Nonhomogeneous Riemannian 3-manifolds with distinct constant Ricci eigenvalues

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Abstract. We extend a construction by K. Yamato [Ya] to obtain new explicit examples of Riemannian 3-manifolds as in the title. Some of these examples have an interesting geometrical interpretation.

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1. Introduction.

According to I.M. Singer [Si] a Riemannian manifold (M, g) is said to be *curvature homogeneous* if, for every two points $p, q \in M$, there is a linear isometry $F : T_pM \to T_qM$ between the corresponding tangent spaces such that $F^*R_q = R_p$ (where R denotes the curvature tensor of (M, g)). Note that a (locally) homogeneous Riemannian manifold is automatically curvature homogeneous. Explicit *nonhomogeneous* examples have been constructed by many authors ([Se1], [T], [Ya], [KTV1–3], [K] — see especially [KTV2] or [KTV3] for full references).

Next, let $(\overline{M}, \overline{g})$ be a homogeneous Riemannian manifold (i.e. such that the isometry group acts transitively on $\overline{M}$). We say that a Riemannian manifold (M, g) has *the same curvature tensor* as $(\overline{M}, \overline{g})$ if, for a fixed point $o \in \overline{M}$ and every point $p \in M$, there is a linear isometry $F : T_pM \to T_o\overline{M}$ such that $F^*\overline{R}_o = R_p$. We also say that $(\overline{M}, \overline{g})$ is *a homogeneous model* for (M, g). If a Riemannian manifold (M, g) possesses a homogeneous model, then it is automatically curvature homogeneous.

For example, all nonhomogeneous Riemannian manifolds whose homogeneous model is a fixed symmetric space have been described explicitly in [BKV].

Until recently, an open problem remained *whether there exist curvature homogeneous spaces without any homogeneous model*. The first example (in dimension four) was based on a construction by K. Tsukada [T], see [KTV3]. The new source of such examples has been now found in the class of all 3-dimensional Riemannian manifolds. In dimension three, the curvature homogeneous spaces are just those with constant Ricci eigenvalues. The new examples are based on the following theorem, which is an easy consequence of an observation by J. Milnor [Mi] and a result by K. Sekigawa [Se2].

Theorem A. *For a homogeneous Riemannian* 3*-manifold* $(\overline{M}, \overline{g})$*, the signature of the Ricci tensor is never equal to* $(+, +, -)$ *or* $(+, 0, -)$*.*

In [K], the present author has proved that all Riemannian 3-manifolds with the prescribed constant Ricci eigenvalues $\varrho_1 = \varrho_2 \neq \varrho_3$ depend (locally) on two arbitrary functions of one variable. This gives a large family of examples with the "forbidden signature" $(+,+,-)$, i.e. curvature homogeneous spaces without a homogeneous model. The author succeeded to find just two explicit examples: the corresponding Ricci eigenvalues are here $(\frac{1}{4}\lambda^2, \frac{1}{4}\lambda^2, -2\lambda^2)$ and $(\frac{2}{9}\lambda^2, \frac{2}{9}\lambda^2, -2\lambda^2)$, respectively.

Recently, A. Spiro and F. Tricerri [ST] have proved the existence of Riemannian 3-manifolds with any prescribed *distinct* constant Ricci eigenvalues. This proves the existence of new curvature homogeneous spaces with both "forbidden signatures" $(+,+,-)$ and $(+,0,-)$, but no explicit example is available.

On the other hand, K. Yamato [Ya] presented a large class of explicit examples of nonhomogeneous and complete Riemannian manifolds with prescribed distinct constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$ *satisfying a specific set of inequalities*. Nevertheless, it has been proved in [KTV3] that each of these examples possesses a homogeneous model, namely a unimodular Lie group with a left invariant metric.

The aim of this note is to extend the Yamato construction to a larger range of the triplets $(\varrho_1, \varrho_2, \varrho_3)$. The new metrics are not more complete and they are given only locally. Yet, they are all explicit and they cover all cases with the "forbidden signatures" of the Ricci tensor! We thus obtain an infinite family of explicitly defined curvature homogeneous spaces without any homogeneous model.

2. The extension of the Yamato's examples.

Let $\varrho_1, \varrho_2, \varrho_3$ be distinct real numbers and consider the following functions (cf. [Ya]):

$$A = \frac{\varrho_1 + \varrho_2 - \varrho_3}{2}, \quad B = \frac{\varrho_1 - \varrho_3}{\varrho_3 - \varrho_2}, \quad C = \frac{-(\varrho_1 + \varrho_2)(\varrho_3 - \varrho_2)^2}{(\varrho_2 - \varrho_1)^2}. \tag{1}$$

The Yamato's result is the following

Theorem B. *Let $A > 0$, $C > 0$, $A + BC > 0$. Then there exists a complete nonhomogeneous metric g on R^3 with the constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$.*

In [Ya] explicit formulas are also provided. We shall now present our extended version; some computational details and explicit formulas will be given later.

We shall need the following convention: let π be any permutation of the index set $\{1, 2, 3\}$. Then A_π, B_π, C_π will denote the functions A, B, C from (1) in which the given permutation of indices is performed. In particular, the corresponding functions for $\pi = (1), (2,3)$ will be denoted as $\overline{A}, \overline{B}, \overline{C}$. Now we have

Main Theorem. *Let $\mathcal{B}$ ("the bad set") denote the set of all triplets $(\varrho_1, \varrho_2, \varrho_3)$ of distinct real numbers such that*

(a) $0 \geq \varrho_1 > \varrho_2 > \varrho_3$,
(b) $\underline{A} \leq 0$,
(c) $\overline{A} + \overline{BC} \leq 0$.

Suppose that a triplet of real numbers $\varrho_1 > \varrho_2 > \varrho_3$ does not belong to $\mathcal{B}$. Then there exists a nonhomogeneous Riemannian metric g on an open set $U \subset R^3$ with the constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$. The metric g is given by explicit formulas involving only elementary functions.

Corollary. *Let $\varrho_1 > \varrho_2 > \varrho_3$ be a triplet of real numbers with the signature $(+,+,-)$ or $(+,0,-)$. Then an explicit example exists of a (nonhomogeneous) Riemannian manifold with the constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$. This is a curvature homogeneous space without any homogeneous model.*

Proof of Corollary: Obviously we have $\varrho_1 > 0$ and thus the Main Theorem can be applied. □

The proof of the Main Theorem.

Starting from the explicit formulas given in [Ya], we shall investigate the class of Riemannian metrics defined on open sets $U \subset R^3[w,x,y]$ by

$$g = \sum_{i=1}^{3} \omega^i \otimes \omega^i \tag{2}$$

where the orthonormal coframe $\{\omega^1, \omega^2, \omega^3\}$ has the form

$$\begin{cases} \omega^1 = dx + P(w,x,y)\,dw, \\ \omega^2 = dy + Q(w,x,y)\,dw, \\ \omega^3 = dw. \end{cases} \tag{3}$$

Here P, Q are unknown functions to be determined.

Recall the standard formulas defining the components ω^i_j of the Riemannian connection form (cf. [KN]):

$$d\omega^i + \sum \omega^i_j \wedge \omega^j = 0, \quad \omega^i_j + \omega^j_i = 0 \quad (i,j = 1,2,3). \tag{4}$$

Now, suppose that, at each point $p \in U$, the orthonormal tangent frame dual to the coframe (3) consists of the unit eigenvectors corresponding to the constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$, respectively. Then all the components of the curvature tensor with at least three distinct indices vanish. Let us denote the sectional curvature in the 2-plane determined by $\omega^i = 0$ as λ_i. Then using the standard formulas for the curvature form we obtain a system of exterior differential equations

$$\begin{cases} d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = \lambda_3 \omega^1 \wedge \omega^2, \\ d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = \lambda_2 \omega^1 \wedge \omega^3, \\ d\omega^2_3 + \omega^2_1 \wedge \omega^1_3 = \lambda_1 \omega^2 \wedge \omega^3. \end{cases} \tag{5}$$

Here

$$\varrho_i = s - \lambda_i \quad \text{for} \quad i = 1,2,3, \quad \text{where} \quad s = \lambda_1 + \lambda_2 + \lambda_3 \tag{6}$$

and hence

$$\lambda_i - \lambda_j = \varrho_j - \varrho_i \quad (i,j = 1,2,3). \tag{7}$$

By differentiating the equations (5) (and after substituting from (5) again) we obtain the following integrability conditions:

$$(\lambda_j - \lambda_i)\omega^i \wedge \omega^k \wedge \omega^i_j + (\lambda_i - \lambda_k)\omega^i \wedge \omega^j \wedge \omega^i_k = 0, \tag{8}$$

where (i, j, k) is any arrangement of the indices $1, 2, 3$.

To express (8) more explicitly, put

$$\omega^i_j = \sum a^i_{jk}\omega^k. \tag{9}$$

Then (8) can be rewritten (using also (7)) in the form

$$\begin{cases} (\varrho_1 - \varrho_3)a^3_{11} + (\varrho_2 - \varrho_3)a^3_{22} = 0, \\ (\varrho_3 - \varrho_2)a^2_{33} + (\varrho_1 - \varrho_2)a^2_{11} = 0, \\ (\varrho_2 - \varrho_1)a^1_{22} + (\varrho_3 - \varrho_1)a^1_{33} = 0. \end{cases} \tag{10}$$

(Recall that $a^i_{jk} + a^j_{ik} = 0$.)

Now, using the conditions (4) and also (10), we obtain

$$\omega^1_2 = \gamma\omega^3, \quad \omega^1_3 = \alpha\omega^1 - \beta\omega^2, \quad \omega^2_3 = -\beta\omega^1 + B\alpha\omega^2, \tag{11}$$

where

$$\alpha = -P'_x = -\frac{1}{B}Q'_y, \quad \beta = \frac{1}{2}(P'_y + Q'_x), \quad \gamma = \frac{1}{2}(P'_y - Q'_x). \tag{12}$$

Here the invariant B from (1) comes in.

Next, we shall express the curvature conditions (5). We obtain the following series of equations (A), (B), (C):

(A1) $\gamma'_x = 0$,
(A2) $\gamma'_y = 0$,
(A3) $\beta^2 = A + B\alpha^2$,
(B1) $Q\alpha'_y + P\alpha'_x - \alpha'_w - \alpha^2 - \beta^2 + 2\beta\gamma = \lambda_2$,
(B2) $\alpha'_y + \beta'_x = 0$,
(B3) $Q\beta'_y + P\beta'_x - \beta'_w - (B+1)\alpha\beta + (B-1)\alpha\gamma = 0$,
(C1) $\equiv$ (B3),
(C2) $B\alpha'_x + \beta'_y = 0$,
(C3) $B(Q\alpha'_y + P\alpha'_x - \alpha'_w) - B^2\alpha^2 - \beta^2 - 2\beta\gamma = \lambda_1$.

Here the new invariant $A = \lambda_3$ from (1) comes in, too.

Now, (B1) and (C3) imply (after the substitution from (A3))

$$A(B-1) - 2(B+1)\beta\gamma = \lambda_1 - B\lambda_2, \tag{13}$$

where $B+1=(\varrho_1-\varrho_2)/(\varrho_3-\varrho_2)\neq 0$. Hence

$$2\beta\gamma=\frac{B\varrho_1-\varrho_2}{B+1}=\varrho_1+(B+1)C, \tag{14}$$

where

$$C=\frac{\varrho_1+\varrho_2}{(B+1)^2} \tag{15}$$

is the last invariant from (1). (A1) and (A2) imply $\gamma=\gamma(w)$ and (14) implies $\beta=\beta(w)$. Then using (B2) and (C2) we infer $\alpha'_y=\alpha'_x=0$, i.e. $\alpha=\alpha(w)$.

We are left with the differential equations (B1), (B3) and with the algebraic equations (A3) and (14), where α,β,γ depend on w only.

Substituting in (B1) for β^2 from (A3) and for $2\beta\gamma$ from (14) we get

$$\alpha'(w)=(B+1)(C-\alpha^2). \tag{16}$$

From (B3) we obtain

$$\beta'(w)+(B+1)\alpha\beta-(B-1)\alpha\gamma=0. \tag{17}$$

Multiplying this by 2β we get

$$(\beta^2)'_w+2(B+1)\alpha\beta^2-2(B-1)\alpha\beta\gamma=0. \tag{18}$$

Substituting again from (A3) and (14) we obtain (because we can assume $\alpha\neq 0$)

$$2(A+BC)(B+1)=(B-1)(\varrho_1+(B+1)C). \tag{19}$$

This is obviously an identity due to the definition of A,B,C. Consequently, if $\alpha(w)$ satisfies the differential equation (16), then the functions $\beta(w)$ and $\gamma(w)$ are defined (up to a sign) by the algebraic equations (A3) and (14). Such a triplet of functions obviously satisfies the system of equations (A), (B), (C).

The differential equation (16) has always a local solution, namely

$$\text{for } C>0,\quad \alpha(w)=\sqrt{C}(e^{2G(w)}-1)/(e^{2G(w)}+1), \tag{20}$$

$$\text{for } C<0,\quad \alpha(w)=\sqrt{|C|}tg(G(w)), \tag{21}$$

where

$$G(w)=\sqrt{|C|}(B+1)w. \tag{22}$$

For $C=0$ we get a solution

$$\alpha(w)=1/((B+1)w). \tag{23}$$

Finally, the functions P, Q can be determined from the differential equations (cf. (12))

$$P'_x = -\alpha, \quad Q'_y = -B\alpha, \tag{24}$$

$$P'_y + Q'_x = 2\beta, \quad P'_y - Q'_x = 2\gamma. \tag{25}$$

We can clearly put

$$P = -\alpha(w)x + [\beta(w) + \gamma(w)]y, \tag{26}$$

$$Q = [\beta(w) - \gamma(w)]x - B\alpha(w)y. \tag{27}$$

We have got an explicit solution of our problem. It remains to prove that the existence of this solution is always guaranteed under the condition $(\varrho_1, \varrho_2, \varrho_3) \notin \mathcal{B}$ and that the corresponding Riemannian metric is not locally homogeneous.

Up to now, we have not used any inequalities between the eigenvalues $\varrho_1, \varrho_2, \varrho_3$. Thus, any permutation of the indices leads to the same kind of computation. We obtain easily (using (A3))

Lemma. *Let $\varrho_1 > \varrho_2 > \varrho_3$ be given real numbers. Then the necessary and sufficient condition for the existence of a local Riemannian metric of the form* (3) *with the constant Ricci eigenvalues $\varrho_1, \varrho_2, \varrho_3$ is the existence of a permutation π of the indices $1, 2, 3$ such that $A_\pi + B_\pi(\alpha_\pi(w))^2 > 0$ holds in some interval of the variable w. Here $\alpha_\pi = \alpha_\pi(w)$ is given by one of the formulas* (20), (21) *or* (23) *in which B and C are replaced by B_π and C_π, respectively.*

(In fact, we have not used the integration constant in the solutions of (16), but this does not change the situation anyway.)

Proof of the Main Theorem: Suppose that the triplet $\varrho_1 > \varrho_2 > \varrho_3$ does not belong to the set $\mathcal{B}$. If one of the conditions (a) or (b) in the Main Theorem does not hold, then $A > 0$, and because any of the functions $\alpha(w)$ given by (20)–(23) is sufficiently small in some interval of the variable w, we have $A + B\alpha^2(w) > 0$ in such interval.

Assume now that (a) and (b) are satisfied but $\overline{A} + \overline{BC} > 0$. We see that $\overline{B} = (\varrho_1 - \varrho_2)/(\varrho_2 - \varrho_3) > 0$. On the other hand, $A \leq 0$ implies $\overline{A} < 0$. Then $\overline{A} + \overline{BC} > 0$ implies $\overline{C} > 0$ and we have to use the formula (20) to calculate the corresponding function $\overline{\alpha}(w)$. Obviously $\overline{\alpha}^2(w) \to \overline{C}$ for $|w| \to \infty$, i.e. there is a constant $N > 0$ such that

$$\overline{\alpha}^2(w) > \overline{C} - (\overline{A} + \overline{BC})\overline{B}^{-1} \tag{28}$$

for $|w| > N$. Hence $\overline{A} + \overline{B}\overline{\alpha}^2 > 0$ outside some interval $(-N, N)$. This concludes the proof of the existence part of our theorem.

The nonhomogeneity of the constructed local metrics can be seen as follows. Using the standard formula

$$\nabla_X E_i = \sum \omega_i^k(X) E_k \tag{29}$$

for the orthonormal frame $\{E_1, E_2, E_3\}$ consisting of the Ricci eigenvectors, we obtain easily the following covariant derivative of the Ricci tensor:

$$(\nabla_{E_1}\varrho)(E_1, E_3) = (\varrho_3 - \varrho_1)\alpha(w). \tag{30}$$

Because (30) is a nonconstant Riemannian invariant, the metric g given by (2), (3), (26) and (27) is not locally homogeneous, q.e.d. □

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