On the injectivity of Boolean algebras

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Abstract. The functor taking global elements of Boolean algebras in the topos **Sh**$\mathfrak{B}$ of sheaves on a complete Boolean algebra $\mathfrak{B}$ is shown to preserve and reflect injectivity as well as completeness. This is then used to derive a result of Bell on the Boolean Ultrafilter Theorem in $\mathfrak{B}$-valued set theory and to prove that (i) the category of complete Boolean algebras and complete homomorphisms has no non-trivial injectives, and (ii) the category of frames has no absolute retracts.

Keywords: sheaves on a complete Boolean algebra, injective Boolean algebra, complete Boolean algebra, injective complete Boolean algebra, absolute frame retract

Classification: 03E25, 03E40, 03G05, 06A23, 06E99

This paper has two separate motivations, each leading to questions concerning the injectivity of Boolean algebras in various settings.

The first of these is the fact, established by Bell [3] that, for any complete Boolean algebra $\mathfrak{B}$ in Zermelo-Fraenkel Set Theory (**ZF**), a certain version of the Boolean Ultrafilter Theorem holds in the $\mathfrak{B}$-valued model of **ZF** iff $\mathfrak{B}$ is injective. Here, we present a stronger result, using the topos **Sh**$\mathfrak{B}$ of sheaves on $\mathfrak{B}$ as the natural, intuitively suggestive realization of the category of $\mathfrak{B}$-valued sets and maps (Higgs [5], Blass-Sčedrov [4]). Indeed, we show a Boolean algebra in **Sh**$\mathfrak{B}$ is injective in the category **BooSh**$\mathfrak{B}$ of all such Boolean algebras iff the Boolean algebra of its global elements is injective in the category **Boo** of ordinary (= set based) Boolean algebras (Proposition 1). In particular, this means the initial Boolean algebra in **Sh**$\mathfrak{B}$ is injective in **BooSh**$\mathfrak{B}$ iff $\mathfrak{B}$ is injective (Corollary 1), which is the present counterpart of Bell's result referred to above.

The second motivation is the following question, raised by Pultr [10]: For a given complete Boolean algebra $\mathfrak{B}$, call any frame extension $L \supseteq \mathfrak{B}$ a $\mathfrak{B}$-*frame* and any homomorphism $L \to \mathfrak{B}$ over $\mathfrak{B}$ a $\mathfrak{B}$-*point* of L. Now, for the case $\mathfrak{B} = \mathbf{2}$ of ordinary frames, it is a classical fact that there are $\mathfrak{B}$-frames without $\mathfrak{B}$-points. Question: Is there a complete Boolean algebra $\mathfrak{B}$ for which every $\mathfrak{B}$-frame has a $\mathfrak{B}$-point? We show the answer is "no", by proving that the category of frames has no non-trivial absolute retracts (Proposition 4). This, in turn, is derived from the result that the category of complete Boolean algebras and complete homomorphisms has no non-trivial injectives (Proposition 3). Our proof of this uses complete Boolean algebras and appropriately defined *atomless* Boolean algebras in **Sh**$\mathfrak{B}$.

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1. Background.

In the following, $\mathfrak{B}$ will be a (non-trivial) complete Boolean algebra, with elements $U, V, W, \ldots$, zero 0 and unit E. $\mathbf{Sh}\mathfrak{B}$ is then the familiar category of sheaves of sets on $\mathfrak{B}$ in "the" category $\mathbf{Ens}$ of sets as provided by Zermelo-Fraenkel set theory not (necessarily) including the Axiom of Choice (**ZF**).

Further, we consider various categories of Boolean algebras, as follows:

Boo — Boolean algebras and their homomorphisms in **Ens**.

$\mathfrak{B}\downarrow$**Boo** — Boolean algebras over $\mathfrak{B}$ in **Ens**, that is, all $\mathfrak{B} \to A$ in **Boo**, with the compatible homomorphisms between their codomains as maps.

BooSh$\mathfrak{B}$ — Boolean algebras and their homomorphisms in **Sh**$\mathfrak{B}$.

For general notation regarding sheaves we follow Banaschewski-Bhutani [2]. In particular, for $A \in \mathbf{BooSh}\mathfrak{B}$ and $U \in \mathfrak{B}$, AU is the component of A at U; if $V \leq U$ in $\mathfrak{B}$, the restriction map $AU \to AV$ is denoted $c \rightsquigarrow c\,|\,V$; and for any homomorphism $h : A \to C$ in $\mathbf{BooSh}\mathfrak{B}$, $h_U : AU \to CU$ will be its component for $U \in \mathfrak{B}$. Recall also that the initial Boolean algebra $\mathbf{2}_\mathfrak{B}$ in $\mathbf{Sh}\mathfrak{B}$ has the components

$$\mathbf{2}_\mathfrak{B} U = \downarrow U = \{V \in \mathfrak{B}\,|\,V \leq U\},$$

for each $U \in \mathfrak{B}$. For any $A \in \mathbf{BooSh}\mathfrak{B}$, $i_A : \mathbf{2}_\mathfrak{B} \to A$ will be the unique homomorphism; then, for any $V \in \downarrow U$, $i_{AU}(V)$ is the element of AU whose restriction to V is the unit of AV while its restriction to the complement of V in $\downarrow U$ is zero. Finally, for any $W \in \mathfrak{B}$, the complete Boolean algebra $\downarrow W$ gives rise to the restriction functor $\mathbf{BooSh}\mathfrak{B} \to \mathbf{BooSh}(\downarrow W)$, taking each $A \in \mathbf{BooSh}\mathfrak{B}$ to its restriction $A|W$ with the same components as A for each $U \leq W$.

The familiar functors $\mathbf{Sh}\mathfrak{B} \to \mathbf{Ens}$ and $\mathbf{Ens} \to \mathbf{Sh}\mathfrak{B}$, associating the set of global elements, that is, the component at E, with each sheaf and the sheaf resulting from the corresponding constant presheaf with each set, respectively, lift to analogous functors $\Gamma : \mathbf{BooSh}\mathfrak{B} \to \mathbf{Boo}$ and $\Delta : \mathbf{Boo} \to \mathbf{BooSh}\mathfrak{B}$, Δ being left adjoint to Γ. The latter may obviously also be viewed as a functor into $\mathfrak{B}\,|\,\mathbf{Boo}$, taking each $A \in \mathbf{BooSh}\mathfrak{B}$ to $\Gamma i_A : \mathfrak{B} \to \Gamma A$ since $\Gamma \mathbf{2}_\mathfrak{B} = \mathbf{2}_\mathfrak{B} E = \mathfrak{B}$; indeed, for any $h : A \to C$ in $\mathbf{BooSh}\mathfrak{B}$, we have $h i_A = i_C$ and consequently $\Gamma h\, \Gamma i_A = \Gamma i_C$.

Each $A \in \mathbf{BooSh}\mathfrak{B}$ determines its *ideal lattice* $\mathfrak{J}A$ in $\mathbf{Sh}\mathfrak{B}$, where $(\mathfrak{J}A)U$ is the lattice, in **Ens**, of all ideals $J \subseteq A|U$, that is, the subsheaves J of $A|U$ in $\mathbf{Sh}(\downarrow U)$ for which each component JV, $V \leq U$, is an ideal of AV. $\mathfrak{J}A$ comes equipped with a lattice homomorphism $[\cdot] : A \to \mathfrak{J}A$, corresponding to the ordinary notion of principal ideal, given by

$$AU \to (\mathfrak{J}A)U$$
$$a \rightsquigarrow [a]_U$$
$$[a]_U V = \downarrow(a\,|\,V) \subseteq AV$$

for any U and $V \leq U$ in $\mathfrak{B}$. Then a Boolean algebra A in $\mathbf{Sh}\mathfrak{B}$ is *complete* iff there exists a map $\bigvee : \mathfrak{J}A \to A$ in $\mathbf{Sh}\mathfrak{B}$ such that, for all $U \in B$, $J \in (\mathfrak{J}A)U$, and $a \in AU$,

$$\textstyle\bigvee_U J \leq a \text{ iff } J \subseteq [a]_U\,.$$

Further, any $A \in \mathbf{BooSh}\mathfrak{B}$ has a *completion*, given by the Boolean algebra $\mathfrak{N}A$ in $\mathbf{Sh}\mathfrak{B}$ of its normal ideals, defined as the equalizer of the identity map $\mathfrak{J}A \to \mathfrak{J}A$ and the map $(\ )^{**} : \mathfrak{J}A \to \mathfrak{J}A$ where $(\cdot)^* : \mathfrak{J}A \to \mathfrak{J}A$ is the pseudocomplementation. It turns out that the lattice homomorphism $[\cdot] : A \to \mathfrak{J}A$ actually maps into $\mathfrak{N}A$, providing an embedding of A into the complete Boolean algebra $\mathfrak{N}A$.

Note that $\mathbf{2}_{\mathfrak{B}}$ is complete, by a result of Johnstone [7] concerning the completeness of the initial Boolean algebra in an arbitrary topos. On the other hand, ΓA is complete for any complete $A \in \mathbf{BooSh}\mathfrak{B}$, by a general result regarding complete partially ordered sets in a topos (Johnstone [6, p. 147]).

Next, we present a useful canonical description of the Boolean algebras in $\mathbf{Sh}\mathfrak{B}$.

For any $U \in \mathfrak{B}$, let U^* be its complement so that the restriction map induce an isomorphism $AE \to AU \times AU^*$ for any $A \in \mathbf{BooSh}\mathfrak{B}$. Further, for any $V \in \mathfrak{B}$, let 0_V and e_V be the zero and unit of AV, and define $s_U \in AE$ as the element corresponding to $(e_U, 0_{U^*})$ by the above isomorphism. Of course, $s_U = i_{AE}(U)$ by the earlier description of the initial homomorphism $i_A : \mathbf{2}_{\mathfrak{B}} \to A$. Now put

$$\tilde{A}U = \{x \in AE \,|\, x \leq s_U\} = {\downarrow} s_U \qquad (U \in \mathfrak{B})$$
$$\tilde{A}U \to \tilde{A}V : x \rightsquigarrow x \wedge s_V \qquad (V \leq U).$$

Clearly, this is a presheaf of Boolean algebras on $\mathfrak{B}$. Moreover, the map $\tilde{A}U \to AU$ by restriction is an isomorphism: it is a Boolean homomorphism since $s_U \,|\, U = e_U$, one-one since $x \leq s_U$ implies $x \,|\, U^* = 0$, and onto because this is the case for the restriction map $AE \to AU$. It now follows immediately that one has an isomorphism $\varphi_A : \tilde{A} \to A$ whose components are given by the restriction maps.

Further, the correspondence $A \rightsquigarrow \tilde{A}$ is functorial: for any $h : A \to C$ in $\mathbf{BooSh}\mathfrak{B}$, h_E maps $\tilde{A}U \subseteq AE$ into $\tilde{C}U \subseteq CE$ and determines $\tilde{h} : \tilde{A} \to \tilde{C}$ since $hi_A = i_C$; moreover, the passage $h \rightsquigarrow \tilde{h}$ clearly preserves composition and identity maps. Finally, one easily checks that the isomorphisms $\varphi_A : \tilde{A} \to A$ are natural in A.

Apart from $\mathbf{Sh}\mathfrak{B}$, we shall occasionally refer to the more general topos $\mathbf{Sh}\mathfrak{L}$ where $\mathfrak{L}$ is an arbitrary frame, that is, a complete lattice in which binary meet distributes over arbitrary joins. Various aspects of the corresponding $\mathbf{BooSh}\mathfrak{L}$ are studied in Banaschewski-Bhutani [2], albeit with the Axiom of Choice assumed for the underlying set theory. For general facts about frames, see Johnstone [8]. Categorical notions will be used here as in Mac Lane [9], and for the details concerning internal satisfaction in a topos we refer to Johnstone [8].

2. Injectivity.

As usual, an object A in any category is called injective if, for any $f : B \to A$ and any monomorphism $h : B \to C$, there exist $g : C \to A$ such that $f = gh$. Here, we are concerned with the relationship between injectives in $\mathbf{BooSh}\mathfrak{B}$ and $\mathbf{Boo}$.

First, a crucial result which shows that $\mathbf{BooSh}\mathfrak{B}$ is essentially determined by what happens at the level of global elements.

Lemma 1. *As a functor into* $\mathfrak{B}{\downarrow}\mathbf{Boo}$, Γ *is full and faithful.*

Proof: For any $f, g : A \to C$ in $\mathbf{BooSh}\mathfrak{B}$, if $\Gamma f = \Gamma g$, that is, $f_E = g_E$, then clearly $\tilde{f} = \tilde{g}$ and hence $f = g$, the latter by the naturality of the isomorphisms φ_A.

On the other hand, for $A, C \in \mathbf{BooSh}\mathfrak{B}$, any $h : \Gamma A \to \Gamma C$ such that $hi_{AE} = i_{CE}$ maps $\tilde{A}U$ into $\tilde{C}U$, because $x \leq i_{AE}(U)$ implies $h(x) \leq hi_{AE}(U) = i_{CE}(U)$, and hence determines a homomorphism $f : \tilde{A} \to \tilde{C}$. Then, for $g = \varphi_C f \varphi_A^{-1} : A \to C$

$$\Gamma g = \Gamma\varphi_C \Gamma f (\Gamma\varphi_A)^{-1} = \Gamma f = h,$$

showing that Γ is full. □

Remark. It would be easy to list conditions which characterize the precise image of $\mathbf{BooSh}\mathfrak{B}$ in $\mathfrak{B}\downarrow\mathbf{Boo}$ by the functor Γ. In particular, the separation property of sheaves that $a = b$ for any $a, b \in AU$ whenever U is covered by the $V \leq U$ for which $a\,|\,V = b\,|\,V$ corresponds to the condition that $h : \mathfrak{B} \to C$ preserve all joins. This does make the relevant presheaf $\tilde{C}$, defined by $\tilde{C}U = \downarrow h(U)$, separated while it is easy to see that $i_{AE} : \mathfrak{B} \to AE$ indeed preservers all joins, for any $A \in \mathbf{BooSh}\mathfrak{B}$.

Concerning injectivity, we now have

Proposition 1. $\Gamma : \mathbf{BooSh}\mathfrak{B} \to \mathbf{Boo}$ *preserves and reflects injectivity.*

Proof: Γ preserves injectivity by a familiar argument based on the fact that its left adjoint Δ preserves monomorphisms. To see that Γ reflects injectivity, let $A \in \mathbf{BooSh}\mathfrak{B}$ be such that ΓA is injective in $\mathbf{Boo}$ and consider any $f : C \to A$ and any monomorphism $h : C \to D$. Since Γ preserves monomorphisms, there exist $g : \Gamma D \to \Gamma A$ in $\mathbf{Boo}$ for which $\Gamma f = g\Gamma h$, by hypothesis. Further, since $fi_C = i_A$ and $hi_C = i_D$,

$$gi_{DE} = g(\Gamma h)i_{CE} = (\Gamma f)i_{CE} = i_{AE},$$

hence g actually belongs to $\mathfrak{B}\downarrow\mathbf{Boo}$, and by Lemma 1, $g = \Gamma k$ for some $k : D \to A$. Then $\Gamma(kh) = g\Gamma h = \Gamma f$, and again by Lemma 1 it follows that $kh = f$ as desired. □

Since $\Gamma\mathbf{2}_{\mathfrak{B}} = \mathfrak{B}$, an obvious special case of the proposition is

Corollary 1. *A complete Boolean algebra* $\mathfrak{B}$ *is injective in* $\mathbf{Boo}$ *iff* $\mathbf{2}_{\mathfrak{B}}$ *is injective in* $\mathbf{BooSh}\mathfrak{B}$.

There is a further consequence of Proposition 1, concerning Sikorski's Theorem by which the complete Boolean algebras are exactly the injectives in $\mathbf{Boo}$. To put this into perspective, recall that Banaschewski-Bhutani [2] proved the Sikorski Theorem for the topos $\mathbf{Sh}\mathfrak{L}$ of sheaves on an arbitrary locale $\mathfrak{L}$, assuming the Axiom of Choice in the underlying set theory. It was then pointed out in [2] that it was unknown whether this could already be proved if one only assumed the Sikorski Theorem. In the present context, however, we have

Corollary 2. *If the Sikorski Theorem holds in* $\mathbf{Ens}$ *then it also holds in* $\mathbf{Sh}\mathfrak{B}$*, for any complete Boolean algebra* $\mathfrak{B}$.

Proof: As noted earlier, if $A \in \mathbf{BooSh}\mathfrak{B}$ is complete then ΓA is complete, hence injective in $\mathbf{Boo}$ by hypothesis, and therefore A is injective in $\mathbf{BooSh}\mathfrak{B}$ by Proposition 1. □

Remark 1. A familiar fact concerning Boolean algebras is that the Boolean Ultrafilter Theorem (BUT), which says every non-trivial Boolean algebra contains an ultrafilter, is equivalent to the condition that the initial Boolean algebra $\mathbf{2}$ is injective in $\mathbf{Boo}$. Hence, the injectivity of $\mathbf{2}_{\mathfrak{B}}$ in $\mathbf{BooSh}\mathfrak{B}$ may be viewed as a form of BUT in $\mathbf{Sh}\mathfrak{B}$, and the above Corollary 1 can then be interpreted as saying that, for any complete Boolean algebra $\mathfrak{B}$, BUT holds in $\mathbf{Sh}\mathfrak{B}$ iff $\mathfrak{B}$ is injective. However, there is another sense in which one might consider the Boolean Ultrafilter Theorem in $\mathbf{Sh}\mathfrak{B}$, namely that of topos theoretical *internal satisfaction* (notation: $\mathbf{Sh}\mathfrak{B} \models$ BUT) which is equivalent to satisfaction in the $\mathfrak{B}$-valued model of set theory. Given the familiar equivalence between ultrafilters and homomorphisms into the initial Boolean algebra, this may be rendered as the condition that, for any $A \in \mathbf{BooSh}\mathfrak{B}$, the statement "If A is non-trivial, then A has a homomorphism into the initial Boolean algebra" has truth value $E \in \mathfrak{B}$. Bell [3] uses his result referred to earlier to show that $\mathbf{Sh}\mathfrak{B} \models$ BUT whenever $\mathfrak{B}$ is injective. In our setting, this appears as an easy consequence of Corollary 1, in view of the following observation: *For any complete Boolean algebra* $\mathfrak{B}$*,* $\mathbf{Sh}\mathfrak{B} \models$ BUT *whenever* $\mathbf{2}_{\mathfrak{B}}$ *is injective.* To see this, let $S \in \mathfrak{B}$ be the truth value of the statement "A is non-trivial" and $T \in \mathfrak{B}$ the truth value of the statement "There exists a homomorphism from A to the initial Boolean algebra". This means S is the largest $U \in \mathfrak{B}$ for which $\mathbf{2}_{\mathfrak{B}}|U \to A|U$ is monic, while T is the join of all $W \in \mathfrak{B}$ such that there exist homomorphisms $A|W \to \mathbf{2}_{\mathfrak{B}}|W$. Now, for $C \in \mathbf{BooSh}\mathfrak{B}$ defined by

$$C|S = A|S \quad \text{and} \quad C|S^* = \mathbf{2}_{\mathfrak{B}}|S^*,$$

$i_C : \mathbf{2}_{\mathfrak{B}} \to C$ is monic, hence by injectivity there exist $h : C \to \mathbf{2}_{\mathfrak{B}}$, and consequently $h|S : A|S \to \mathbf{2}_{\mathfrak{B}}|S$. Thus $S \leq T$ so that $S \to T = E$ for the truth value in question, as claimed.

In conclusion, we note that the precise meaning of $\mathbf{Sh}\mathfrak{B} \models$ BUT is that whenever $\mathbf{2}_{\mathfrak{B}} \to A$ is monic then the $U \in \mathfrak{B}$ for which there exist $A \mid U \to \mathbf{2}_{\mathfrak{B}} \mid U$ have join E. We do not know whether this "local" injectivity implies the formally stronger condition that $\mathbf{2}_{\mathfrak{B}}$ be injective.

Remark 2. As in the case of BUT just discussed, one can also consider the Sikorski Theorem in $\mathbf{Sh}\mathfrak{B}$ in the sense of internal satisfaction rather than in the external meaning of Corollary 2, and a similar argument then shows that the external Sikorski Theorem implies the internal one. We note this provides an alternative proof for the final result of Bell [3] that satisfaction of the Sikorski Theorem is preserved by Boolean-valued models.

Remark 3. For sheaves on a non-Boolean locale $\mathfrak{L}$, Proposition 1 becomes false: although Γ always preserves injectivity it need not reflect it. There are obvious counter-examples in the case that $\mathfrak{L}$ is the three-element chain.

3. Complete homomorphisms.

The correspondence $A \rightsquigarrow \mathfrak{J}A$, for $A \in \mathbf{BooSh}\mathfrak{B}$, is obviously functorial, the lattice homomorphism $\mathfrak{J}h : \mathfrak{J}A \to \mathfrak{J}C$ induced by a homomorphism $h : A \to C$ taking any ideal $J \subseteq A|U$ to the ideal of $C|U$ determined by the images $h_V[JV] \subseteq$

CV, $V \le U$. For complete $A, C \in \mathbf{BooSh}\mathfrak{B}$, a homomorphism $h : A \to C$ is then *complete* iff the following square

$$\begin{tikzcd} \mathfrak{J}A \arrow[r, "\mathfrak{J}h"] \arrow[d] & \mathfrak{J}C \arrow[d] \\ A \arrow[r, "h"'] & C \end{tikzcd}$$

commutes where the vertical maps are the respective join maps. It is clear that, in the special case of $\mathbf{Ens} = \mathbf{Sh2}$, this amounts to the usual notation of completeness, that is, preservation of arbitrary joins.

The following describes the relation between completeness, both, of Boolean algebras and their homomorphisms, in $\mathbf{Sh}\mathfrak{B}$ and in $\mathbf{Ens}$.

Proposition 2. $\Gamma : \mathbf{BooSh}\mathfrak{B} \to \mathbf{Boo}$ *preserves and reflects completeness.*

PROOF: (1) We have to show, for any $A \in \mathbf{BooSh}\mathfrak{B}$, that A is complete iff ΓA is complete. As already noted, ΓA is complete whenever A is, and hence we only need to prove the converse. For this, it is convenient to use the isomorphism $A \cong \tilde{A}$. In the following, we use "sup" for the join in AE. For any ideal $J \subseteq \tilde{A}|U$, put

$$\textstyle\bigvee_U J = \sup \bigcup\{JW \,|\, W \le U\}.$$

Then, for any $a \in \tilde{A}U$,

$$\begin{aligned} J \subseteq [a]_U \ &\text{iff} \ \ JW \subseteq [a]_U W, \ \ \text{for all} \ \ W \le U, \\ &\text{iff} \ \ JW \subseteq {\downarrow}(a \wedge s_W), \ \ \text{for all} \ \ W \le U, \\ &\text{iff} \ \ x \le a, \ \ \text{for all} \ \ x \in JW, \ W \le U, \\ &\text{iff} \ \ \textstyle\bigvee_U J \le a. \end{aligned}$$

Hence $\bigvee_U$ has the correct property at U. To see this defines a sheaf map we have to check for any $V \le U$ that the corresponding square

$$\begin{tikzcd} (\mathfrak{J}\tilde{A})U \arrow[r] \arrow[d] & \tilde{A}U \arrow[d] \\ (\mathfrak{J}\tilde{A})V \arrow[r] & \tilde{A}V \end{tikzcd}$$

commutes. Here, for any ideal $J \subseteq \tilde{A}|U$, its image in $\tilde{A}V$ through $\tilde{A}U$ is

$$(\textstyle\bigvee_U J) \wedge i_{AE}(V) = \sup\{x \wedge i_{AE}(V) \mid x \in JW \ \text{ for some } \ W \le U\}$$

while its image through $(\mathfrak{J}\tilde{A})V$ is

$$\textstyle\bigvee_V (J|V) = \sup\{y \in AE \mid y \in JS \ \text{ for some } \ S \le V\}.$$

Now, $x \in JW$ implies $x \wedge i_{AE}(V) \in J(W \wedge V)$ and hence the latter is one of the y. On the other hand, each y is trivially of this form, so the two sets involved are the same, and hence their joins are equal.

(2) Given any complete $A, C \in \mathbf{BooSh}\mathfrak{B}$, we have to show that a homomorphism $h : A \to C$ is complete iff $\Gamma h : \Gamma A \to \Gamma C$ is complete. For ($\Rightarrow$), we use that the join of any ideal $J \subseteq AE$ can be represented by the join map of A as follows: Let $\overline{J}$ be the ideal of A such that, for all $U \in \mathfrak{B}$ and $x \in AU$, $x \in \overline{J}U$ iff there exists a cover $U = \bigvee U_i$ and corresponding $a_i \in H$ for which $x\,|\,U_i \leq a_i\,|\,U_i$, for all i. Then, one easily checks that $\overline{J} \subseteq [c]$ iff $J \subseteq {\downarrow}c$, for all $c \in AE$, and this means $\bigvee_E \overline{J}$ is the join of J in AE. Further, if I is the ideal of CE generated by $h_E[J]$ then $(\mathfrak{J}h)_E(\overline{J}) = \overline{I}$ for the corresponding ideal $\overline{I}$ of C, and hence

$$h_E(\sup J) = h_E(\bigvee_E \overline{J}) = \bigvee_E (\mathfrak{J}h)_E(\overline{J}) = \bigvee_E \overline{I} = \sup I = \sup h_E[J],$$

which shows h_E is complete.

For ($\Leftarrow$), note first that h is complete iff the corresponding $\tilde{h} : \tilde{A} \to \tilde{C}$ is complete so that it suffices to prove the latter. Moreover, for this we can use the expression established in (1) for the join maps of $\tilde{A}$ and $\tilde{C}$. Thus, for any ideal $J \subseteq \tilde{A}\,|\,U$,

$$\begin{aligned}\tilde{h}(\bigvee_U J) &= h_E(\sup \bigcup \{JW\,|\,W \leq U\}) \\ &= \sup \bigcup \{h_E[JW]\,|\,W \leq U\},\end{aligned}$$

the latter by the completeness of h_E, whereas

$$\bigvee_U (\mathfrak{J}\tilde{h})_U(J) = \sup \bigcup \{\mathfrak{J}\tilde{h}_U(J)W\,|\,W \leq U\}.$$

Now for any $c \in CE$, $c \in (\mathfrak{J}\tilde{h})_U(J)W$ iff there exists a cover $W = \bigcup W_i$ and corresponding $a_i \in JW_i$ such that $c \wedge i_{CE}(W_i) \leq h_E(a_i)$, and this immediately shows the above two joins are equal, as desired. □

Recall that $\mathbf{2}_{\mathfrak{B}}$ is complete and that, by the remark after Lemma 1, any $i_{AE} : \mathfrak{B} \to AE$ preserves all joins. Thus, for complete A, i_{AE} is a complete homomorphism, and hence we have the

Corollary. *For any complete $A \in \mathbf{BooSh}\mathfrak{B}$, $i_A : \mathbf{2}_{\mathfrak{B}} \to A$ is complete.*

Remark. Proposition 2 also no longer holds for sheaves on non-Boolean locales; again, there are obvious counter-examples in the case of the three-element chain. On the other hand, the corollary actually holds under the most general conditions possible, namely, for any topos in which the initial Boolean algebra is complete.

4. Complete injectives.

Recall that, in any category, an object A is called an absolute retract if any monomorphism $A \to B$ has a left inverse. Of course, every injective object is an absolute retract; the converse, while not true in general, holds whenever the category has the property that any $f : A \to B$ and any monomorphism $h : A \to C$ are part of a commuting square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "\overline{f}"] \\ B \arrow[r, "\overline{h}"'] & D \end{tikzcd}$$

with monic $\overline{h}$. We note, as a companion to the corresponding well-known fact concerning **Boo**, that the category **CBoo** of complete Boolean algebras and complete homomorphisms is of this kind: Given $f : A \to B$ and monic $h : A \to C$ in **CBoo**, one has a diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "\ell"] & \\ B \arrow[r, "k"'] & L \arrow[r, "g"'] & M \end{tikzcd}$$

where the square is the pushout in the category **Frm** of frames and $g : L \to M$ any frame embedding into a complete Boolean algebra. Now, by Banaschewski [1], k is monic, and thus $\overline{f} = g\ell$ and $\overline{h} = gk$ provide the desired commuting square in **CBoo**.

The main aim of this section is to establish that **CBoo** has no non-trivial injectives. The following notion will be useful tool for this purpose.

A Boolean algebra A in $\mathbf{Sh}\mathfrak{B}$ will be called *atomless* if

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, a \in A)\ ((0 < a) \to (\exists\, b \in A)\ ((0 < b) \wedge (b < a)))$$

in the usual sense of satisfaction in the internal logic of $\mathbf{Sh}\mathfrak{B}$, where $b < a$ is understood as $(b \leq a) \wedge \neg(b = a)$. To explicate this condition, we note: $[0 < a]_A$ is the largest subsheaf of A disjoint from the singleton subsheaf $\{0\}$, which means that, for any $U \in B$,

$$[0 < a]_A\ U = \{x \in AU \,|\, 0 < x|V \ \text{ for all } V \leq U,\ V \neq 0\}.$$

We shall call the $x \in AU$ which satisfy the stated condition *persistently positive*. Similarly, $[b < a]_A$ is the subsheaf of $A \times A$ with the components

$$\{(x, y) \in AU \times AU \,|\, x|V < y|V \ \text{ for all }\ V \leq U,\ V \neq 0\}.$$

For pairs (x, y) which satisfy this condition we shall say that $x < y$ *persistently*. Hence $A \in \mathbf{BooSh}\mathfrak{B}$ is atomless iff, for any persistently positive $x \in AU$, U is covered by the $W \leq U$ such that there exist persistently positive $y \in AW$ for which $y < x|W$ persistently.

Note that, for the case $\mathbf{Ens} = \mathbf{Sh2}$, atomlessness just means what the naive connotation amounts to.

We need a number of lemmas concerning atomless Boolean algebras in $\mathbf{Sh}\mathfrak{B}$.

Lemma 3. *The completion of any atomless $A \in$ **BooSh**$\mathfrak{B}$ is again atomless.*

Proof: We use the fact that, for the completion $C \supseteq A$,

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, c \in C)\ ((c = 0) \leftrightarrow (\forall\, a \in A)\ ((a \leq c) \rightarrow (a = 0)))$$

which is easily derived from the description of the completion of A in terms of the Boolean algebra of normal ideals of A. Now, by Booleanness, this implies

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, c \in C)\ ((0 < c) \rightarrow (\exists\, a \in A)\ ((0 < a) \wedge (a \leq c)))$$

which in turn yields

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, c \in C)\ ((0 < c) \rightarrow (\exists\, a, b \in A)\ ((0 < b) \wedge (b < a) \wedge (a \leq c)))$$

by the hypothesis on A. Clearly, this shows C is atomless. □

Lemma 4. *For any atomless $A \in$ **Boo**, ΔA is atomless.*

Proof: We use the familiar description of $(\Delta A)U$, for any $U \in \mathfrak{B}$, as the set of all maps $s : A \rightarrow \downarrow U$ such that $s(a) \wedge s(b) = 0$ whenever $a \neq b$ and $U = \bigvee\{s(a) \,|\, a \in A\}$. The restriction maps $(\Delta A)U \rightarrow (\Delta A)V$ are then given by $(s \mid V)(a) = s(a) \wedge V$. Particular elements of $(\Delta A)U$ are the constants, defined for each $a \in A$ and $U \in \mathfrak{B}$ by

$$\underset{\sim}{a}_U(x) = \begin{cases} U & (x = a) \\ 0 & (x \neq a) \end{cases}$$

Then $\underset{\sim}{a}_U \mid V = \underset{\sim}{a}_V$ whenever $V \leq U$. Also, for any $s \in (\Delta A)U$ and $a \in A$, $s \,|\, s(a) = \underset{\sim}{a}_{s(a)}$, as easy calculation shows. Further, from the general description of binary meet in $(\Delta A)U$ by the formula

$$(s \wedge t)(x) = \bigvee\{s(y) \wedge t(z) \,|\, y \wedge z = x\},$$

one sees that $\underset{\sim}{a}_U \wedge \underset{\sim}{b}_U = ((a \wedge b)_\sim)_U$ for any $a, b \in A$ and $U \in \mathfrak{B}$. In particular, if $a \leq b$ ($a < b$) then $\underset{\sim}{a}_U \leq \underset{\sim}{b}_U$ ($\underset{\sim}{a}_U < \underset{\sim}{b}_U$ persistently).

Note that $s \in (\Delta A)U$ is persistently positive iff $s(0) = 0$. Given the latter and $S \,|\, V = 0$ for some $V \leq U$, we have

$$0 = s \,|\, s(a) \wedge V = \underset{\sim}{a}_{s(a)} \,|\, s(a) \wedge V$$

for any non-zero $a \in A$, hence $s(a) \wedge V = 0$, and since these $s(a)$ cover U it follows that $V = 0$. Conversely, if $s(0) > 0$ then $s \,|\, s(0) = \underset{\sim}{0}_{s(0)} = 0$ shows that s fails to be persistently positive.

Now, consider any $s \in (\Delta A)U$ such that $s(0) = 0$. Then $a > 0$ for any $a \in A$ for which $s(a) > 0$, and by hypothesis there exist $b \in A$ such that $0 < b < a$. Hence, for $W = s(a)$, $0 < \underset{\sim}{b}_W < \underset{\sim}{a}_W$ persistently while $\underset{\sim}{a}_W = s \,|\, W$. Since U is the join of these W, this shows ΔA is atomless. □

Remark. Actually, Lemma 4 is a special case of a very general principle: For any algebra A in **Ens**, of arbitrary (finitary) type, and any sentence φ in the corresponding first order language, if A satisfies φ then ΔA satisfies φ in $\mathbf{Sh}\mathfrak{B}$.

Lemma 5. *Any complete atomless* $C \in \mathbf{BooSh}\mathfrak{B}$ *has no complete homomorphism* $h : C \to \mathbf{2}_{\mathfrak{B}}$.

PROOF: Otherwise, let $K \subseteq C$ be the kernel of h, that is, the ideal with the components $KU = \{x \in CU \mid h_U(x) = 0\}$ and put $s = \bigvee_E K$ with complement $a \in CE$. Then, $h_E(s) = 0$ by completeness, so that $h_U(a \,|\, U) = U$ and therefore $a|U > 0$ for all $U \neq 0$. This shows $\mathbf{Sh}\mathfrak{B} \models (0 < a)$ and hence

$$\mathbf{Sh}\mathfrak{B} \models (\exists\, c \in C)\ ((0 < c) \wedge (c < a))$$

by hypothesis. On the other hand, by the definition of s and a,

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, c \in C)\ ((h(c) = 0) \to (c \leq s))$$

and

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, c \in C)\ (h(c) = 1) \to (a \leq c),$$

and usual deduction then leads to a contradiction in view of the basic fact that

$$\mathbf{Sh}\mathfrak{B} \models (\forall\, c \in C)\ ((h(c) = 0) \vee (h(c) = 1)).$$

□

These lemmas now lead to the following crucial result concerning complete Boolean algebras and complete homomorphisms in $\mathbf{Sh}\mathfrak{B}$:

Lemma 6. $\mathbf{2}_{\mathfrak{B}}$ *is not injective in* $\mathbf{CBooSh}\mathfrak{B}$.

PROOF: For any non-trivial $A \in \mathbf{Boo}$, $i_{\Delta A} : \mathbf{2}_{\mathfrak{B}} \to \Delta A$ is obviously monic. Hence, if A is also atomless then ΔA is atomless by Lemma 4, its completion C is atomless by Lemma 3, and there are no complete homomorphisms $C \to \mathbf{2}_{\mathfrak{B}}$ by Lemma 5, while $i_C : \mathbf{2}_{\mathfrak{B}} \to C$ is monic. □

Remark. Perhaps it should be pointed out that there is indeed a large supply of atomless Boolean algebras in **ZF**, for instance: the Boolean algebras free on some infinite set, the Boolean algebras of open-closed subsets of Boolean spaces without isolated points, such as the Cantor space, and the Boolean algebras of regular open subsets of regular Hausdorff spaces without isolated points, such as the real line.

Proposition 3. **CBoo** *has no non-trivial injectives.*

PROOF: For any non-trivial $\mathfrak{B} \in \mathbf{CBoo}$, if $C \in \mathbf{CBooSh}\mathfrak{B}$ is, as in the preceding proof, such that $i_C : \mathbf{2}_{\mathfrak{B}} \to C$ is monic but there is no $h : C \to \mathbf{2}_{\mathfrak{B}}$, then $i_{CE} : \mathfrak{B} \to CE$ is monic and complete but without complete left inverse, by Lemma 1 and Proposition 2. □

Corollary. **CBooSh$\mathfrak{B}$** *has no non-trivial objectives.*

Proof: We show that, for any injective $A \in$ **CBooSh$\mathfrak{B}$**, AE is injective in **CBoo** which then makes it trivial by the proposition and this, in turn, makes A trivial.

Actually, we only prove that AE is an absolute retract in **CBoo**, but by the opening remark of this section, this will be sufficient. Consider, then, any monic $h : AE \to C$ in **CBoo** and define $\overline{C}$ by $\overline{C}U = \downarrow hi_{AE}(U)$. One readily checks this is a Boolean algebra in **Sh$\mathfrak{B}$**, complete by Proposition 2 since $\overline{C}E = C$ is complete, with a homomorphism $k : \tilde{A} \to \overline{C}$ determined by h, also complete by Proposition 2 since h is complete. Now, k is monic and $\tilde{A} \cong A$ injective, hence k has a left inverse, and this provides the desired left inverse for $k_E = h$. $\square$

Proposition 4. **Frm** *has no non-trivial absolute retracts.*

Proof: Any such frame L must be Boolean since every frame has an embedding into a Boolean frame. Thus, $L \in$ **CBoo** which makes it an absolute retract, and hence injective, in the subcategory **CBoo** of **Frm**. By Proposition 3, it then follows that L is trivial, as claimed. $\square$

Since the usual embedding of a frame into a Boolean frame is a constructive procedure, it also applies to the category **FrmSh$\mathfrak{B}$** of frames in **Sh$\mathfrak{B}$**. Hence, the argument just used also applies to this category, and by the corollary of Proposition 3 we then obtain:

Corollary. **FrmSh$\mathfrak{B}$** *has no non-trivial absolute retracts.*

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