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Köthe dual of Banach sequence spaces $\ell_p[X]$ $(1 \le p < \infty)$ and Grothendieck space

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Abstract. In this paper, we show the representation of Köthe dual of Banach sequence spaces $\ell_p[X]$ $(1 \le p < \infty)$ and give a characterization of that the spaces $\ell_p[X]$ $(1 < p < \infty)$ are Grothendieck spaces.

Keywords: vector-valued sequence space; Köthe dual; GAK-space; Grothendieck space
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Let X be a Banach space and X^* its topological dual, and let B_X denote the closed unit ball of X. For $1 \le p < \infty$, let

$$\ell_p(X) = \Big\{ \overline{x} = (x_j) \in X^{\mathbb{N}} : \|\overline{x}\|_{\ell_p} = \Big(\sum_{i=1}^{\infty} \|x_i\|^p \Big)^{1/p} < \infty \Big\},$$

$$\ell_p[X] = \Big\{ \overline{x} = (x_j) \in X^{\mathbb{N}} : \text{ for each } f \in X^*, \ \sum_{i \ge 1} |f(x_i)|^p < \infty \Big\}.$$

And for each $\overline{x} \in \ell_p[X]$, let

$$\|\overline{x}\|_{(\ell_p)} = \sup \Big\{ \Big(\sum_{i \ge 1} |f(x_i)|^p \Big)^{1/p} : f \in B_{X^*} \Big\}.$$

Then $(\ell_p(X), \|\cdot\|_{\ell_p})$ and $(\ell_p[X], \|\cdot\|_{(\ell_p)})$ are Banach spaces (see [1], [2], [3]). For $\overline{x} \in X^{\mathbb{N}}$, let

$$\overline{x}\ (i \le n) = (x_1, \dots, x_n, 0, 0, \dots),$$
$$\overline{x}\ (i > n) = (0, \dots, 0, x_{n+1}, x_{n+2}, \dots).$$

And let

$$\ell_p[X]_r = \{ \overline{x} \in \ell_p[X] : \lim_n \|\overline{x}\ (i > n)\|_{(\ell_p)} \} = 0.$$

If $\ell_p[X]_r = \ell_p[X]$, then $\ell_p[X]$ is said to be a GAK-space [4].

For a vector-valued sequence space $S(X)$ from X, define its Köthe dual with respect to the dual pair (X, X^*) (see [4]) as follows:

$$S(X)^{\times} \mid_{(X,X^*)} = \Big\{ \overline{f} = (f_j) \in X^{*\mathbb{N}} : \text{ for each } \overline{x} = (x_j) \in S(X), \ \sum_{i \ge 1} |f_i(x_i)| < \infty \Big\}.$$

We denote $S(X)^{\times} \mid_{(X,X^*)}$ by $S(X)^{\times}$ simply if the meaning is clear from the context.

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Lemma 1. *For $1 \le p < \infty$, $(\ell_p[X]_r)^\times = \ell_p[X]^\times$.*

Proof: It is easy to see that $\ell_p[X]^\times \subseteq (\ell_p[X]_r)^\times$. So we only need to prove that $(\ell_p[X]_r)^\times \subseteq \ell_p[X]^\times$.

For $\overline{x} = (x_j) \in \ell_p[X]$ and $t = (t_j) \in c_0$, let $t\overline{x} = (t_j x_j)$. Then $\|t\overline{x}\ (i > n)\|_{(\ell_p)} \le \|\overline{x}\|_{(\ell_p)} \sup_{i>n} |t_i|$ implies that $t\overline{x} \in \ell_p[X]_r$. So for $\overline{f} = (f_j) \in (\ell_p[X]_r)^\times$, we have

$$\sum_{i\ge 1} |f_i(t_i x_i)| < \infty\,.$$

It follows from the fact that $t \in c_0$ was taken arbitrary that

$$\sum_{i\ge 1} |f_i(x_i)| < \infty\,.$$

Thus, $\overline{f} \in \ell_p[X]^\times$ and the proof is completed. □

Lemma 2. (1) *For $1 \le p < \infty$, $\ell_p[X]^\times \subseteq (\ell_p[X], \|\cdot\|_{(\ell_p)})^*$ and $(\ell_p[X]_r)^\times = (\ell_p[X]_r, \|\cdot\|_{(\ell_p)})^*$.*

(2) *Let $\|\cdot\|^*_{(\ell_p)}$ denote the dual norm of $\|\cdot\|_{(\ell_p)}$ on the dual space $(\ell_p[X], \|\cdot\|_{(\ell_p)})^*$. Then for each $\overline{x} \in \ell_p[X]$, we have*

$$\|\overline{x}\|_{(\ell_p)} = \sup\{|\langle \overline{x}, \overline{f}\rangle| : \overline{f} \in \ell_p[X]^\times,\ \|\overline{f}\|^*_{(\ell_p)} \le 1\}\,,$$

where $\langle \overline{x}, \overline{f}\rangle = \sum_{i\ge 1} f_i(x_i)$.

Proof: See Theorem 2.3 in [3]. □

Lemma 3. *Every weak* unconditionally Cauchy series in X^* is weak unconditionally Cauchy series.*

Proof: See the proof of p. 49, Corollary 11 in [5]. □

Lemma 4. *For $1 \le p < \infty$,*

$$\ell_p[X^*] = \Big\{\overline{f} = (f_j) \in X^{*\mathbb{N}} :\ \ \textit{for each}\ \ x \in X,\ \sum_{i\ge 1} |f_i(x)|^p < \infty\Big\}\,.$$

Proof: Let

$$\Delta = \Big\{\overline{f} = (f_j) \in X^{*\mathbb{N}} :\ \ \text{for each}\ \ x \in X,\ \sum_{i\ge 1} |f_i(x)|^p < \infty\Big\}\,.$$

By definition, we only need to prove that $\Delta \subseteq \ell_p[X^*]$.

Let $\overline{f} \in \Delta$ and $t_j \in \ell_q (1/p + 1/q = 1)$. Then $\sum_{i\ge 1} |f_i(t_i x)| < \infty$ for each $x \in X$. So the series $\sum_j t_j f_j$ is weak* unconditionally Cauchy in X^* and hence, it is weak unconditionally Cauchy by Lemma 3. That is, $\sum_{i\ge 1} |F(t_i f_i)| < \infty$ for each $F \in X^{**}$. Since (t_j) is arbitrary in ℓ_q, $\sum_{i\ge 1} |F(f_i)|^p < \infty$ and $\overline{f} = (f_j) \in \ell_p[X^*]$. The proof is completed. □

Lemma 5 (the principle of local reflexivity, [6]). *Let X be a normed space and Z^{**} a finite dimensional subspace of X^{**}. For $\{F_i\}_1^n \subseteq Z^{**}$, $\{f_i\}_1^n \subseteq X^*$ and $\varepsilon > 0$, there exists a linear map $T : Z^{**} \to X$ such that $\|T\| \le 1$ and*

$$|f_i(TF_i) - F_i(f_i)| < \varepsilon, \quad i = 1, 2, \ldots, n.$$

Proposition 6. $\ell_p[X^{**}]^\times |_{(X^{**},X^*)} = \ell_p[X]^\times |_{(X,X^*)}$ $(1 \le p < \infty)$.

PROOF: It is easy to see that $\ell_p[X] \subseteq \ell_p[X^{**}]$ implies that

$$\ell_p[X^{**}]^\times |_{(X^{**},X^*)} \subseteq \ell_p[X]^\times |_{(X,X^*)} \ .$$

So we only need to prove that

$$\ell_p[X]^\times |_{(X,X^*)} \subseteq \ell_p[X^{**}]^\times |_{(X^{**},X^*)} \ .$$

Let $\overline{f} = (f_j) \in \ell_p[X]^\times |_{(X,X^*)}$ and $\overline{F} = (F_j) \in \ell_p[X^{**}]$. For a fixed $n \in \mathbb{N}$, by Lemma 5, there exists a linear map $T_n : \operatorname{span}\{F_i\}_1^n \to X$ such that $\|T_n\| \le 1$ and

$$|F_i(f_i)| \le |f_i(T_nF_i)| + 1/n, \quad i = 1, 2, \ldots, n.$$

Now we prove that $\{(T_nF_1, \ldots, T_nF_n, 0, 0, \ldots)\}_{n=1}^\infty$ is a bounded subset of $\ell_p[X]$. By Theorem 1.5 in [2], we have

$$\begin{aligned}
&\|(T_nF_1, \ldots, T_nF_n, 0, 0, \ldots)\|_{(\ell_p)} \\
&= \sup\Big\{\|\sum_{i=1}^n s_iT_nF_i\| : s = (s_j) \in B_{\ell_q}\Big\} \ (1/p + 1/q) = 1 \\
&\le \sup\Big\{\|T_n\|\|\sum_{i=1}^n s_iF_i\| : s \in B_{\ell_q}\Big\} \\
&\le \sup\Big\{\|\sum_{i=1}^\infty s_iF_i\| : s \in B_{\ell_q}\Big\} = \|\overline{F}\|_{(\ell_p)} \,.
\end{aligned}$$

So $\{(T_nF_1, \ldots, T_nF_n, 0, 0, \ldots)\}_{n=1}^\infty$ is a bounded subset of $\ell_p[X]$ and hence, $\sigma(\ell_p[X], \ell_p[X]^\times |_{(X,X^*)})$-bounded. Thus, we have

$$\sum_{i=1}^n |F_i(f_i)| \le \sum_{i=1}^n |f_i(T_nF_i)| + 1 \le \sup_{n\ge 1}\Big\{\sum_{i=1}^n |f_i(T_nF_i)|\Big\} + 1.$$

Because $n \in \mathbb{N}$ is arbitrary, it follows that

$$\sum_{i=1}^\infty |F_i(f_i)| < \infty \,.$$

So we prove that $\overline{f} = (f_j) \in \ell_p[X^{**}]^\times |_{(X^{**},X^*)}$ and this completes the proof. $\square$

Proposition 7. $(\ell_p[X]^\times |_{(X,X^*)})^\times |_{(X^*,X^{**})}= \ell_p[X^{**}]$ $(1 \le p < \infty)$.

Proof: By Proposition 6, it is easy to see that

$$\begin{aligned}\ell_p[X^{**}] &\subseteq (\ell_p[X^{**}]^\times |_{(X^{**},X^*)})^\times |_{(X^*,X^{**})} \\ &= (\ell_p[X]^\times |_{(X,X^*)})^\times |_{(X^*,X^{**})} \;.\end{aligned}$$

So we only need to prove that

$$(\ell_p[X^{**}]^\times |_{(X^{**},X^*)})^\times |_{(X^*,X^{**})} \subseteq \ell_p[X^{**}] \,.$$

Let $\overline{F} = (F_j) \in (\ell_p[X^{**}]^\times |_{(X^{**},X^*)})^\times |_{(X^*,X^{**})}$. Since $f \in X^*$ and $t = (t_j) \in \ell_q$ $(1/p + 1/q = 1)$ implies that $(t_j f) \in \ell_p[X^{**}]^\times |_{(X^{**},X^*)}$, $\sum_{i\ge 1} |F_i(t_i f)| < \infty$. Thus, $\sum_{i\ge 1} |F_i(f)|^p < \infty$ and hence, $\overline{F} \in \ell_p[X^{**}]$ by Lemma 4. The proof is completed. □

Theorem 8. *For $1 \le p < \infty$, $\ell_p \overset{\vee}{\otimes} X$, the injective tensor product of ℓ_p and X, is isometrically isomorphic to the space $(\ell_p[X]_r, \|\cdot\|_{(\ell_p)})$.*

Proof: For each $u = \sum_{i=1}^n t^{(i)} \otimes x_i \in \ell_p \otimes X$ $(t^{(i)} \in \ell_p,\ x_i \in X)$, define $\overline{x}_u = (\sum_{i=1}^n t_1^{(i)} x_i, \sum_{i=1}^n t_2^{(i)} x_i, \dots)$. Then

$$\begin{aligned}\|\overline{x}_u\|_{(\ell_p)} &= \sup\Big\{ \Big| \sum_{k\ge 1} s_k f\Big(\sum_{i=1}^n t_k^{(i)} x_i\Big)\Big| : f \in B_{X^*},\ s \in B_{\ell_q}\Big\} \\ &= \sup\Big\{ \Big| \sum_{i=1}^n f(x_i)\langle t^{(i)}, s\rangle\Big| : f \in B_{X^*},\ s \in B_{\ell_q}\Big\} \\ &= \lambda(u) \quad \text{(see [7, p. 223])} \ \ (1/p + 1/q = 1).\end{aligned}$$

Let $M = \sup_{1\le i\le n} \|x_i\|$. It follows from the above equality that

$$\begin{aligned}\|\overline{x}_u\ (j>k)\|_{(\ell_p)} &= \sup\Big\{ \Big| \sum_{i=1}^n f(x_i)\langle t^{(i)}, s\ (j>k)\rangle\Big| : f \in B_{X^*},\ s \in B_{\ell_q}\Big\} \\ &\le M \sup\Big\{ \sum_{i=1}^n |\langle t^{(i)}, s\ (j>k)\rangle| : s \in B_{\ell_q}\Big\} \,.\end{aligned}$$

Since B_{ℓ_q} is weak* compact, Theorem 6.11 in [8] implies that

$$\lim_k \|\overline{x}_u\ (j>k)\|_{(\ell_p)} = 0.$$

So, $\overline{x}_u \in \ell_p[X]_r$ and we can define a map $\varphi : \ell_p \otimes X \to \ell_p[X]_r$ by $\varphi(u) = \overline{x}_u$. It is easy to see that φ is a linear isometrically isomorphic map from $\ell_p \otimes X$ to $\ell_p[X]_r$. Next, we only need to prove that φ is surjective.

For $\overline{x} = (x_1, \dots, x_n, 0, 0, \dots)$, if we let $u = \sum_{i=1}^{n} e_i \otimes x_i$ (where $e_i = (0, \dots, 0, 1^{(i)}, 0, 0, \dots)$), then $\overline{x} = \varphi(u)$. Notice that $\lim_n \overline{x}\ (j \le n) = \overline{x}$ for each $\overline{x} \in \ell_p[X]_r$. So φ is surjective and the proof is completed. $\square$

For two Banach spaces X and Y, let $\mathcal{B}^\wedge(X,Y)$, $I(X,Y)$ and $N(X,Y)$ denote the class of integral bilinear functionals on $X \times Y$, the class of integral operators from X to Y and the class of nuclear operators from X to Y respectively (see p. 232 and p. 170 in [7]).

Theorem 9. *Let $1 \le p < \infty$ and $1/p + 1/q = 1$. Then $\overline{f} = (f_j) \in \ell_p[X]^\times |_{(X,X^*)}$ if and only if there exist an $r = (r_j) \in \ell_1$ a bounded sequence $\{s^{(n)}\}_{n=1}^\infty$ of ℓ_q and a bounded sequence $\{h_n\}_{n=1}^\infty$ of X^* such that*

$$f_i = \sum_{n \ge 1} r_n s_i^{(n)} h_n, \quad i = 1, 2, \dots .$$

Proof: Necessity. Let $\overline{f} = (f_j) \in \ell_p[X]^\times$. By Lemma 1 and Lemma 2, $\overline{f} \in (\ell_p[X]_r, \|\cdot\|_{(\ell_p)})^*$. So Theorem 8 implies that there is an $\psi^* \in (\ell_p \overset{\vee}{\otimes} X)^*$ corresponding to $\overline{f}$. By Definition 6 in [7, p. 232], there is an $\psi \in \mathcal{B}^\wedge(\ell_p, X)$ corresponding to ψ^*. Furthermore, by Corollary 12 in [7, p. 237], there exists a $T_\psi \in I(\ell_p, X^*)$ corresponding to ψ. Since Corollary 10 in [7, p. 235] and Theorem 6 in [7, p.248] guarantee that $I(\ell_p, X^*) = N(\ell_p, X^*)$, there are an $r = (r_j) \in \ell_1$, a bounded sequence $\{s^{(n)}\}_{n=1}^\infty$ of ℓ_q and a bounded sequence $\{h_n\}_{n=1}^\infty$ of X^* such that

$$T_\psi(t) = \sum_{n \ge 1} r_n \langle t, s^{(n)} \rangle h_n, \qquad \text{for } \ t \in \ell_p.$$

Now for each $i \ge 1$ and each $x \in X$, by the above corresponding relations, we have

$$T_\psi(e_i)(x) = \psi(e_i, x) = \psi^*(e_i \otimes x) = \langle \varphi(e_i \otimes x), \overline{f} \rangle = f_i(x).$$

Thus

$$f_i = T_\psi(e_i) = \sum_{n \ge 1} r_n s_i^{(n)} h_n, \quad i = 1, 2, \dots .$$

Sufficiency. Let $M = \sup_{n \ge 1} \|s^{(n)}\|_q$ and $N = \sup_{n \ge 1} \|h_n\|$. Then, for each $\overline{x} = (x_j) \in \ell_p[X]$, we have

$$\sum_{i \ge 1} |s_i^{(n)} h_n(x_i)| \le MN \|\overline{x}\|_{(\ell_p)}, \qquad \text{for } \ n \ge 1.$$

And so

$$\sum_{i \ge 1} |f_i(x_i)| \le \sum_{n \ge 1} |r_n| \sum_{i \ge 1} |s_i^{(n)} h_n(x_i)| < \infty.$$

Therefore, $\overline{f} \in \ell_p[X]^\times$ and the proof is completed. $\square$

Theorem 10. *For* $1 < p < \infty$, $(\ell_p[X]^\times, \|\cdot\|^*_{(\ell_p)})$ *is a* GAK-*space.*

Proof: Let $\overline{f} = (f_j) \in \ell_p[X]^\times$. Then by Theorem 9, there exist an $r = (r_j) \in \ell_1$, a bounded sequence $\{s^{(n)}\}_1^\infty$ of ℓ_q and a bounded sequence $\{h_n\}_1^\infty$ of X^* such that

$$f_i = \sum_{n\geq 1} r_n s_i^{(n)} h_n, \quad i = 1, 2, \dots .$$

Without loss of generality, we can assume that $\|s^{(n)}\|_q \leq 1$ and $\|h_n\| \leq 1$ for $n \geq 1$. Thus, for $\overline{x} \in \ell_p[X]$ with $\|\overline{x}\|_{(\ell_p)} \leq 1$, we have

$$\sum_{i\geq 1} |s_i^{(n)} h_n(x_i)| \leq \|\overline{x}\|_{(\ell_p)} \leq 1 \quad \text{for } \ n \geq 1.$$

So

$$\Big\{ \Big(\sum_{i\geq 1} |s_i^{(n)} h_n(x_i)| \Big)_{n\geq 1} : \|\overline{x}\|_{(\ell_p)} \leq 1 \Big\} \subseteq B_{\ell_\infty}.$$

Let $\varepsilon > 0$. Then B_{ℓ_∞} is weak* compact implies that there exists an $n_0 \in \mathbb{N}$ such that

$$\sum_{n>n_0} |r_n| \sum_{i\geq 1} |s_i^{(n)} h_n(x_i)| < \varepsilon/2, \quad \overline{x} \in \ell_p[X], \quad \|\overline{x}\|_{(\ell_p)} \leq 1.$$

Since B_{ℓ_p} is weakly compact set and

$$\Big\{ (h_n(x_i))_{i\geq 1} : \overline{x} \in \ell_p[X], \ \|\overline{x}\|_{(\ell_p)} \leq 1, \ n \geq 1 \Big\} \subseteq B_{\ell_p},$$

there is a $k_0 \in \mathbb{N}$ such that for each $k > k_0$,

$$\sum_{i>k} |s_i^{(n)} h_n(x_i)| < \varepsilon/2\|r\|_1$$

for $\overline{x} \in \ell_p[X]$ with $\|\overline{x}\|_{(\ell_p)} \leq 1$ and $n = 1, 2, \dots, n_0$. Thus, for each $\overline{x} \in \ell_p[X]$ with $\|\overline{x}\|_{(\ell_p)} \leq 1$ and each $k > k_0$, we have

$$\begin{aligned}
\sum_{i>k} |f_i(x_i)| &\leq \sum_{n=1}^{n_0} |r_n| \sum_{i>k} |s_i^{(n)} h_n(x_i)| + \sum_{n>n_0} |r_n| \sum_{i>k} |s_i^{(n)} h_n(x_i)| \\
&\leq \Big(\sum_{n=1}^{\infty} |r_n| \Big) \varepsilon/2\|r\|_1 + \sum_{n>n_0} |r_n| \sum_{i\geq 1} |s_i^{(n)} h_n(x_i)| < \varepsilon.
\end{aligned}$$

So for $k > k_0$,

$$\begin{aligned}
\|\overline{f}\ (j > k)\|^*_{(\ell_p)} &= \sup\Big\{ |\langle \overline{x}, \overline{f}\ (j > k)\rangle| : \overline{x} \in \ell_p[X], \ \|\overline{x}\|_{(\ell_p)} \leq 1 \Big\} \\
&= \sup\Big\{ |\sum_{i>k} f_i(x_i)| : \|\overline{x}\|_{(\ell_p)} \leq 1 \Big\} < \varepsilon.
\end{aligned}$$

Therefore, $\lim_k \|\overline{f}\ (j > k)\|^*_{(\ell_p)} = 0$ and $\overline{f} \in (\ell_p[X], \|\cdot\|^*_{(\ell_p)})_r$. □

For $1 < p < \infty$, by Theorem 10 and [4, Proposition 4.9], we have

$$(*) \qquad (\ell_p[X]^\times\ |_{(X,X^*)})^\times\ |_{(X^*,X^{**})} = (\ell_p[X]^\times\ |_{(X,X^*)}, \|\cdot\|^*_{(\ell_p)})^*\,.$$

Now, if we let $\|\cdot\|^{**}_{(\ell_p)}$ denote the dual norm of $\|\cdot\|^*_{(\ell_p)}$ on the dual space $(\ell_p[X]^\times\ |_{(X,X^*)}$, $\|\cdot\|^*_{(\ell_p)})^*$, then by Proposition 7 and Lemma 2, the norm $\|\cdot\|_{(\ell_p)}$ on the space $\ell_p[X^{**}]$ is equal to the norm $\|\cdot\|^{**}_{(\ell_p)}$.

Similarly as the proof of Theorem 3.6 in [3], we have the following two propositions.

Proposition 11. *Let $\overline{f}^{(n)} \in \ell_p[X]^\times$ $(1 \leq p < \infty)$. Then that*

$$\sigma(\ell_p[X]^\times\ |_{(X,X^*)}, (\ell_p[X]^\times\ |_{(X,X^*)})^\times\ |_{(X^*,X^{**})}) - \lim_n \overline{f}^{(n)} = 0$$

is equivalent to

(a) $\sigma(X^*, X^{**}) - \lim_n f_i^{(n)} = 0$ *for* $i \geq 1$*; and*
(b) $\sup_{n\geq 1} \|\overline{f}^{(n)}\|^*_{(\ell_p)} < \infty$

if and only if $((\ell_p[X]^\times\ |_{(X,X^*)})^\times\ |_{(X^*,X^{**})}, \|\cdot\|^{**}_{(\ell_p)})$ *is a* GAK*-space.*

Proposition 12. *Let $\overline{f}^{(n)} \in (\ell_p[X]_r)^*$ $(1 \leq p < \infty)$. Then*

$$\sigma((\ell_p[X]_r)^*,\ \ell_p[X]_r) - \lim \overline{f}^{(n)} = 0$$

if and only if $\sigma(X^*, X) - \lim_n f_i^{(n)} = 0$ *for* $i \geq 1$ *and* $\sup_{n\geq 1} \|\overline{f}^{(n)}\|^*_{(\ell_p)} < \infty$.

We say a Banach space X to be a Grothendieck space if every weak* null sequence on X^* is weak null sequence (see [7, p. 179]). Leonard [1] has proved that $\ell_p(X)$ $(1 < p < \infty)$ is a Grothendieck space if and only if X is a Grothendieck space. Now we have

Theorem 13. *For $1 < p < \infty$. The Banach space $(\ell_p[X]_r, \|\cdot\|_{(\ell_p)})$ is a Grothendieck space if and only if*

(i) *X is a Grothendieck space; and*
(ii) *$(\ell_p[X^{**}], \|\cdot\|_{(\ell_p)})$ is a* GAK*-space.*

Proof: Sufficiency. By (ii), $(\ell_p[X], \|\cdot\|_{(\ell_p)})$ is a GAK-space, i.e. $\ell_p[X]_r = \ell_p[X]$.

Let $\overline{f}^{(n)} \in (\ell_p[X], \|\cdot\|_{(\ell_p)})^*$ such that

$$\sigma(\ell_p[X]^*, \ell_p[X]) - \lim_n \overline{f}^{(n)} = 0.$$

By Proposition 12, we have

$$\sigma(X^*, X) - \lim_n f_i^{(n)} = 0, \quad i = 1, 2, \ldots$$

and

$$\sup_{n\geq 1} \|\overline{f}^{(n)}\|^*_{(\ell_p)} < \infty.$$

By (i), we have

$$\sigma(X^*, X^{**}) - \lim_n f_i^{(n)} = 0, \quad i = 1, 2, \ldots .$$

By (ii) and Propositions 2, 6, 7, the space $((\ell_p[X]^*)^\times |_{(X^*,X^{**})}, \|\cdot\|_{(\ell_p)})$ is a GAK-space. So Proposition 11 guarantees that

$$\sigma(\ell_p[X]^*, (\ell_p[X]^*)^\times) - \lim_n \overline{f}^{(n)} = 0.$$

It follows from $(*)$ that

$$\sigma(\ell_p[X]^*, \ell_p[X]^{**}) - \lim_n \overline{f}^{(n)} = 0.$$

and completes the sufficiency.

Necessity. To prove (i), let $f_n \in X^*$ $(n \geq 1)$ such that

$$\sigma(X^*, X) - \lim_n f_n = 0.$$

Let $\overline{f}^{(n)} = (f_n, 0, 0, \ldots)$ for $n \geq 1$. Then $\overline{f}^{(n)} \in (\ell_p[X]_r)^*$ and

$$\sigma((\ell_p[X]_r)^*, \ell_p[X]_r) - \lim_n \overline{f}^{(n)} = 0.$$

So

$$\sigma((\ell_p[X]_r)^*, (\ell_p[X]_r)^{**}) - \lim_n \overline{f}^{(n)} = 0$$

and hence, $\sigma(X^*, X^{**}) - \lim_n f_n = 0$. (i) follows.

For (ii), let $\overline{f}^{(n)} \in \ell_p[X]^\times |_{(X,X^*)}$ such that

$$\sigma(X^*, X^{**}) - \lim_n f_i^{(n)} = 0, \quad i = 1, 2, \ldots$$

and

$$\sup_{n\geq 1} \|\overline{f}^{(n)}\|^*_{(\ell_p)} < \infty.$$

By Lemmas 1, 2 and Proposition 12, we have

$$\sigma((\ell_p[X]_r)^*, \ell_p[X]_r) - \lim_n \overline{f}^{(n)} = 0.$$

And hence,

$$\sigma((\ell_p[X]_r)^*, (\ell_p[X]_r)^{**}) - \lim_n \overline{f}^{(n)} = 0.$$

It follows from $(*)$ that

$$\sigma(\ell_p[X]^\times |_{(X,X^*)}, (\ell_p[X]^\times |_{(X,X^*)})^\times |_{(X^*,X^{**})}) - \lim_n \overline{f}^{(n)} = 0.$$

So Propositions 6, 7, 11 imply that $(\ell_p[X^{**}], \|\cdot\|_{(\ell_p)})$ is a GAK-space and (ii) follows. The proof is completed. $\square$

Corollary 14. *If $\ell_p[X]_r$ $(1 < p < \infty)$ is a Grothendieck space, then $\ell_p[X]$ is a GAK-space.*

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