_m \,\}$ forms a decomposition of $K$.

**Lemma 2.** *Consider $K \in \mathcal{K}$, a Borel set $V \subseteq [a,b] \subset (0,1)$ and a continuous function $\gamma\colon [0,1] \to [0,1]$ such that $\gamma(p) \le \lambda_p(K)$ for all $p \in V$. Then to each $\varepsilon > 0$ there is a finite Borel measurable decomposition $\{A_1, \dots, A_t\}$ of $V$ and the sets $K \supseteq F_i \in \mathcal{K}$ such that*

$$0 \le \gamma(p) - \lambda_p(F_i) \le \varepsilon$$

*holds for each $p \in A_i$ and $1 \le i \le t$.*

Proof: Since $p \mapsto \lambda_p(K)$ is a continuous function defined on $(0,1)$ we get that $\gamma(p) \le \lambda_p(K)$ holds for all $p \in \overline{V}$. Fix a $p \in \overline{V}$. Lemma 1 provides a set $K \supseteq D_p \in \mathcal{K}$ such that

$$0 \le \gamma(p) - \lambda_p(D_p) \le \frac{1}{2}\varepsilon\,.$$

Let $V_p$ be an open neighbourhood of $p$ such that

$$0 \le \gamma(q) - \lambda_q(D_p) \le \varepsilon \quad \text{for all} \quad q \in V_p\,.$$

Now, let $V_{p_1}, \dots, V_{p_t}$ be a covering of the compact set $\overline{V}$. It is easy to see that the sets

$$A_1 = V_{p_1} \cap V\,,\ A_2 = V_{p_2} \cap A_1^c \cap V\,,\ \dots\,,\ A_t = V_{p_t} \cap A_1^c \cap \dots \cap A_{t-1}^c \cap V\,,$$
$$F_1 = D_{p_1}\,, \dots\,,\ F_t = D_{p_t}$$

provide the desired construction. $\square$

**Lemma 3.** *Let $[a,b] \subset (0,1)$ and let $f\colon [a,b] \to [0,1]$ be a Borel measurable function. Then there exists a Borel set $B \subseteq (0,1)$ such that $f(p) = \lambda_p(B)$ for all $p \in [a,b]$.*

Proof: Consider a nondecreasing sequence of simple functions $0 \le f_n \le 1$ such that $f_n \to f$ uniformly on $[a,b]$. Denote by $\{U_{n,1}, \dots, U_{n,r(n)}\}$ a Borel measurable decomposition of $[a,b]$ such that

$$f_n(p) = \sum_{j=1}^{r(n)} c_{n,j}\, I_{U_{n,j}}(p)\,, \qquad p \in [a,b]\,,$$

where $c_{n,j} \in [0,1]$. By induction, we shall construct sequences

$$\mathcal{W}_n = \{W_{n,1}, \dots, W_{n,\alpha(n)}\} \subset \mathcal{B}(0,1)\,,\ \mathcal{H}_n = \{H_{n,1}, \dots, H_{n,\alpha(n)}\} \subset \mathcal{K}\,,$$

such that for all $n \ge 0$:

(i) $\mathcal{W}_n$ is a Borel measurable decomposition of the interval $[a,b]$;
(ii) $\mathcal{W}_n \preccurlyeq \mathcal{W}_{n-1} \preccurlyeq \dots \preccurlyeq \mathcal{W}_0$;
(iii) if $W_{0,i_0} \in \mathcal{W}_0$, $W_{1,i_1} \in \mathcal{W}_1$, $\dots$, $W_{n,i_n} \in \mathcal{W}_n$ and $W_{0,i_0} \supseteq W_{1,i_1} \supseteq \dots \supseteq W_{n,i_n}$, then the sets $H_{0,i_0}$, $H_{1,i_1}$, $\dots$, $H_{n,i_n}$ are pairwise disjoint;
(iv) the inequality $0 \le f_n(p) - \hat{f}_n(p) \le n^{-1}$ holds for all $p \in [a,b]$, where

$$\hat{f}_n(p) = \sum_{k=0}^{n} \sum_{i=1}^{\alpha(n)} \lambda_p(H_{k,i})\, I_{W_{k,i}}(p)\,.$$

Put $f_0 \equiv 0$, $\hat{f}_0 \equiv 0$, $\mathcal{W}_0 = \{[a,b]\}$ and $\mathcal{H}_0 = \{\emptyset\}$. Assume that $\mathcal{W}_1$, $\mathcal{H}_1$, …, $\mathcal{W}_{m-1}$, $\mathcal{H}_{m-1}$ have been already constructed such that (i), (ii), (iii), (iv) hold for some $m \in \mathbb{N}$ and $n = 0, 1, \dots, m-1$. Choose a finite Borel measurable decomposition $\mathcal{V}_m = \{V_{m,1}, \dots, V_{m,s(m)}\}$ of $[a,b]$ such that $\mathcal{V}_m \preccurlyeq \{U_{m,1}, \dots, U_{m,r(m)}\}$ and $\mathcal{V}_m \preccurlyeq \mathcal{W}_{m-1}$. Fix a $V_{m,g} \in \mathcal{V}_m$ and let $U_{m,j} \in \{U_{m,1}, \dots, U_{m,r(m)}\}$ be the unique set for which $V_{m,g} \subseteq U_{m,j}$ holds. By (ii), there exists an uniquely determined sequence of positive integers $i_0$, $i_1$, … , $i_{m-1}$ such that $[a,b] = W_{0,i_0} \supseteq W_{1,i_1} \supseteq \cdots \supseteq W_{m-1,i_{m-1}} \supseteq V_{m,g}$. It follows easily from (iii) and (iv) that

$$\begin{aligned} 0 \le & f_{m-1}(p) - \hat{f}_{m-1}(p) \le f_m(p) - \hat{f}_{m-1}(p) = c_{m,j} - \sum_{k=0}^{m-1} \lambda_p(H_{k,i_k}) \\ = & c_{m,j} - \lambda_p(\bigcup_{k=0}^{m-1} H_{k,i_k}) \le 1\,, \qquad p \in V_{m,g}\,. \end{aligned}$$

Since $c_{m,j} - \sum_{k=0}^{m-1} \lambda_p(H_{k,i_k})$ is a polynomial (because $H_{k,i_k} \in \mathcal{K}$), there exists a continuous function $\gamma\colon [0,1] \to [0,1]$ such that

$$\gamma(p) = f_m(p) - \hat{f}_{m-1}(p) \le \lambda_p(K_g)\,, \qquad p \in V_{m,g}\,,$$

where

$$K_g = (0,1) - \bigcup_{k=0}^{m-1} H_{k,i_k}\,.$$

Thus, for each $1 \le g \le s(m)$ there exists by Lemma 2 a finite Borel measurable decomposition $\{A_1^{m,g}, \dots, A_{t(g)}^{m,g}\}$ of $V_{m,g}$ and the sets $F_1^{m,g}$, … , $F_{t(g)}^{m,g} \in \mathcal{K}$ such that $F_1^{m,g} \subseteq K_g$, … , $F_{t(g)}^{m,g} \subseteq K_g$ and

$$0 \le f_m(p) - \hat{f}_{m-1}(p) - \lambda_p(F_i^{m,g}) \le m^{-1}\,, \quad p \in A_i^{m,g}\,,\ 1 \le i \le t(g)\,. \tag{2}$$

Putting

$$\begin{aligned} \mathcal{W}_m &= \{A_i^{m,g} \,|\, g = 1, \dots, s(m); i = 1, \dots, t(g)\}\,, \\ \mathcal{H}_m &= \{F_i^{m,g} \,|\, g = 1, \dots, s(m); i = 1, \dots, t(g)\}\,, \end{aligned}$$

it is easy to verify (i), (ii), (iii), (iv) for $\mathcal{W}_1$, $\mathcal{H}_1$, … ,$\mathcal{W}_m$, $\mathcal{H}_m$ using (2).

For each $n \in \mathbb{N}$ put

$$C_n = \bigcup_{k=1}^{n} \bigcup_{i=1}^{\alpha(k)} \left(H_{k,i} \cap \Lambda(W_{k,i})\right)\,.$$

By (i), (ii), (iii) and by (1) we have $\lambda_p(C_n) = \hat{f}_n(p)$ for all $p \in [a,b]$ and, consequently, $\lambda_p(C_n) \to f(p)$ uniformly on $[a,b]$ by (iv). Since $C_n \subseteq C_{n+1}$ for all $n \in \mathbb{N}$, we may put

$$B = \bigcup_{n=1}^{\infty} C_n$$

to get that $f(p) = \lambda_p(B)$ for all $p \in [a,b]$. $\square$

Now, to prove our Theorem it is sufficient to verify the implication (a)$\Rightarrow$(b): Let $f\colon (0,1) \to [0,1]$ be a Borel measurable function. By Lemma 3, there exists a Borel set $B_n \subseteq (0,1)$ such that $f(p) = \lambda_p(B_n)$ for all $p \in [\frac{1}{n}, \frac{n-1}{n}]$ and all $n \geq 3$. Thus, it is sufficient to put

$$B = \bigcup_{n=3}^{\infty} \left(B_n \cap \Lambda(J_n)\right) ,$$

where

$$J_3 = [\tfrac{1}{3}, \tfrac{2}{3}] , \quad J_n = [\tfrac{1}{n}, \tfrac{1}{n-1}) \cup (\tfrac{n-2}{n-1}, \tfrac{n-1}{n}] , \quad n \geq 4 .$$

As the contrary implication is standard, the proof is completed.

**3. Corollaries.**

In the sequel, $F \circ \nu$ denotes the image measure of a measure $\nu$ w.r.t. a measurable map $F$, i.e. $(F \circ \nu)(A) = \nu(F^{-1}(A))$ for all measurable sets $A$. Also, if necessary, we identify for each $p \in (0,1)$ the probability space $((0,1), \mathcal{B}(0,1), \lambda_p)$ with the product $(\{0,1\}^{\mathbb{N}}, \mathcal{B}(\{0,1\}^{\mathbb{N}}), \mu_p = \bigotimes_1^{\infty} (1-p)\varepsilon_0 + p\varepsilon_1)$. The identification is obviously "good enough" for all our purposes, as the measure $\mu_p$ is the image of $\lambda_p$ w.r.t. the dyadic expansion map $x \to (x_1, x_2, \dots)$ which has the measurable inverse defined almost surely w.r.t. $\mu_p$.

**Corollary 1.** *For each Borel measurable function $f\colon (0,1) \to (0,1)$ there exists a Borel measurable function $H_f\colon (0,1) \to (0,1)$ such that $H_f \circ \lambda_p = \lambda_{f(p)}$ for all $p \in (0,1)$.*

Proof: By Theorem there exists a Borel set $B_f \subseteq \{0,1\}^{\mathbb{N}}$ such that $f(p) = \lambda_p(B_f)$ for all $p \in (0,1)$. Let $\{i_{n,k}\}_{k=1}^{\infty} \subseteq \mathbb{N}$, $n \in \mathbb{N}$, are increasing sequences such that $i_{n,k}$ are distinct integers for all $(n,k) \in \mathbb{N}^2$. Define a mapping $\rho_n\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ for each $n \in \mathbb{N}$ by

$$\rho_n(x) = (x_{i_{n,1}}, x_{i_{n,2}}, \dots) , \qquad x \in \{0,1\}^{\mathbb{N}} ,$$

and put $B_f^n = \rho_n^{-1}(B_f)$. The indicator functions $I_{B_f^1}$, $I_{B_f^2}$, $\dots$ are i.i.d. random variables w.r.t. each probability measure $\lambda_p$ such that $\lambda_p[\, I_{B_f^n} = 1\,] = \lambda_p(B_f^n) = \lambda_p(B_f) = f(p)$ holds. Thus, the function $H_f$ defined by

$$H_f(x) = (I_{B_f^1}(x), I_{B_f^2}(x), \dots) , \qquad x \in \{0,1\}^{\mathbb{N}} ,$$

has the desired property. $\square$

**Corollary 2.** *For each $\alpha \in (0,1)$ there exists a Borel measurable function*

$$H_\alpha\colon (0,1) \to (0,1)$$

*such that $H_\alpha \circ \lambda_p = \lambda_\alpha$ holds for all $p \in (0,1)$.*

Recall that a probability measure $\nu$ on $((0,1), \mathcal{B}(0,1))$ is called symmetric, if

$$\nu(A) = \nu\left(\{x \in (0,1) \,|\, (x_{\pi(1)}, \dots, x_{\pi(n)}, x_{n+1}, x_{n+2}, \dots) \in A\}\right)$$

holds for each $A \in \mathcal{B}(0,1)$, for each $n \in \mathbb{N}$ and for each permutation $\pi\colon \{1,\dots,n\} \to \{1,\dots,n\}$. Equivalently, a measure $\nu$ on $((0,1),\mathcal{B}(0,1))$ is symmetric iff $\nu$ is the distribution of a random variable

$$Y = \sum_{n=1}^{\infty} 2^{-n} X_n \,,$$

where $\{X_n\}_{n=1}^{\infty}$ is a sequence of exchangeable 0–1 random variables. For example, each measure $\lambda_p$, $p \in (0,1)$, is symmetric.

**Corollary 3.** *For each Borel probability measure $\mu$ on $\mathbb{R}$ there exists a Borel measurable function $H_\mu\colon (0,1) \to \mathbb{R}$ such that $H_\mu \circ \nu = \mu$ holds for all symmetric probability measures $\nu$ defined on $((0,1),\mathcal{B}(0,1))$.*

Proof: It is easy to see that it suffices to treat the case $\mu = \lambda_{1/2}$. A well-known de Finetti's result says that for each symmetric probability measure $\nu$ on $((0,1), \mathcal{B}(0,1))$ there exists a probability measure $Q$ on $((0,1),\mathcal{B}(0,1))$ such that

$$\nu(A) = \int_0^1 \lambda_p(A)\ \ Q(dp)$$

holds for all $A \in \mathcal{B}(0,1)$ (see e.g. [1, p. 225]). Now, the assertion follows easily applying Corollary 2 with $\alpha = \frac{1}{2}$. $\square$

## References

[1] Feller W., *An Introduction to Probability Theory and its Applications. Volume II.*, John Wiley & Sons, Inc., New York, London and Sydney, 1966.
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(*Received* September 8, 1992)