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Short proofs of two theorems in topology

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Abstract. We present short and elementary proofs of the following two known theorems in General Topology:

(i) [H. Wicke and J. Worrell] A T_1 weakly $\delta\theta$-refinable countably compact space is compact.

(ii) [A. Ostaszewski] A compact Hausdorff space which is a countable union of metrizable spaces is sequential.

Keywords: countably compact, initially κ-compact, weakly $\delta\theta$-refinable, κ-refinable, sequential

Classification: 54D30, 54D20, 54D55

Throughout this note, κ denotes an infinite cardinal number and all topological spaces are assumed to be T_1.

A space X is called *κ-refinable* if every open cover γ of X has an open refinement λ such that $\lambda = \bigcup_{\alpha<\kappa} \lambda_\alpha$ and for each $x \in X$, there exists $\alpha < \kappa$ such that $1 \leq |\{V \in \lambda_\alpha : x \in V\}| \leq \kappa$. An example of a (hereditary) κ-refinable space is any space that can be represented as a union of $\leq \kappa$ metrizable subspaces.

The ω_0-refinable spaces are the same as weakly $\delta\theta$-refinable spaces, the spaces introduced by H. Wicke and J. Worrell. In 1976 they proved that countably compact weakly $\delta\theta$-refinable spaces are compact [WW]. A slightly different proof of this theorem appears in [B]. See also [A] for a generalization of weak $\delta\theta$-refinability and yet another proof of this theorem. Below, we present a proof which is shorter and much more elementary than these proofs. Moreover, the theorem is more general than that of Wicke and Worrell's.

Recall that a topological space is called *initially κ-compact* if every open cover of it of cardinality $\leq \kappa$ has a finite subcover. Note that 'initially ω_0-compact' is the same as 'countably compact'. The reader is referred to [S] for a survey of initially κ-compact spaces.

Theorem 1. *An initially κ-compact κ-refinable space is compact.*

Proof: Assume the contrary, and let X be an initially κ-compact κ-refinable space which is not compact. Let γ be a maximal open cover of X without a finite subcover. Let $\lambda = \bigcup_{\alpha<\kappa} \lambda_\alpha$ be an open refinement of γ which witnesses the κ-refinability of X. For each $\alpha < \kappa$, and for each $x \in X$, let $\lambda_\alpha(x) = \{V \in \lambda_\alpha : x \in V\}$ and $X_\alpha = \{x \in X : 1 \leq |\lambda_\alpha(x)| \leq \kappa\}$. Then $X = \bigcup_{\alpha<\kappa} X_\alpha$. Since X is initially κ-compact, there exists β such that X_β cannot be covered by κ or less members of γ. Let $W = \bigcup \lambda_\beta$. Since $X_\beta \subseteq W$, $W \notin \gamma$. By the maximality of γ, there exists $U \in \gamma$ such that $X = W \cup U$. Then $X_\beta \setminus U$ cannot be covered by κ or less members of γ.

By induction, we choose a sequence $x_1, x_2, \dots$ of points in $X_\beta \setminus U$ as follows: let $x_1 \in X_\beta \setminus U$ be arbitrary. If $x_1, \dots, x_n$ have already been chosen, then, since $\left|\bigcup_{i=1}^n \lambda_\beta(x_i)\right| \le \kappa$, $X_\beta \setminus U$ is not contained in $\bigcup \left(\bigcup_{i=1}^n \lambda_\beta(x_i)\right)$. Choose $x_{n+1} \in (X_\beta \setminus U) \setminus \bigcup \left(\bigcup_{i=1}^n \lambda_\beta(x_i)\right)$.

Let $S = \{x_1, x_2, \dots\}$. Then $S \subseteq X \setminus U$ and, since $X \setminus U \subseteq W$, no point of $X \setminus U$ is a limit point of S. This is a contradiction, since $X \setminus U$ is countably compact. $\square$

A topological space X is called *sequential* if every nonclosed subset A of X contains a sequence converging to a point in $X \setminus A$.

In [O], A. Ostaszewski proved that a countably compact regular space which can be represented as a union of countably many metrizable spaces is sequential. The proof consists of about four printed pages. Below, we present a short proof based on the Wicke-Worrell Theorem.

Theorem 2. *A countably compact regular space which can be represented as a countable union of metrizable spaces is sequential* (*and compact*).

Proof: Let X be a countably compact regular space, and let $X = \bigcup_{i=1}^\infty X_i$, where each X_i is metrizable. Let A be a non-closed subset of X. Since X is hereditary ω_0-refinable (i.e. hereditarily weakly $\delta\theta$-refinable), A cannot be countably compact. Therefore, there exists a sequence $S = \{x_1, x_2, \dots\}$ in A which has no cluster point in A. Let $Y = \overline{S} \setminus S$. Since Y is non-empty and compact, $Y \cap X_i$ is not nowhere dense in Y , for some i. Hence, $Y \cap X_i$ contains a point which has countable character in Y and thus in $\overline{S}$ as well. Therefore, S contains a subsequence converging to a point in Y. $\square$

The last part of the above proof shows that any countably compact regular space which can be represented as a union of countably many first countable spaces contains a point of countable character. This is essentially the same as Theorem 3 of [O] attributed to M.E. Rudin and K. Kunen there. We have a much stronger theorem of this type which we prove by a different method.

Theorem 3. *Let X be a regular initially ω_1-compact space which can be represented as a union of $\le \omega_1$ subspaces of countable pseudocharacter. Then every non-empty G_δ subset of X contains a point of countable character in X.*

Proof: Let $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$, where each X_α has countable pseudocharacter. Let U be a non-empty G_δ subset of X and suppose that no point of U has countable character in X. By induction, we choose a decreasing sequence $\{F_\alpha : \alpha < \omega_1\}$ of non-empty closed G_δ subsets of X as follows:

If $U \cap X_0 = \emptyset$, let F_0 be an arbitrary non-empty closed G_δ subset of X such that $F_0 \subseteq U$. If $U \cap X_0 \neq \emptyset$, let $x \in U \cap X_0$. Then there exists a G_δ subset V of X such that $V \cap X_0 = \{x\}$. Since X is countably compact and x does not have countable character in X, $\{x\}$ is not G_δ in X. Therefore, $\emptyset \neq U \cap V \neq \{x\}$. Let

F_0 be a non-empty closed G_δ subset of X such that $F_0 \subseteq (U \cap V) \setminus \{x\}$. Then $F_0 \cap X_0 = \emptyset$.

If $\beta < \omega_1$, and for each $\alpha < \beta$, we have chosen F_α, then, since $\bigcap_{\alpha<\beta} F_\alpha$ is a G_δ subset of X, by repeating the above argument with $\bigcap_{\alpha<\beta} F_\alpha$ in place of U and X_β in place of X_0 we can find a non-empty closed G_δ subset F_β of X such that $F_\beta \subseteq \bigcap_{\alpha<\beta} F_\alpha$ and $F_\beta \cap X_\beta = \emptyset$.

Let $F = \bigcap\{F_\alpha : \alpha < \omega_1\}$. Since X is initially ω_1-compact, $F \neq \emptyset$. On the other hand, since $F \cap X_\alpha = \emptyset$, for each $\alpha < \omega_1$, $F = \emptyset$. This is a contradiction. □

References

[A] Arhangel'skiĭ A.V., *The star method, new classes of spaces and countable compactness*, Soviet Math. Dokl. **21** (1980), 550–554.

[B] Burke D.K., *Covering properties*, Handbook of Set Theoretic Topology, North Holland, 1984, pp. 347–422.

[O] Ostaszewski A.J., *Compact σ-metric Hausdorff spaces are sequential*, Proc. Amer. Math. Soc. **68** (1978), 339–343.

[S] Stephenson R.M., Jr., *Initially κ-compact and related spaces*, Handbook of Set Theoretic Topology, North Holland (1984), pp. 603–632.

[WW] Wicke H.H., Worrell J.M., Jr., *Point countability and compactness*, Proc. Amer. Math. Soc. **55** (1976), 427–431.

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