On the topological structure of compact 5-manifolds

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Abstract. We classify the genus one compact (PL) 5-manifolds and prove some results about closed 5-manifolds with free fundamental group. In particular, let M be a closed connected orientable smooth 5-manifold with free fundamental group. Then we prove that the number of distinct smooth 5-manifolds homotopy equivalent to M equals the 2-nd Betti number (mod 2) of M.

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1. Preliminaries.

In this paper we work in the piecewise linear (PL) category (see for example [9]). All considered manifolds will be compact and connected. We also use edge-colored graphs to represent manifolds according to [2], [4] and [5]. Here we recall the basic definitions. An <u>edge-coloration</u> c on a multigraph $G = \big(V(G), E(G)\big)$ is a map $c : E(G) \longrightarrow \mathcal{C}_G$ (where $\mathcal{C}_G$ is a finite set, called the <u>color set</u> of G) such that $c(e) \neq c(f)$ for any two adjacent edges e, $f \in E(G)$. The pair (G, c) is said to be an $(n+1)$-<u>colored graph</u> if G is regular of degree $n+1$ and $\mathcal{C}_G = \{0, 1, \dots, n\}$. For any $B = \{b_1, b_2, \dots, b_k\} \subset \mathcal{C}_G$, we set $G_B = \big(V(G), c^{-1}(B)\big)$ and denote by $\alpha_{b_1 b_2 \dots b_k}$ the number of components of G_B. An n-<u>pseudocomplex</u> $K = K(G)$ can be associated with (G, c) as follows: 1) take an n-simplex $\sigma^n(v)$ for each vertex $v \in V(G)$ and label its vertices by $\mathcal{C}_G$; 2) if v and w are joined in G by an i-colored edge, then identify the $(n-1)$-faces of $\sigma^n(v)$ and $\sigma^n(w)$ opposite to the vertex labelled by i so that equally labelled vertices coincide. We say that (G, c) <u>represents</u> the polyhedron $|K(G)|$ and every homeomorphic space. We note that each component θ of the subgraph G_B uniquely corresponds to an $(n-k)$-simplex σ_θ (card $B = k$) of $K(G)$, whose vertices are labelled by $\mathcal{C}_G \backslash B$. The polyhedron $|K(\theta)|$ is said to be the <u>disjoined link</u> of σ_θ in K, written $lkd(\sigma_\theta, K)$. Actually $|K|$ is a closed n-manifold if and only if $|K(G_{\hat{i}})|$ is an $(n-1)$-sphere, $\hat{i} = \mathcal{C}_G \backslash \{i\}$, $i \in \mathcal{C}_G$. A <u>crystallization</u> of a closed n-manifold M is an $(n+1)$-colored graph (G, c) representing M such that $G_{\hat{i}}$ is connected for each $i \in \mathcal{C}_G$. Any bipartite (resp. non-bipartite) $(n+1)$-colored graph (G, c) admits a particular 2-cell imbedding (see [15])

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$f_\epsilon : |G| \longrightarrow F_\epsilon$, where F_ϵ denotes the orientable closed (resp. non-orientable) surface of Euler-characteristic

$$\chi(F_\epsilon) = \sum_{i \in \mathbb{Z}_{n+1}} \alpha_{\epsilon_i \epsilon_{i+1}} + (1-n)p/2\,.$$

Here p is the order of G and $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n)$ is a cyclic permutation of the color set $\mathcal{C}_G$. We set $g_\epsilon(G) = 1 - \chi(F_\epsilon)/2$, i.e. $g_\epsilon(G)$ is the genus (resp. half of the genus) of F_ϵ if G is bipartite (resp. non-bipartite). Then the genus $g(M)$ of a closed n-manifold M is the minimum $g_\epsilon(G)$ over all crystallizations G of M and cyclic permutations ϵ of $\mathcal{C}_G$. It is known that the n-sphere $\mathbb{S}^n$ is the only closed n-manifold of genus zero (see for example [5]). In [4] all closed 4-manifolds of genus one are proved to be (PL) homeomorphic to $\mathbb{S}^1 \otimes \mathbb{S}^3$. Here $\mathbb{S}^1 \otimes \mathbb{S}^3$ denotes either the topological product $\mathbb{S}^1 \times \mathbb{S}^3$ or the twisted $\mathbb{S}^3$-bundle over $\mathbb{S}^1$. In the present paper we classify all compact 5-manifolds of genus one. Then we obtain some results about closed orientable 5-manifolds with free fundamental group. We also conjecture that the genus characterizes the simply-connected closed 5-manifolds.

2. Main results.

From now on, let (G, c) be a crystallization of a closed 5-manifold M, $K = K(G)$ the triangulation of M represented by G, $\{v_i \,/\, i \in \mathcal{C}_G\}$ the vertex-set of K and (i, j, h, r, s, t) an arbitrary permutation of the color-set $\mathcal{C}_G$. We may always assume that v_i corresponds to the subgraph $G_{\hat{\imath}}$ for each color $i \in \mathcal{C}_G$. If $B \subset \mathcal{C}_G$, then $K(B)$ denotes the subcomplex of $K = K(G)$ generated by the vertices v_i's, $i \in B$. Obviously the number of $(k-1)$-simplexes of $K(B)$, $\mathrm{card}B = k$, equals the number $\alpha_{\mathcal{C}_G \setminus B}$ of components of $G_{\mathcal{C}_G \setminus B}$. If SdK is the first barycentric subdivision of K, then $H(i,j)$ (resp. $H(i,j,h)$) is the largest subcomplex of SdK, disjoint from $SdK(i,j) \cup SdK(h,r,s,t)$ (resp. $SdK(i,j,h) \cup SdK(r,s,t)$). Then the polyhedron $|H(i,j)|$ (resp. $|H(i,j,h)|$) is a closed 4-manifold $F = F(i,j)$ (resp. $F(i,j,h)$) which splits M into two complementary 5-manifolds $V = N(i,j)$, $V' = N(h,r,s,t)$ (resp. $N = N(i,j,h)$, $N' = N(r,s,t)$) having F as common boundary. Further the Mayer-Vietoris exact sequences of the triples (M, V, V') and (M, N, N') give $0 \longrightarrow H_5(M) \longrightarrow H_4(F) \longrightarrow 0$, hence M is orientable if and only if F is. Finally V and V' (resp. N and N') are regular neighbourhoods of $|SdK(i,j)|$ and $|SdK(h,r,s,t)|$ (resp. $|SdK(i,j,h)|$ and $|SdK(r,s,t)|$) in M respectively.

Lemma 1. *Let (G, c) be a crystallization of a closed 5-manifold M. Then we have the following relations*

$$2\alpha_{rst} = \alpha_{rs} + \alpha_{st} + \alpha_{tr} - p/2 \tag{1}$$

$$\sum_{i,j,h} \alpha_{ijh} = 2 \sum_{i,j} \alpha_{ij} - 5p \tag{2}$$

$$\sum_{i,j,h,r} \alpha_{ijhr} = \sum_{i,j} \alpha_{ij} - 3p + 6 \tag{3}$$

Proof: (1). Let T be a triangle of the 2-dimensional subcomplex $K(i,j,h)$. Then the Euler-Poincaré characteristic χ_T of $lkd(T,K)$ is given by

$$\chi_T = \chi(\mathbb{S}^2) = 2 = q_3(T) - q_4(T) + q_5(T),$$

where $q_k(T)$ is the number of k-simplexes of K containing T as their face. If $B \subset \mathcal{C}_G$, let $q_k(B)$ denotes the number of k-simplexes of K containing vertices labelled by B. Then it is easy to check that

$$q_3(i,j,h) = q_3(i,j,h,r) + q_3(i,j,h,s) + q_3(i,j,h,t) = \alpha_{st} + \alpha_{rt} + \alpha_{rs},$$
$$q_4(i,j,h) = q_4(i,j,h,r,s) + q_4(i,j,h,r,t) + q_4(i,j,h,t,s) = \alpha_t + \alpha_s + \alpha_r = \frac{3}{2}p$$

and

$$q_5(i,j,h) = p.$$

Summation over all the triangles of $K(i,j,h)$ gives

$$\begin{aligned} 2\alpha_{rst} = 2q_2(i,j,h) &= q_3(i,j,h) - q_4(i,j,h) + q_5(i,j,h) = \\ &= \alpha_{st} + \alpha_{rt} + \alpha_{rs} - (3/2)\,p + p = \alpha_{st} + \alpha_{rt} + \alpha_{rs} - p/2 \end{aligned}$$

as requested.

(2). It is a direct consequence of (1).

(3). Now call q_k, $k \in \mathcal{C}_G$, the number of k-simplexes of K. By construction we have

$$q_0 = 6, \qquad q_1 = \sum_{i,j,h,r} \alpha_{ijhr}, \qquad q_2 = \sum_{i,j,h} \alpha_{ijh}$$
$$q_3 = \sum_{i,j} \alpha_{ij}, \qquad q_4 = 3p \text{ and} \qquad q_5 = p.$$

Then the Euler-Poincarè characteristic $\chi(M)$ of $M = |K|$ is given by

$$\begin{aligned} \chi(M) = \sum_k (-1)^k q_k &= 6 - \sum_{i,j,h,r} \alpha_{ijhr} + \sum_{i,j,h} \alpha_{ijh} - \sum_{i,j} \alpha_{ij} + 2p \\ &= 6 - \sum_{i,j,h,r} \alpha_{ijhr} + \sum_{i,j} \alpha_{ij} - 3p = 0 \qquad \text{(use (2))}. \end{aligned}$$

The proof is completed. □

Now we assume that (G,c) regularly imbeds into the closed surface of genus $g = g(M)$ and of Euler-Poincarè characteristic

$$\alpha_{01} + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{45} + \alpha_{50} - 2p = 2 - 2g. \tag{4}$$

Each subgraph $G_{\hat{i}}$, $i \in \mathcal{C}_G$, regularly imbeds into an orientable closed surface since $G_{\hat{i}}$ represents the combinatorial 4-sphere $lkd(v_i, K)$. Then we can define the non negative integer $g_{\hat{i}}$, $i \in \mathcal{C}_G$, as follows:

$$\alpha_{i+1\ i+2} + \alpha_{i+2\ i+3} + \alpha_{i+3\ i+4} + \alpha_{i+4\ i+5} + \alpha_{i+5\ i+1} = 2 - 2g_{\hat{i}} + \frac{3}{2}p \qquad i \in \mathcal{C}_G, \quad \text{indices mod } 6. \tag{5}$$

By substituting (5) into (4) and by using (1) we get

$$\alpha_{jh} = \alpha_{ijh} + g - g_{\hat{i}} \qquad i \in \mathcal{C}_G, \quad j \equiv i+1 \pmod 6, \quad h \equiv i-1 \pmod 6. \tag{6}$$

As a direct consequence, we have also proved that $g \geq g_{\hat{i}}$ for each color $i \in \mathcal{C}_G$.

Lemma 2. *With the above notation, we have*

$$\alpha_{135} = 1 + 2g - g_{\hat{0}} - g_{\hat{2}} - g_{\hat{4}} \tag{7}$$

$$\alpha_{024} = 1 + 2g - g_{\hat{1}} - g_{\hat{3}} - g_{\hat{5}} \tag{8}$$

$$\alpha_{02} + \alpha_{13} + \alpha_{15} + \alpha_{24} + \alpha_{35} + \alpha_{04} = 4 + 8g + p - 2\sum_i g_{\hat{i}} \tag{9}$$

Proof: We get the formula (7) (resp. (8)) of the statement by simply adding the relations obtained from (6) for $i = 0, 2, 4$ (resp. $i = 1, 3, 5$) and by using (1) and (4). Adding (7) and (8) and making use of (1) we obtain the formula (9). □

Theorem 3. *Let M be a closed connected 5-manifold. Then $g(M) = 1$ if and only if M is (PL) homeomorphic to $\mathbb{S}^1 \otimes \mathbb{S}^4$.*

Proof: If M is (PL) homeomorphic to $\mathbb{S}^1 \otimes \mathbb{S}^4$, then $g(M) = 1$ (see for example [5]). Now we prove the converse implication. For convenience, we work in the orientable case. If $g = 1$, then (7) and (8) of Lemma 2 imply that α_{135} and α_{024} belong to the set $\{1, 2, 3\}$. We apply the inequalities $g(M) \geq rk\,\Pi_1(M) \geq rk\,H_1(M)$ (see [2]). Here FH_* (resp. TH_*) denotes the free (resp. torsional) part of the homology group H_*. By symmetry we have to consider the following three cases:

(1) $\alpha_{135} = 1$
(2) $\alpha_{135} = 2$
(3) $\alpha_{135} = \alpha_{024} = 3$.

Case (1). Since $\alpha_{135} = 1$, the complex $K(0, 2, 4)$ consists of exactly one triangle. However $K(0, 2, 4)$ might have other edges besides the ones of the named triangle. Thus the regular neighborhood $N = N(0, 2, 4)$ of $K(0, 2, 4)$ is (PL) homeomorphic to a boundary connected sum $\#_k \mathbb{S}^1 \times B^4$, B^4 being a closed 4-ball (if $k = 0$, then we set $N = B^5$). Thus we have $\partial N \simeq_{PL} \partial N' \simeq_{PL} \#_k \mathbb{S}^1 \times \mathbb{S}^3$, where $N' = N(1, 3, 5)$.

Since N' collapses onto the 2-dimensional complex $K(1,3,5)$, the Mayer-Vietoris sequence of the triple (M, N, N') implies that

$$0 \longrightarrow H_4(M) \longrightarrow H_3(\partial N) \simeq \oplus_k \mathbb{Z} \longrightarrow 0 \tag{10}$$

$$0 \longrightarrow H_3(M) \longrightarrow H_2(\partial N) \simeq 0 \tag{11}$$

$$\begin{aligned} 0 \longrightarrow H_2(N') \longrightarrow H_2(M) \longrightarrow H_1(\partial N) \simeq \oplus_k \mathbb{Z} \to \\ \longrightarrow H_1(N) \oplus H_1(N') \simeq \oplus_k \mathbb{Z} \oplus H_1(N') \longrightarrow H_1(M) \longrightarrow 0. \end{aligned} \tag{12}$$

By (11) we have $0 \simeq H_3(M) \simeq H^2(M) \simeq FH_2(M) \oplus TH_1(M)$, i.e. $FH_2(M) \simeq TH_1(M) \simeq 0$. Since $H_2(N')$ is free, (12) implies that $0 \longrightarrow H_2(N') \longrightarrow FH_2(M) \simeq 0$, hence $H_2(N') \simeq 0$ and $H_2(M)$ is free, i.e. $H_2(M) \simeq 0$. Thus (12) splits as $H_1(M)$ is free. This gives $H_1(M) \simeq H_1(N') \simeq \oplus_k \mathbb{Z}$. Because $g = 1 \geq rk\, H_1(M)$, it follows that either $k = 0$ or $k = 1$, hence either $\partial N \simeq \mathbb{S}^4$ or $\partial N \simeq \mathbb{S}^1 \times \mathbb{S}^3$ respectively. In the first case we have $H_1(M) \simeq \Pi_1(M) \simeq 0$ and $H_2(M) \simeq 0$, so M is (PL) homeomorphic to $\mathbb{S}^5$ by the classification theorem of simply-connected spin 5-manifolds (see [1] and [13]). This is a contradiction since the genus of $\mathbb{S}^5$ is zero. In the second case we have $H_1(M) \simeq \Pi_1(M) \simeq H_4(M) \simeq \mathbb{Z}$ and $H_2(M) \simeq H_3(M) \simeq 0$. Further M is obtained by attaching two disjoint copies of $\mathbb{S}^1 \times B^4$ along their boundaries (use $H_2(N') \simeq 0$ and $H_1(N') \simeq H_1(M) \simeq \mathbb{Z}$). Then M is homotopy equivalent to $\mathbb{S}^1 \times \mathbb{S}^4$, hence $M \simeq_{PL} \mathbb{S}^1 \times \mathbb{S}^4$ by the Shaneson theorem (see [10]).

Case (2). If $\alpha_{135} = 2$, then (7) implies that $g_{\hat{0}} + g_{\hat{2}} + g_{\hat{4}} = 1$, hence for example $g_{\hat{0}} = 1$. Now the relation (6), for $i = 0$, gives $\alpha_{15} = \alpha_{015}$. Thus $K(0,2,3,4)$ consists of as many 3-simplexes as there are triangles in $K(2,3,4)$. Therefore $K(0,2,3,4)$ collapses onto the 2-dimensional complex $K(2,3,4)$, i.e. the polyhedron $V' = N(0,2,3,4)$ collapses onto a 2-polyhedron. We also have $V = N(1,5) \simeq \#_k(\mathbb{S}^1 \times B^4)$ and $\partial V \simeq \partial V' \simeq \#_k(\mathbb{S}^1 \times \mathbb{S}^3)$ since $K(1,5)$ consists of two vertices joined by $k+1$ edges for some non-negative integer k. Now we can repeat the arguments of Case (1) by replacing the pair (N, N') with (V, V').

Case (3). If $\alpha_{135} = \alpha_{024} = 3$, then $g_{\hat{i}} = 0$ for each color $i \in \mathcal{C}_G$ by (7) and (8). Then the relation (6) gives $\alpha_{15} = \alpha_{015} + 1$, i.e. $K(0,2,3,4)$ has one more 3-simplex than there are triangles in $K(2,3,4)$. Call σ_1, σ_2 the two 3-simplexes of $K(0,2,3,4)$ which have a common triangle $T \in K(2,3,4)$ as their face. If $\partial\sigma_1 \neq \partial\sigma_2$, then $K(0,2,3,4)$ collapses to a 2-dimensional subcomplex, hence the pair (V, V'), $V = N(1,5)$, $V' = N(0,2,3,4)$, satisfies the conditions of Case (2). If $\partial\sigma_1 = \partial\sigma_2$, then $H_3(V') \simeq \mathbb{Z}$. We prove that this case gives a contradiction. First of all we observe that

$$\partial V' \simeq \partial V \simeq \partial N(1,5) \simeq \#_k \mathbb{S}^1 \times \mathbb{S}^3$$

for some integer $k \geq 0$. Indeed, the Mayer-Vietoris sequence of the triple (M, V, V') yields

$$0 \longrightarrow H_5(M) \longrightarrow H_4(\partial V) \longrightarrow 0,$$

hence M is orientable if and only if ∂V is. Furthermore $K(1,5)$ is the one-dimensional subcomplex of $K = K(G)$ which consists of all edges with vertices v_1 and v_5. Thus the regular neighborhood $V = N(1,5)$ of $K(1,5)$ is PL homeomorphic to a boundary connected sum $\#_k \mathbb{S}^1 \times B^4$, hence $\partial V \simeq \#_k \mathbb{S}^1 \times \mathbb{S}^3$ as claimed.

Now, the exact sequence of the pair $(V', \partial V')$ gives

$$(13) \quad 0 = H_2(\partial V) \longrightarrow H_2(V') \longrightarrow H_2(V', \partial V') \to \\ \to H_1(\partial V') \simeq \oplus_k \mathbb{Z} \longrightarrow H_1(V') \longrightarrow H_1(V', \partial V') \simeq 0$$

and

$$(14) \quad 0 = H_4(V') \longrightarrow H_4(V', \partial V') \longrightarrow H_3(\partial V') \simeq \oplus_k \mathbb{Z} \to \\ \to H_3(V', \partial V') \longrightarrow H_2(\partial V') \simeq 0$$

since $H_1(V', \partial V') \simeq H^4(V') \simeq 0$. The isomorphism $H^4(V') \simeq 0$ follows from the fact that V' collapses onto the 3-dimensional complex $K(0,2,3,4)$. By Lefschetz duality we also have $H_2(V', \partial V') \simeq H^3(V') \simeq FH_3(V') \oplus TH_2(V') \simeq \mathbb{Z} \oplus TH_2(V')$, $H_4(V', \partial V') \simeq H^1(V') \simeq FH_1(V')$ and $H_3(V', \partial V') \simeq H^2(V') \simeq FH_2(V') \oplus TH_1(V')$. Thus (13) and (14) become

$$(13') \qquad 0 \longrightarrow H_2(V') \longrightarrow \mathbb{Z} \oplus TH_2(V') \longrightarrow \oplus_k \mathbb{Z} \longrightarrow H_1(V') \longrightarrow 0$$

and

$$(14') \qquad 0 \longrightarrow FH_1(V') \longrightarrow \oplus_k \mathbb{Z} \longrightarrow FH_2(V') \oplus TH_1(V') \longrightarrow 0$$

hence we obtain

$$(15) \qquad \beta_2(V') - 1 + k - \beta_1(V') = 0$$

and

$$(16) \qquad \beta_1(V') - k + \beta_2(V') = 0\,,$$

where $\beta_k(V')$ denotes the k-th Betti number of V'. From (15) and (16) we have that

$$2\beta_2(V') = 1\,,$$

which is a contradiction. □

Corollary 4. $g(\#_k \mathbb{S}^1 \otimes \mathbb{S}^4) = k$.

Proof: Use $g(M) \geq rk\, \Pi_1(M)$, Theorem 3 and the subadditivity of the genus. □

The concept of genus can be extended to boundary case in a natural way (see for example [5]). By slightly modifying the proof of Theorem 3 we obtain the following result

Theorem 5. *Let M be a compact 5-manifold with (possibly empty) connected boundary ∂M. Then $g(M) = 1$ if and only if M is (PL) homeomorphic to either $\mathbb{S}^1 \otimes \mathbb{S}^4$ or $\mathbb{S}^1 \otimes \mathbb{S}^4 \backslash$(open 5-ball) or $\mathbb{S}^1 \otimes B^4$. Here $\mathbb{S}^1 \otimes B^4$ denotes either $\mathbb{S}^1 \times B^4$ or the twisted B^4-bundle over $\mathbb{S}^1$.*

3. Free fundamental groups.

In this section we consider closed orientable 5-manifolds M with free fundamental group $\Pi_1(M) \simeq *_g\mathbb{Z}$, $g \geq 1$. If $g = 1$, then J.L. Shaneson proved that the number of closed smooth 5-manifolds of the same homotopy type as M is finite and at most equals the number of elements of $H_2(M;\mathbb{Z}_2)$. Here we extend this result for $g > 1$ by using (PL) surgery theory in dimension five (see [6] and [14]). For convenience, we recall some definitions listed in the quoted papers. Firstly we note that it follows from $Wh(\mathbb{Z}) \simeq 0$ and $Wh(\Pi * \Pi') = Wh(\Pi) \oplus Wh(\Pi')$ (see [8]) that "s-cobordant" is equivalent to "h-cobordant" in our case. Let M^n be a closed orientable (PL) n-manifold with fundamental group $\Pi_1 = \Pi_1(M)$ and let ξ^k be a linear bundle over M. Then $\Omega_n^+(M,\xi)$ denotes the set of bordism classes of normal maps (X, f, b) where X is a (PL) n-manifold, $f : X \longrightarrow M$ a map of degree one, $b : \nu_X^k \longrightarrow \xi^k$ a linear bundle map covering f and ν_X^k is the stable normal bundle of $X^n \longrightarrow \mathbb{S}^{n+k}$, $k \gg n$. Let $\mathcal{N}_n(M)$ be the union of all $\Omega_n^+(M,\xi)$ over all k-plane bundle ξ^k over M modulo the additional equivalence relation that $(X_0, f_0, b_0) \in \Omega_n^+(M, \xi_1)$ is equivalent to $(X_1, f_1, b_1) \in \Omega_n^+(M, \xi_2)$ if and only if (X_0, f_0, b_0) is normally cobordant to (X_1, f_1, b_1) for some linear bundle automorphism $\xi_1 \longrightarrow \xi_0$ (see [6, p. 74]). The elements of $\mathcal{N}_n(M)$ are called the <u>normal invariants</u> of M. Let $\mathcal{S}_n(M)$ denote the set of equivalence classes of pairs (X, h), where X is a compact (PL) n-manifold, $h : X \longrightarrow M$ is an orientation preserving simple homotopy equivalence and $(X, h) \sim (X', h')$ if and only if there is an orientation preserving (PL) homeomorphism $\gamma : X \longrightarrow X'$ such that $h' \circ \gamma$ is homotopic to h. Finally, denote by $L_n(\Pi_1)$ the n-th Wall group in the orientable case, n=dim M and $\Pi_1 = \Pi_1(M)$ (see [6, p. 77] and [14]). Recall that if $h : X \longrightarrow M$ represents an element of $\mathcal{S}_n(M)$ there exists an obvious forgetful map

$$\eta_n : \mathcal{S}_n(M) \longrightarrow \mathcal{N}_n(M)$$

which associates to (X, h) the class of (X, h, h^*) in $\mathcal{N}_n(M)$, h^* being the obvious map on stable normal bundles induced by h. Further, there is a map

$$\sigma_n : \mathcal{N}_n(M) \longrightarrow L_n(\Pi_1)$$

which associates to any normal invariant (X, f, b) the surgery obstruction $\sigma_n(X, f, b)$ (see [6, p. 77]). Finally we denote by

$$\omega_n : L_{n+1}(\Pi_1) \longrightarrow \mathcal{S}_n(M)$$

the map induced by the action of $L_{n+1}(\Pi_1)$, $n+1$ =dim $(M \times I)$, $I = [0,1]$, $\Pi_1 = \Pi_1(M \times I) \simeq \Pi_1(M)$, on $\mathcal{S}_n(M)$ (see [6, p. 80]). By [6, Theorem 5.11] and

[14, Theorem 10.8], there is an exact sequence

$$\mathcal{S}_{n+1}\big(M\times I,\partial(M\times I)\big)\xrightarrow{\eta_{n+1}}\mathcal{N}_{n+1}\big(M\times I,\partial(M\times I)\big)\xrightarrow{\sigma_n}$$
$$\to L_{n+1}(\Pi_1)\xrightarrow{\omega_n}\mathcal{S}_n(M)\xrightarrow{\eta_n}\mathcal{N}_n(M).$$

We prove the following

Theorem 6. *Let M^5 be a closed connected orientable smooth (or PL) 5-manifold with fundamental group $\Pi_1(M)=*_g\mathbb{Z}$. Then the map*

$$\eta_5:\mathcal{S}_5(M)\longrightarrow\mathcal{N}_5(M)$$

is injective and $\operatorname{Im}\eta_5\simeq H_2(M;\mathbb{Z}_2)$, i.e. the number of distinct smooth 5-manifolds homotopy equivalent to M equals the 2-nd Betti number (mod 2) of M.

Proof: We prove that

(1) σ_5 and σ_6 are epimorphisms.
(2) $\mathcal{N}_5(M)\simeq H_2(M;\mathbb{Z}_2)\oplus H_1(M)$
(3) σ_5 is injective on the summand $H_1(M)$.

(1) Since $L_6(\Pi_1)=L_6(*_g\mathbb{Z})\simeq\mathbb{Z}_2$ (see [3, Theorem 1.6, p. 28]), the map

$$L_6(1)\simeq\mathbb{Z}_2\longrightarrow L_6(*_g\mathbb{Z})\simeq\mathbb{Z}_2$$

is an isomorphism, hence one can represent the non-trivial element of L_6 by a degree one normal map $(\mathbb{S}^3\times\mathbb{S}^3,f,b)$ with $f:\mathbb{S}^3\times\mathbb{S}^3\longrightarrow\mathbb{S}^6$ (see [11], [12]). Then the action of L_6 on $\mathcal{S}_6(M\times I,M\times\partial I)$ is defined by taking an element $k:(K,\partial K)\longrightarrow(M\times I,M\times\partial I)$ in $\mathcal{S}_6(M\times I,M\times\partial I)$ and forming the connected sum in the interior $k\#f:K\#\mathbb{S}^3\times\mathbb{S}^3\longrightarrow M\times I=M\times I\#\mathbb{S}^6$. Using the additivity of surgery obstructions and the fact $\sigma_6(k)=0$, we have that $\sigma_6(k\#f)=\sigma_6(f)$ is the generator of $L_6(\Pi_1)$ and

$$\big(K\#\mathbb{S}^3\times\mathbb{S}^3,k\#f,(k\#f)^*\big)\in\Omega_6^+\big(M\times I,M\times\partial I,\xi\big)\subset\mathcal{N}_6\big(M\times I,M\times\partial I\big),$$

i.e. σ_6 is surjective. This implies that the sequence

$$0\to\mathcal{S}_5(M)\xrightarrow{\eta_5}\mathcal{N}_5(M)\xrightarrow{\sigma_5}L_5(\Pi_1)$$

is exact, i.e. η_5 is injective. Now we prove that σ_5 is surjective. Since M is orientable, any imbedded 1-sphere $\tilde f:\mathbb{S}^1\longrightarrow M$ has trivial normal bundle, i.e. $\tilde f$ extends to an imbedding $f:\mathbb{S}^1\times B^4\longrightarrow M$. Let $f_1,f_2,\dots,f_g:\mathbb{S}^1\times B^4\longrightarrow M$ be disjoint imbeddings such that $\tilde f_i=f_i|_{\mathbb{S}^1\times 0}$ represent a set of generators of $\Pi_1(M)$ (by general position this is always possible).

Let N_i, $i=1,2,\dots,g$, be the 5-manifold obtained by deleting $f_i(\mathbb{S}^1\times\overset{\circ}{B}{}^4)$ from M and substituting $(\mathbb{S}^1\times\|E_8\|)\backslash(\mathbb{S}^1\times\overset{\circ}{B}{}^4)$ by an obvious identification of their boundaries. Here $\|E_8\|$ represents the simply-connected Poincaré 4-complex realizing the form E_8 as constructed in [6, pp. 22–23]. Note that $\mathbb{S}^1 \times \|E_8\|$ is a 5-manifold. Using an appropriate normal map

$$\mathbb{S}^1 \times \|E_8\| \longrightarrow \mathbb{S}^1 \times \mathbb{S}^4 ,$$

we obtain a normal map of degree one

$$\xi_i : N_i \longrightarrow M = M \backslash f_i(\mathbb{S}^1 \times \overset{\circ}{B}{}^4) \bigcup_{\mathbb{S}^1 \times \mathbb{S}^3} (\mathbb{S}^1 \times \mathbb{S}^4 \backslash \mathbb{S}^1 \times \overset{\circ}{B}{}^4)$$

hence $(N_i, \xi_i, \xi_i^*) \in \Omega_5^+(M, \xi) \subset \mathcal{N}_5(M)$. Furthermore, the surgery obstruction $\sigma_5(N_i, \xi_i, \xi_i^*)$ is exactly the i-th generator of $L_5(\Pi_1) = L_5(*_g\mathbb{Z}) \cong \oplus_g\mathbb{Z}$ (use [3, Theorem 1.6, p. 28]), i.e. σ_5 is epi. Thus we have the exact sequence

$$0 \longrightarrow \mathcal{S}_5(M) \xrightarrow{\eta_5} \mathcal{N}_5(M) \xrightarrow{\sigma_5} L_5(\Pi_1) \simeq \oplus_g\mathbb{Z} \longrightarrow 0. \tag{17}$$

Now D. Sullivan proved that there is a bijection between $\mathcal{N}_n(M)$ and the group $[M, G/TOP]$ of the homotopy classes of maps from M to the H-space G/TOP (see for example [6, Theorem 5.4, p. 77]). Since $\Pi_2(G/TOP) \simeq \mathbb{Z}_2$, $\Pi_3(G/TOP) \simeq \Pi_5(G/TOP) \simeq 0$ and $\Pi_4(G/TOP) \simeq \mathbb{Z}$ with vanishing k-invariant in $H^5\big(K(\mathbb{Z}_2, 2)\big)$, the Postnikov resolution of G/TOP gives an H-map

$$G/TOP \longrightarrow K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}, 4)$$

which is a 5-equivalence. In particular, for any topological closed 5-manifold M, we have

$$\begin{aligned}\mathcal{N}_5(M) \simeq [M, G/TOP] \simeq [M, K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}, 4)] \simeq \\ H^2(M; \mathbb{Z}_2) \oplus H^4(M) \simeq H_2(M; \mathbb{Z}_2) \oplus H_1(M) \simeq \\ H_2(M; \mathbb{Z}_2) \oplus \oplus_g\mathbb{Z} \simeq H_2(M; \mathbb{Z}_2) \oplus L_6(\Pi_1).\end{aligned}$$

Thus we have Ker $\sigma_5 \simeq$Im $\eta_5 \simeq H_2(M; \mathbb{Z}_2)$ by (17) as requested. □

As a direct consequence of Theorem 6 (see also [10]), we obtain the following

Corollary 7.

(1) *If M has the homotopy type of $\#_g\mathbb{S}^1 \times \mathbb{S}^4$, then M is diffeomorphic to $\#_g\mathbb{S}^1 \times \mathbb{S}^4$.*
(2) *Any h-cobordism of $\#_g\mathbb{S}^1 \times \mathbb{S}^4$ with itself is a product.*
(3) *Let L be a disjoint union of g copies of $\mathbb{S}^3$ and let $\psi : L \longrightarrow \mathbb{S}^5$ be a smooth imbedding. Then ψ is ambient isotopic to the standard inclusion $L \subset \mathbb{S}^5$ if and only if $\mathbb{S}^5 \backslash \psi(L)$ has the homotopy type of the wedge $\vee_g\mathbb{S}^1$.*

Now we use (1) of Corollary 7 to prove the following result.

Corollary 8. *Let M be a closed orientable smooth (or PL) 5-manifold with $\Pi_1(M) \simeq *_g\mathbb{Z}$ and $H_2(M) \simeq 0$. Suppose that there exists a crystallization (G,c) of M for which at least one of α_{ijhr}'s equals $g+1$. Then M is (PL) homeomorphic to $\#_g\mathbb{S}^1 \times \mathbb{S}^4$.*

Proof: First we note that a finite presentation $\langle X : R\rangle$ of the fundamental group $\Pi_1(M)$ can be directly obtained from the crystallization (G,c) of M (for details see [5]). Here we briefly recall the construction. If $\mathcal{C}_G = \{i,j,h,r,s,t\}$ is the color set of G, then the generators of X are in bijection with the connected components of the subgraph $G_{\{i,j,h,r\}}$, but one, while the relators of R are in bijection with the $\{s,t\}$-colored cycles of G. This implies that the inequality

$$\alpha_{ijhr} - 1 \geq \operatorname{rk}\Pi_1(M) = g$$

holds. Suppose for example $\alpha_{0234} = g+1$. Then the pseudocomplex $K(1,5)$ consists of two vertices joined by exactly $1+g$ edges, hence its regular neighborhood $N = N(1,5)$ is (PL) homeomorphic to $\#_g\mathbb{S}^1 \times B^4$. Further we have that $H_4(M) \simeq H^1(M) \simeq \oplus_g\mathbb{Z}$ and $H_3(M) \simeq H^2(M) \simeq FH_2(M) \oplus TH_1(M) \simeq 0$. Then the Mayer-Vietoris sequence of the triple (M,N,N'), $N' = N(0,2,3,4)$, implies that

$$0 \longrightarrow H_4(M) \simeq \oplus_g\mathbb{Z} \longrightarrow H_3(\partial N) \simeq \oplus_g\mathbb{Z} \longrightarrow H_3(N') \longrightarrow 0,$$
$$0 \longrightarrow H_2(N') \longrightarrow H_2(M) \simeq 0,$$

$$0 \longrightarrow H_1(\partial N) \simeq \oplus_g\mathbb{Z} \longrightarrow H_1(N) \oplus H_1(N') \simeq \oplus_g\mathbb{Z} \oplus H_1(N') \to$$
$$\longrightarrow H_1(M) \simeq \oplus_g\mathbb{Z} \longrightarrow 0,$$

hence $H_1(N') \simeq \oplus_g\mathbb{Z}$ and $H_2(N') \simeq 0$. Furthermore $H_3(N')$ is free since $N' = N(0,2,3,4)$ collapses onto the 3-dimensional pseudocomplex $K(0,2,3,4)$. Thus the first exact sequence splits, i.e. $H_3(N') \simeq 0$. This implies that there do not exist two 3-simplexes in $K(0,2,3,4)$ with common boundary (notice that any ball of a pseudocomplex is abstractly isomorphic to the standard simplex of the same dimension). Therefore any 3-simplex of $K(0,2,3,4)$ can be retracted, by deformation, on a 2-dimensional subcomplex, i.e. $K(0,2,3,4)$ collapses onto a 2-dimensional subcomplex, say $\tilde{K}$. Moreover, $\tilde{K}$ is still a pseudocomplex, so any two faces of a simplex of $\tilde{K}$ do not identify together. Thus the conditions $H_2(N') \simeq H_2(\tilde{K}) \simeq 0$ and $H_1(\tilde{K}) \simeq H_1(N') \simeq \oplus_g\mathbb{Z}$ imply that $\tilde{K}$ (and whence $K(0,2,3,4)$) collapses to a one-dimensional subcomplex formed by two vertices joined by exactly $1+g$ edges (use the same argument as above). Then N' is also (PL) homeomorphic to $\#_g\mathbb{S}^1 \times B^4$. The manifold M is obtained by attaching two disjoint copies of $\#_g\mathbb{S}^1 \times B^4$ along their boundaries. Since $\Pi_1(M) \simeq *_g\mathbb{Z}$, M is homotopy equivalent to $\#_g\mathbb{S}^1 \times \mathbb{S}^4$, hence $M \simeq_{PL} \#_g\mathbb{S}^1 \times \mathbb{S}^4$ by (1) of Corollary 7. □

We conjecture that $\Pi_1(M) \simeq *_g\mathbb{Z}$ and $g(M) = g$ imply the hypothesis of Corollary 8.

We complete the section with the following

Proposition 9. *Let M be a closed orientable spin smooth (or PL) 5-manifold with free fundamental group. If $H_2(M)$ has no torsion, then M is null cobordant.*

Proof: Let $\psi_i : \mathbb{S}^1 \times B^4 \longrightarrow M$ be disjoint imbeddings such that the homotopy class $\left[\ \psi_i|_{\mathbb{S}^1 \times 0}\right]$ is the i-th generator of $\Pi_1(M) \simeq *_g\mathbb{Z}$, $i = 1, 2, \ldots, g$. We set $M_0 = M \setminus \cup_{i=1}^{g} \psi_i(\mathbb{S}^1 \times \overset{\circ}{B}{}^4)$ and consider the cobordism

$$W^6 = M \times I \cup_\psi \bigcup_{i=1}^{g} B^2 \times B^4$$

between M and $M' = M_0 \cup \bigcup_{i=1}^{g} B^2 \times \mathbb{S}^3$. Here we set $I = [0,1]$ and $\psi = \{\psi_i : i = 1, 2, \ldots, g\}$ as usual. Obviously M' is a simply-connected 5-manifold obtained from M by killing the generators of $\Pi_1(M)$ according to ψ. Further the pairs (M, M_0) and (M', M_0) are homology equivalent (by excision) to the disjoint unions $\cup_{i=1}^{g}(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times \mathbb{S}^3)$ and $\cup_{i=1}^{g}(B^2 \times \mathbb{S}^3, \mathbb{S}^1 \times \mathbb{S}^3)$ respectively. The following diagram easily implies that $H_2(M) \simeq H_2(M_0) \simeq H_2(M')$:

$$\begin{tikzcd}
& H_3(M', M_0) \simeq 0 \arrow[d] & & \\
0 \simeq H_3(M, M_0) \arrow[r] & H_2(M_0) \arrow[r, "\text{iso}"] \arrow[d] & H_2(M) \arrow[r] & H_2(M, M_0) \simeq 0 \\
& H_2(M') \arrow[d] & & \\
& H_2(M', M_0) \simeq \oplus_g\mathbb{Z} \arrow[d] & & \\
0 \simeq H_2(M, M_0) \arrow[r] & H_1(M_0) \arrow[r, "\text{iso}"] \arrow[d] & H_1(M) \simeq \oplus_g\mathbb{Z} \arrow[r] & H_1(M, M_0) \simeq 0 \\
& H_1(M') \simeq 0 & &
\end{tikzcd}$$

We also recall that the Stiefel-Whitney numbers are invariant under surgery (see [7]), hence $w_2(M) \simeq w_2(M') \simeq 0$. Since $H_2(M')$ is free, M' is diffeomorphic to $\#_k\mathbb{S}^2 \times \mathbb{S}^3$ by the classification theorem of simply connected spin 5-manifolds (see [13]). Thus W is a cobordism between M and $\#_k\mathbb{S}^2 \times \mathbb{S}^3$, where $k = rkH_2(M)$. Let $\hat{W}$ be a compact 6-manifold obtained from W by capping the boundary component $\#_k\mathbb{S}^2 \times \mathbb{S}^3$ by $\#_k\mathbb{S}^2 \times B^4$. Since M bounds $\hat{W}$, the proof is completed. $\square$

We conjecture that if $\Pi_1(M) \simeq *_g\mathbb{Z}$ and $g(M) = g$, then M bounds exactly $\#_g\mathbb{S}^1 \times B^5$, i.e. $M \simeq_{PL} \#_g\mathbb{S}^1 \times \mathbb{S}^4$.

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