Necessary and sufficient conditions for weak convergence of random sums of independent random variables

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Abstract. Let $\{X_n,\, n \geq 1\}$ be a sequence of independent random variables such that $EX_n = a_n$, $E(X_n - a_n)^2 = \sigma_n^2$, $n \geq 1$. Let $\{N_n,\, n \geq 1\}$ be a sequence od positive integer-valued random variables. Let us put $S_{N_n} = \sum_{k=1}^{N_n} X_k$, $L_n = \sum_{k=1}^{n} a_k$, $s_n^2 = \sum_{k=1}^{n} \sigma_k^2$, $n \geq 1$. In this paper we present necessary and sufficient conditions for weak convergence of the sequence $\{(S_{N_n} - L_n)/s_n,\, n \geq 1\}$, as $n \to \infty$. The obtained theorems extend the main result of M. Finkelstein and H.G. Tucker (1989).

Keywords: random sums, weak convergence, stable law, nonrandom centering, measure of dependence between σ-fields

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1. Introduction.

Let $\{X_n,\, n \geq 1\}$ be a sequence of independent random variables, defined on a probability space $(\Omega, \mathcal{A}, P)$, such that $EX_n = a_n$, $E(X_n - a_n)^2 = \sigma_n^2 < \infty$, $n \geq 1$. Let us put

$$S_n = \sum_{k=1}^{n} X_k, \qquad L_n = \sum_{k=1}^{n} a_k, \qquad s_n^2 = \sum_{k=1}^{n} \sigma_k^2, \qquad n \geq 1.$$

Let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables, defined on the same probability space $(\Omega, \mathcal{A}, P)$.

Recently many authors have studied limit behaviour of the following sequences:

$$\{(S_{N_n} - L_{N_n})/s_{N_n},\ n \geq 1\}, \qquad \{(S_{N_n} - EL_{N_n})/\sigma(S_{N_n}),\ n \geq 1\},$$
$$\{(S_{N_n} - L_n)/s_n,\ n \geq 1\},$$

under the assumption that for each $n \geq 1$ the random variables $N_n, X_1, X_2, \ldots$ are independent. Also the rate of convergence to the obtained limit law has extensively been studied (cf. [3], [6], [9], [4], [5], [8] and the references given there).

The limit distribution of the sequence $\{(S_{N_n} - L_n)/s_n,\, n \geq 1\}$ is presented in [3]. Namely, M. Finkelstein and H.G. Tucker [3] have obtained the following very interesting result.

Theorem A. *Let $\{X_n,\, n \geq 1\}$ be a sequence of independent and identically distributed random variables such that $EX_1 = \mu \neq 0$ and $E(X_1 - \mu)^2 = \sigma^2 > 0$. If $\{N_n,\, n \geq 1\}$ is a sequence of positive integer-valued random variables independent of X_n, $n \geq 1$, then the condition*

$$(S_{N_n} - n\mu)/\sigma\sqrt{n} \xrightarrow{D} \text{(some) } Z \tag{1.1}$$

holds if and only if the condition

$$(N_n - n)/\sqrt{n} \xrightarrow{D} \text{(some) } U \tag{1.2}$$

holds, in which case the distribution of Z is that of $X + Y$, where X and Y are independent random variables, X being $N(0,1)$ and Y having the same distribution as $\mu U/\sigma$.

The main aim of this paper is to extend Theorem A in the following directions:

(i) We consider the random variables X_n, $n \geq 1$, not necessarily identically distributed.

(ii) We omit the assumption that the random variables X_n, $n \geq 1$, have finite moments, and therefore we consider weak convergence to the Levy class distribution functions.

(iii) We do not assume that the random variables N_n, $n \geq 1$, are independent of X_n, $n \geq 1$. We study limit distribution of the sequence $\{(S_{N_n} - L_n)/s_n,\, n \geq 1\}$, under the assumption that for some $1 \leq q \leq \infty$

$$r(n) = \mathbb{R}_{1,q}(\sigma\{N_k, k \geq 1\}, \sigma\{X_k, k \geq n\}) \to 0 \tag{1.3}$$

or

$$\mathbb{R}(n) = \mathbb{R}_{1,q}(\sigma\{N_n\}, \sigma\{X_k, k \geq 1\}) \to 0 \tag{1.4}$$

as $n \to \infty$, where $\mathbb{R}_{p,q}(\mathcal{F}, \mathcal{G})$ denotes the measure of dependence between σ-fields $\mathcal{F}$ and $\mathcal{G}$ introduced in [2] (cf. (1.1)). Namely, for $1 \leq p$, $q \leq \infty$

$$\mathbb{R}_{p,q}(\mathcal{F}, \mathcal{G}) = \sup |Efg - Ef\ Eg| / \|f\|_p \|g\|_q\,,$$

where the sup is taken over all f and g such that f is simple, real-valued, and $\mathcal{F}$-measurable and g is simple, real-valued, and $\mathcal{G}$-measurable. (0/0 is presented to be 0.) Of course, $\mathbb{R}_{p,q}$ is simply a norm of the bilinear form covariance.

In Section 2 we present the results. In Section 3 some auxiliary lemmas are given. The proofs of the main results are presented in Section 4.

2. Results.

Theorem 1. *Let $\{X_n,\, n \geq 1\}$ be a sequence of independent random variables and let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying (1.3) or (1.4) for some $1 \leq q \leq \infty$. Let $\{L_n,\, n \geq 1\}$ and $\{s_n,\, n \geq 1\}$ be sequences of real numbers and positive real numbers, respectively. Denote*

$$a_n = L_n - L_{n-1}, \quad S_n = \sum_{k=1}^{n} X_k,$$

$$S_{N_n} = \sum_{k=1}^{N_n} X_k, \quad L_{N_n} = \sum_{k=1}^{\infty} L_k I[N_n = k], \quad s_{N_n} = \sum_{k=1}^{\infty} s_k I[N_n = k], \quad n \geq 1.$$

Assume

$$\max_{1\leq k\leq n} P[|X_k - a_k| \geq \varepsilon s_n] \to 0 \ \text{ as } \ n \to \infty,$$

and

$$(S_n - L_n)/s_n \xrightarrow{D} F(\cdot) \ \text{ as } \ n \to \infty, \tag{2.1}$$

where

$$\int e^{itx} F(dx) = \exp\{i\gamma t + \oint (e^{itx} - 1 - itx/(1+x^2))\ (1+x^2)/x^2\, G(dx)\}, \tag{2.2}$$

γ is a real number, $G(\cdot)$ is nondecreasing bounded function ($\oint$ means that the integrand is equal to $-t^2/2$ for $x = 0$) and not identically equal to a constant, and

$$N_n \xrightarrow{P} \infty \ \text{ as } \ n \to \infty \tag{2.3}$$

or for every $k, n \in \mathbb{N}$ and some constant $C > 0$

$$|L_n - L_k| \geq C|n-k| \ \text{ and } \ n/s_n \to \infty \ \text{ as } \ n \to \infty, \tag{2.4}$$

or

$$L_n/s_n \to \infty \ \text{ or } \ L_n/s_n \to \infty, \ \text{ as } \ n \to \infty, \tag{2.5}$$

If

$$(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n) \to \ \text{(some)} \ A(\cdot,\cdot), \tag{2.6}$$

where A is a two-dimensional distribution function, then

$$(S_{N_n} - L_n)/s_n \xrightarrow{D} \ \text{(some)} \ \Psi(\cdot), \tag{2.7}$$

where

$$(2.8) \quad \int e^{itx}\,\Psi(dx) =$$
$$= \iint\limits_{\mathbb{R}^2} \exp\{i\gamma ty + itz + \oint (e^{i(ty)x} - 1 - i(ty)x/(1+x^2))\,(1+x^2)/x^2\,G(dx)\}\,A(dy,\,dz).$$

If (2.7) *holds with some distribution function* $\Psi(\cdot)$, *then the sequence* $\{(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n),\ n \geq 1\}$ *is tight.*

It is known that the set of possible weak limits of sums of independent random variables (cf. for e.g. [7, IV, § 3]) is the class of Levy distribution function $F(\cdot)$ which may be characterized by (2.2) and: For every $0 < \alpha < 1$, there exists the characteristic function $f_\alpha(t)$ such that

$$\int e^{itx}\,F(dx) = \int e^{it\alpha x}\,F(dx)\ f_\alpha(t), \quad t \in \mathbb{R}.$$

Furthermore, by Lemma 11 [7, IV, § 3], (2.1) implies

$$(2.9) \qquad s_{n+1}/s_n \to 1, \quad \text{and} \quad s_n \to \infty \quad \text{as} \quad n \to \infty.$$

The condition that $G(\cdot)$ is not identically equal to a constant implies

$$\oint (e^{itx} - 1 - itx/(1+x^2))\ (1+x^2)/x^2\ G(dx) \neq 0$$

so that $F(\cdot)$ in (2.1) is not a degenerate distribution function.

We note that the condition (2.3) may be expressed as follows:

For some sequence $\{\alpha(n),\ n \geq 1\}$ such that $\alpha(n) \to \infty$ as $n \to \infty$,

$$(2.10) \qquad P(N_n < \alpha(n)) \to 0 \quad \text{as} \quad n \to \infty.$$

Let us observe that if for each $n \geq 1$, the random variables $N_n, X_1, X_2, \ldots$ are independent, then (1.3) and (1.4) hold with $r(n) = \mathbb{R}(n) = 0$ for every $q \geq 1$.

The next result deals with the convergence to the stable limit law. Assume

$$(2.11) \qquad P(X_n > x)/P(|X_n| > x) \to c_{1,n}/(c_{1,n} + c_{2,n}) \quad \text{as} \quad x \to \infty,$$

where $\{c_{j,n},\ n \geq 1\}$, $j = 1, 2$, are some sequences of nonnegative numbers such that $c_{1,n} + c_{2,n} > 0$, $n \geq 1$.

For some $0 \le \alpha \le 2$ we define

$$e_1 = \int_0^\infty u^{-\alpha} \sin(u)\, du, \quad e_2 = \begin{cases} -\int_0^\infty u^{-\alpha} \cos(u)\, du, & \text{if } \alpha < 1, \\ 1 & \text{if } \alpha = 1 \\ \int_0^\infty u^{-\alpha}(1-\cos(u))\, du, & \text{otherwise} \end{cases}$$

$$\sigma_n^\alpha = (c_{1,n} + c_{2,n})e_1, \quad s_n^\alpha = \sum_{i=1}^n \sigma_i^\alpha, \quad s_0 = 1,$$

$$a_n = \begin{cases} 0, & \text{if } \alpha < 1, \\ EX_n, & \text{if } \alpha > 1, \\ \int_0^1 d_n(x)\, dx + \int_1^\infty (d_n(x) - (c_{2,n} - c_{1,n})/x)\, dx + & \\ \qquad + \sum_{i=1}^{n-1} (c_{1,i} - c_{2,i}) e_2 \ln(s_i/s_{i-1}) + & \\ \qquad +(c_{1,n} - c_{2,n}) e_2 \ln(s_n) + (c_{2,n} - c_{1,n})\gamma, & \text{otherwise} \end{cases}$$

$$L_n = \sum_{i=1}^n a_i, \quad n \ge 1,$$

where $d_n(x) = P(X_n > x) - P(X_n < -x)$, γ is the Euler's constant and $s_n = (s_n^\alpha)^{1/\alpha}$. Furthermore, let

$$\beta_n = \sum_{i=1}^n (c_{1,i} - c_{2,i}) e_2 \,.$$

Let $G_{\alpha,\beta,\nu,\lambda}(\cdot)$ denote the stable law with parameters $\alpha, \beta, \nu, \lambda$, $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\lambda > 0$, $\nu \in \mathbb{R}$, i.e.

$$\int e^{itx}\, G_{\alpha,\beta,\nu,\lambda}(dx) = \exp\{i\nu t - \lambda |t|^\alpha (1 + i \operatorname{sgn}(t) \omega(t,\alpha,\beta))\},$$

where $\omega(t,\alpha,\beta) = \beta tg(\pi\alpha/2)$ for $\alpha \ne 1$ and $\omega(t,\alpha,\beta) = -(2\beta/\pi)\ln|t|$ for $\alpha = 1$.

Theorem 2. *Let $\{X_n,\, n \ge 1\}$ be a sequence of independent random variables satisfying* (2.11)*. Assume, for some $\alpha \in (0,2]$,*

$$\beta_n / s_n^\alpha \to \beta \ \text{ as } \ n \to \infty \tag{2.12}$$

and

$$(S_n - L_n)/s_n \xrightarrow{D} G_{\alpha,\beta,0,1}(\cdot) \ \text{ as } \ n \to \infty, \tag{2.13}$$

hold.

Let $\{N_n,\, n \ge 1\}$ be a sequence of positive integer-valued random variables satisfying (1.3) *or* (1.4) *and* (2.3) *or* (2.4)*. If*

$$(s_{N_n}^\alpha / s_n^\alpha, (\beta_{N_n} - \beta_n)/s_n^\alpha, (L_{N_n} - L_n)/s_n) \xrightarrow{D} \ \text{(some) } A(\cdot,\cdot,\cdot), \tag{2.14}$$

where A is a three-dimensional distribution function, then

$$(S_{N_n} - L_n)/s_n \xrightarrow{D} \Psi(\cdot) \quad as \quad n \to \infty, \tag{2.15}$$

where

$$\int e^{itx}\,\Psi(dx) = \iiint\limits_{\mathbb{R}^3} \exp\{-|t|^\alpha(x + i\,\mathrm{sgn}(t)\omega(\alpha,\beta,t)) - \\ - |t|^\alpha i\,\mathrm{sgn}(t)\omega(\alpha,1,t)y + itz\}\,A(dx,\,dy,\,dz).$$

If (2.15) *holds with some distribution function Ψ, then the sequence $\{(s^\alpha_{N_n}/s^\alpha_n, (\beta_{N_n} - \beta_n)/s^\alpha_n, (L_{N_n} - L_n)/s_n),\ n \ge 1\}$ is tight.*

Note that for $\alpha < 1$ we have $L_k = 0$ for all k, hence $(L_{N_n} - L_n) = 0$, $n \ge 1$. The given result seems to be interesting in case of i.i.d. random variables, but because in case $\alpha < 1$ the centralization is equal to 0, we formulate two corollaries for $\alpha > 1$ and $\alpha = 1$ only.

Corollary 1. *Let $\{X_n,\ n \ge 1\}$ be a sequence of independent and identically distributed random variables. Assume X_1 belongs to the area of attraction of a stable law $G_{\alpha,\beta,0,\lambda}(\cdot)$, $\alpha \in (1, 2]$. Let $\{N_n,\ n \ge 1\}$ be a sequence of positive integer-valued random variables satisfying* (1.3) *or* (1.4). *If*

$$(N_n - n)/n^{1/\alpha} \xrightarrow{D} \ \text{(some)}\ \ A(\cdot), \quad \text{as} \quad n \to \infty, \tag{2.16}$$

then

$$(S_{N_n} - nEX_1)/(n\lambda)^{1/\alpha} \xrightarrow{D} \ \text{(some)}\ \ \Psi(\cdot), \quad \text{as} \quad n \to \infty, \tag{2.17}$$

where $\lambda = e_1(c_{1,1} + c_{2,1})$, and

$$\int e^{itx}\,\Psi(dx) = \exp\{-|t|^\alpha \lambda(1 + i\,\mathrm{sgn}(t)\omega(\alpha,\beta,t))\} \int_{\mathbb{R}} \exp\{-x|t|^\alpha i\,\mathrm{sgn}(t)\ \omega(\alpha,\beta,t) \\ (c_{1,1} - c_{2,1})e_2/\lambda^{1/\alpha} + itxEX_1/\lambda^{1/\alpha}\}\,A(dx).$$

If (2.17) *holds, then the sequence given in* (2.16) *is tight.*

Corollary 2. *Let $\{X_n,\ n \ge 1\}$ be a sequence of independent and identically distributed random variables. Assume X_1 belongs to the area of attraction of Cauchy law. Let $\{N_n,\ n \ge 1\}$ be a sequence of positive integer-valued random variables satisfying* (1.3) *or* (1.4). *If*

$$(N_n/n, (N_n \ln(N_n) - n\ln(n))/n) \xrightarrow{D} \ \text{(some)}\ \ A(\cdot,\cdot), \quad \text{as} \quad n \to \infty, \tag{2.18}$$

then

$$(S_{N_n} - n\mu - rn\ln(n))/(n\lambda) \xrightarrow{D} \ \text{(some)}\ \ \Psi(\cdot), \quad \text{as} \quad n \to \infty, \tag{2.19}$$

where $\lambda = e_1(c_{1,1}+c_{2,1})$, $r = e_1(c_{1,1}-c_{2,1})$,

$$\mu = \int_0^1 d_1(x)\,dx + \int_1^\infty (d_1(x) - (c_{1,1}-c_{2,1})/x)\,dx + (c_{1,1}-c_{2,1})\ [\gamma + e_2\ln(\lambda)],$$

and

$$\int e^{itx}\,\Psi(dx) =$$

$$= \iint\limits_{\mathbb{R}^2} \exp\{-|t|\lambda(x+(2x-1)i\,\mathrm{sgn}(t)\omega(1,\beta,t))+it(x+1)\mu+it\beta(y+1)\}\,A(dx,\,dy).$$

If (2.19) *holds, then the sequence given in* (2.18) *is tight.*

The next result deals with the central limit theorem. Here we can formulate a stronger result than in Theorems 1 and 2 (cf. Lemma 6 in Section 3).

Theorem 3. *Let* $\{X_n,\ n\ge 1\}$ *be a sequence of independent random variables such that* $EX_n = a_n$ and $E(X_n-a_n)^2 = \sigma_n^2 < \infty$, $n \ge 1$. *Let* $\{N_n,\ n\ge 1\}$ *be a sequence of positive integer-valued random variables satisfying* (1.3) *or* (1.4). *Let us put*

$$S_n = \sum_{k=1}^{n} X_k, \quad L_n = \sum_{k=1}^{n} a_k, \quad s_n^2 = \sum_{k=1}^{n}\sigma_k^2,$$
$$S_{N_n} = \sum_{k=1}^{N_n} X_k, \quad L_{N_n} = \sum_{k=1}^{N_n} a_k, \quad s_{N_n}^2 = \sum_{k=1}^{N_n}\sigma_k^2, \quad n \ge 1.$$

If

$$(S_n - L_n)/s_n \xrightarrow{D} N(0,1) \ \text{ as } \ n\to\infty, \tag{2.20}$$

and (2.3) *or* (2.4) *or* (2.5) *hold, then the following conditions are equivalent:*

$$(s_{N_n}^2/s_n^2, (L_{N_n}-L_n)/s_n) \xrightarrow{D} \ \text{(some)}\ \ A(\cdot,\cdot), \tag{2.21}$$

where A *is a two-dimensional distribution function,*

$$(S_{N_n} - L_n)/s_n \xrightarrow{D} \ \text{(some)}\ \ \Psi(\cdot), \tag{2.22}$$

where Ψ *is a distribution function.*

The distribution functions A *and* Ψ *are such that*

$$\int_{-\infty}^{\infty} \exp(itx)\,\Psi(dx) = \iint\limits_{\mathbb{R}^2} \exp(-t^2x/2 + ity)\,A(dx,\,dy). \tag{2.23}$$

Corollary 3. *Let $\{X_n,\, n \geq 1\}$ be a sequence of independent random variables such that $EX_n = \mu \neq 0$, $E(X_n-\mu)^2 = \sigma^2 < \infty$, $n \geq 1$, and*

$$(2.24)\qquad (S_n - n\mu)/\sigma\sqrt{n} \xrightarrow{D} N(0,1) \text{ as } n \to \infty.$$

Let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying (1.3) *or* (1.4). *Then the following conditions are equivalent:*

$$(2.25)\qquad (N_n - n)/\sqrt{n} \xrightarrow{D} \text{ (some) } G(\cdot), \text{ as } n \to \infty,$$

and

$$(2.26)\qquad (S_{N_n} - n\mu)/\sigma\sqrt{n} \xrightarrow{D} \text{ (some) } \Psi(\cdot), \text{ as } n \to \infty.$$

The distribution functions G and Ψ are such that

$$\int e^{itx}\, \Psi(dx) = \exp(-t^2/2) \int e^{it\mu x/\sigma}\, G(dx).$$

Let us observe that if, in addition, X_n, $n \geq 1$, are identically distributed, then (2.24) holds. Thus Corollary 3, under the assumption that the random variables $N_n, X_1, X_2, \ldots$ are independent for each $n \geq 1$, gives Theorem A.

3. Auxiliary lemmas.

In the proofs of the main results we need some lemmas. Let $\mathfrak{L}(X)$ denote the distribution of the random variable X.

Lemma 1. *Let $\{X_n,\, n \geq 1\}$ be a sequence of independent random variables and let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying* (1.3) *or* (1.4). *Let $\{Y_n,\, n \geq 1\}$ be a sequence of independent random variables and independent of $\{X_n,\, n \geq 1\}$ and $\{N_n,\, n \geq 1\}$ such that $\mathfrak{L}(X_n) = \mathfrak{L}(Y_n)$, $n \geq 1$. Let $\{s_n,\, n \geq 1\}$ and $\{L_n,\, n \geq 1\}$ be sequences of real numbers such that $s_n > 0$, $n \geq 1$, and $s_n \to \infty$ as $n \to \infty$. Let $Z_n = Y_1 + \cdots + Y_n$, $n \geq 1$. Assume* (2.1) *holds. Then the following conditions are equivalent:*

$$(3.1)\qquad (S_{N_n} - L_n)/s_n \xrightarrow{D} \text{ (some) } \Psi(\cdot), \text{ as } n \to \infty.$$

and

$$(3.2)\qquad (Z_{N_n} - L_n)/s_n \xrightarrow{D} \text{ (some) } G(\cdot), \text{ as } n \to \infty.$$

in which case $\Psi(\cdot) \equiv G(\cdot)$.

Proof: Let us observe that $\mathfrak{L}(S_n) = \mathfrak{L}(Z_n)$, $n \geq 1$, but in general $\mathfrak{L}(S_{N_n}) \neq \mathfrak{L}(Z_{N_n})$, $n \geq 1$, since N_n is independent of Y_n, $n \geq 1$, but may be dependent of X_n, $n \geq 1$.

Assume (1.4) holds. Then

$$
\begin{aligned}
I_n(t) &= |E\exp\{it(S_{N_n}-L_n)/s_n\} - E\exp\{it(Z_{N_n}-L_n)/s_n\}| = \\
&= |\sum_{m=1}^{\infty}[EI(N_n=m)\exp\{it(S_m-L_n)/s_n\} - \\
&\quad - EI(N_n=m)E\exp\{it(Z_m-L_n)/s_n\}]| \le \\
&\le \sum_{m=1}^{\infty}\mathbb{R}(n)P(N_n=m) = \mathbb{R}(n) \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}
\tag{3.3}
$$

Thus (3.1) holds if and only if (3.2) holds and $\Psi(\cdot)\equiv G(\cdot)$. We remark that under the assumption (1.4) we did not use (2.1).

Assume now (1.3) holds. Then by (2.1), for every $\varepsilon>0$, there exists a positive number K_ε such that for every $n\ge 1$

$$P(|S_n-L_n|/s_n \ge K_\varepsilon)\le\varepsilon.$$

Furthermore, we may and do assume $0<\varepsilon_1<\varepsilon_2$ implies $K_{\varepsilon_1}\ge K_{\varepsilon_2}$ and that $K_\varepsilon\to\infty$ as $\varepsilon\to 0$.

Let us put

$$
\begin{aligned}
\psi(n) &= \max\{k: s_k\le s_n^{1/2}\},\\
\varepsilon(n) &= 2\inf\{\varepsilon>0: K_\varepsilon < s_n^{1/4}, \varepsilon > s_n^{-1/4}\},\\
\varrho(n) &= \min\{\psi(n), (\varepsilon(n))^{-1/2}\}.
\end{aligned}
$$

We have $s_n\to\infty$, hence $\varepsilon(n)\to 0$, $\psi(n)\to\infty$ and $K_{\varepsilon(n)}\to\infty$ as $n\to\infty$. Furthermore, for every $1\le i\le\varrho(n)$

$$
\begin{aligned}
P(|S_i-L_i|/s_n > s_n^{-1/4}) &\le P(|S_i-L_i|/s_i s_n^{1/2} > s_n^{-1/4}) \le P(|S_i-L_i|/s_i > s_n^{1/4}) \le \\
&\le P(|S_i-L_i|/s_i > K_{\varepsilon(n)}) \le \varepsilon(n)
\end{aligned}
$$

and, in consequence,

$$P(|S_i-L_i|/s_n > s_n^{-1/4}, N_n=i)\le\varepsilon(n).$$

Thus, for every t such that $|t|<s_n^{1/8}$, we get

$$
\begin{aligned}
&|E(\exp\{it(S_{N_n}-L_n)/s_n\} - \exp\{it(L_{N_n}-L_n)/s_n\})\ I[N_n\le\varrho(n)]| \le \\
&\le E|(\exp\{it(S_{N_n}-L_{N_n})/s_n\}-1)|\ I[N_n\le\varrho(n), \max_{1\le i\le\varrho(n)}|S_i-L_i|/s_n\le s_n^{-1/4}] + \\
&+ \sum_{i\le\varrho(n)} 2P(|S_i-L_i|/s_n > s_n^{-1/4}) \le 2|t|s_n^{-1/4} + 2\varrho(n)\varepsilon(n) \le 4(\varepsilon(n))^{1/2}.
\end{aligned}
$$

Similarly, replacing S_i by Z_i, we get

$$|E(\exp\{it(Z_{N_n} - L_n)/s_n\} - \exp\{it(L_{N_n} - L_n)/s_n\})\ I[N_n \le \varrho(n)]| \le 4(\varepsilon(n))^{1/2},$$

so that

$$|E(\exp\{it(S_{N_n} - L_n)/s_n\} - \exp\{it(Z_{N_n} - L_n)/s_n\})\ I[N_n \le \varrho(n)]| \le 8(\varepsilon(n))^{1/2}.$$

On the other hand, step by step as in above, for $|t| < s_n^{1/8}$ we also get

$$E|\exp\{it(S_{[\varrho(n)]} - L_{[\varrho(n)]})/s_n\} - 1| \le 4(\varepsilon(n))^{1/2},$$

and

$$E|\exp\{it(Z_{[\varrho(n)]} - L_{[\varrho(n)]})/s_n\} - 1| \le 4(\varepsilon(n))^{1/2},$$

where $[x]$ denotes the integral part of x. Hence, taking into account the inequalities obtained above and using the triangle inequality, for $|t| \le s_n^{1/8}$ we have

$$\begin{aligned}
I_n(t) &\le |E(\exp\{it(S_{N_n} - L_n)/s_n\} - \\
&\quad - \exp\{it(Z_{N_n} - L_n)/s_n\})\ I[N_n > \varrho(n)]| + 8(\varepsilon(n))^{1/2} \le \\
&\le |E(\exp\{it(S_{N_n} - L_n - S_{[\varrho(n)]} + L_{[\varrho(n)]})/s_n\} - \\
&\quad - \exp\{it(Z_{N_n} - L_n - S_{[\varrho(n)]} + L_{[\varrho(n)]})/s_n\})\ I[N_n > \varrho(n)]| + 8(\varepsilon(n))^{1/2} \le \\
&\le |E(\exp\{it(S_{N_n} - L_n - S_{[\varrho(n)]} + L_{[\varrho(n)]})/s_n\} - \\
&\quad - \exp\{it(Z_{N_n} - L_n - Z_{[\varrho(n)]} + L_{[\varrho(n)]})/s_n\})\ I[N_n > \varrho(n)]| + \\
&\quad + |E(\exp\{it(Z_{N_n} - L_n - S_{[\varrho(n)]} + L_{[\varrho(n)]})/s_n\} - \\
&\quad - \exp\{it(Z_{N_n} - L_n - Z_{[\varrho(n)]} + L_{[\varrho(n)]})/s_n\})\ I[N_n > \varrho(n)]| + 8(\varepsilon(n))^{1/2} \le \\
&\le |E(\exp\{it(S_{N_n} - S_{[\varrho(n)]} - L_n + L_{[\varrho(n)]})/s_n\} - \\
&\quad - \exp\{it(Z_{N_n} - Z_{[\varrho(n)]} - L_n + L_{[\varrho(n)]})/s_n\})\ I[N_n > \varrho(n)]| + 16(\varepsilon(n))^{1/2} \le \\
&\le \sum_{k > \varrho(n)} |E\exp\{it(S_k - S_{[\varrho(n)]} - L_n + L_{[\varrho(n)]})/s_n\}\ I[N_k = k] - \\
&\quad - E\exp\{it(S_k - S_{[\varrho(n)]} - L_n + L_{[\varrho(n)]})/s_n\}\ P[N_n = k]| + 16(\varepsilon(n))^{1/2} \le \\
&\le r([\varrho(n)] + 1) + 16(\varepsilon(n))^{1/2} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus the proof of Lemma 1 is finished. □

Lemma 2. *If $\{X_n,\, n \ge 1\}$ and $\{Y_n,\, n \ge 1\}$ are tight sequences of random variables, then the following sequences are also tight:*

(a) $\{X_n + Y_n,\, n \ge 1\}$,
(b) $\{X_n Y_n,\, n \ge 1\}$,
(c) $\{(X_n, Y_n),\, n \ge 1\}$.

The proof is simple and therefore omitted.

Lemma 3. *Let $\{X_n,\, n \geq 1\}$ be a sequence of independent random variables and let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying* (1.3) *or* (1.4)*. Let $\{s_n,\, n \geq 1\}$ and $\{L_n,\, n \geq 1\}$ be sequences of real numbers such that $0 < s_n$, $n \geq 1$, and $s_n \to \infty$ as $n \to \infty$. Assume* (2.1) *and* (3.1) *hold with nondegenerate distribution function $F(\cdot)$, then the sequence $\{s_{N_n}/s_n,\, n \geq 1\}$ is tight.*

Proof: Let $\{Y_n,\, n \geq 1\}$ and $\{V_n,\, n \geq 1\}$ be independent sequences of independent random variables and independent of the sequences $\{X_n,\, n \geq 1\}$ and $\{N_n,\, n \geq 1\}$, such that $\mathfrak{L}(X_n) = \mathfrak{L}(Y_n) = \mathfrak{L}(V_n)$, $n \geq 1$. Let us put

$$Z_n = \sum_{k=1}^{n} Y_k, \quad U_n = \sum_{k=1}^{n} V_k, \quad n \geq 1.$$

Then

$$(Z_n - L_n)/s_n \xrightarrow{D} F(\cdot), \quad (U_n - L_n)/s_n \xrightarrow{D} F(\cdot), \qquad \text{as } n \to \infty,$$

and, by Lemma 1,

$$(Z_{N_n} - L_n)/s_n \xrightarrow{D} \Psi(\cdot), \quad (U_{N_n} - L_n)/s_n \xrightarrow{D} \Psi(\cdot), \qquad \text{as } n \to \infty.$$

By Lemma 2 (a) the sequences $\{(Z_{N_n} - U_{N_n})/s_n,\, n \geq 1\}$ and $\{(Z_n - U_n)/s_n,\, n \geq 1\}$ are tight. Moreover,

$$(Z_n - L_n)/s_n \xrightarrow{D} \int_{-\infty}^{\infty} F(x + \cdot)\, F(dx) \;\; \text{as } \; n \to \infty.$$

Because $F(\cdot)$ is nondegenerate distribution function and $\int F(x+\cdot)\, F(dx)$ is symmetric distribution function so that there exists $c > 0$ such that $\int F(x + c)\, F(dx) > 0$.

Assume that $\{s_{N_n}/s_n,\, n \geq 1\}$ is not tight. Thus, for some $\varepsilon > 0$ there exist the sequences $\{k_n,\, n \geq 1\}$ and $\{l_n,\, n \geq 1\}$ such that $k_n \in \{1, 2, \dots\}$, $k_n \to \infty$, $l_n \to \infty$ as $n \to \infty$, and $P(s_{N_{k_n}}/s_{k_n} > l_n) > \varepsilon$, $n \geq 1$. Hence, for sufficiently large n,

$$\begin{aligned}
P(Z_{N_{k_n}} - U_{N_{k_n}} \geq c l_n s_{k_n}) &\geq \sum_{m: s_m > l_n s_{k_n}} P(Z_m - U_m \geq c l_n s_{k_n})\; P(N_{k_n} = m) \geq \\
&\geq \sum_{m: s_m > l_n s_{k_n}} P(Z_m - U_m \geq c s_m)\; P(N_{k_n} = m) \geq \\
&\geq (1 - \int_{-\infty}^{\infty} F(x + c)\, F(dx))\; P(s_{N_{k_n}} \geq l_n s_{k_n})/2 \geq \\
&\geq (1/4)(1 - \int_{-\infty}^{\infty} F(x + c)\, F(dx))\; \varepsilon > 0.
\end{aligned}$$

Thus we get a contradiction, and this ends the proof. □

Lemma 4. *Let $\{Y_n,\, n \geq 1\}$ be a sequence of independent random variables and let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables independent of $\{Y_n,\, n \geq 1\}$. If $\{L_n,\, n \geq 1\}$ and $\{s_n,\, n \geq 1\}$ are sequences of real numbers such that $0 < s_n$, $n \geq 1$, $s_n \to \infty$ as $n \to \infty$ and the sequences $\{(Z_n - L_n)/s_n,\, n \geq 1\}$ and $\{s_{N_n}/s_n,\, n \geq 1\}$ are tight, then the sequence $\{(Z_{N_n} - L_{N_n})/s_n,\, n \geq 1\}$ is tight, too.*

Proof: We have

$$P(|Z_{N_n} - L_{N_n}|/s_{N_n} > K) = \sum_{m=1}^{\infty} P(|Z_m - L_m|/s_m > K)\; P(N_n = m) \leq \varepsilon$$

provided, for every $m \geq 1$, $P(|Z_m - L_m|/s_m > K) \leq \varepsilon$. Thus the sequence $\{(Z_{N_n} - L_{N_n})/s_{N_n},\, n \geq 1\}$ is tight, so that the sequence $\{(Z_{N_n} - L_{N_n})/s_n = ((Z_{N_n} - L_{N_n})/s_{N_n})(s_{N_n}/s_n),\, n \geq 1\}$ is tight by Lemma 2 (b). □

Lemma 5. *Let $\{L_n,\, n \geq 1\}$ and $\{s_n,\, n \geq 1\}$ be sequences of real numbers such that $0 < s_n$, $n \geq 1$, and $s_n \to \infty$ as $n \to \infty$. Let $\{N_n,\, n \geq 1\}$ be a sequence of positive integer-valued random variables such that the sequence $\{(L_{N_n} - L_n)/s_n,\, n \geq 1\}$ is tight. Then (2.3) or (2.4) or (2.5) implies (2.10).*

Proof: Assume (2.4) holds. Then

$$\begin{aligned} P(|L_{N_n} - L_n|/s_n > K) &\geq P(|N_n - n|/s_n > K/C) \geq \\ &\geq P((N_n - n)/s_n < -K/C) = P(N_n < s_n(n/s_n - K/C)). \end{aligned}$$

Thus, taking into account the tightness of $\{(L_{N_n} - L_n)/s_n,\, n \geq 1\}$ and the second part of (2.4), we get

$$P(N_n < s_n(n/s_n - n/2s_n)) = P(N_n < n/2) \to 0, \quad \text{as} \quad n \to \infty,$$

so that (2.10) holds with $\alpha(n) = n/2$, $n \geq 1$. Let us suppose (2.5). If $L_n/s_n \to \infty$ as $n \to \infty$, then

$$P(|L_{N_n} - L_n|/s_n > K) \geq P((L_{N_n} - L_n)/s_n < -K) = P(L_{N_n} < s_n(L_n/s_n - K)).$$

Now the tightness and $L_n/s_n \to \infty$ as $n \to \infty$ imply

$$P(L_{N_n} < s_n(L_n/s_n - L_n/2s_n)) = P(L_{N_n} < L_n/2) \to 0, \quad \text{as} \quad n \to \infty,$$

so that (2.10) holds with $\alpha(n) = \inf\{k \in \mathbb{N} : L_k \geq L_n/2\}$, $n \geq 1$. Of course, since $L_n \to \infty$ as $n \to \infty$, we get $\alpha(n) \to \infty$ as $n \to \infty$.

If $L_n/s_n \to \infty$ as $n \to \infty$, the proof of (2.10) is the same. The equivalence of (2.3) and (2.10) has been explained after Theorem 1. □

Lemma 6. *Let $A(\cdot,\cdot)$ and $A'(\cdot,\cdot)$ be two distribution functions. If for every $t \in \mathbb{R}$*

$$\int\!\!\int \exp(-t^2x/2 + ity)\, A'(dx,\, dy) = \int\!\!\int \exp(-t^2x/2 + ity)\, A(dx,\, dy),$$

then $A = A'$

The proof is easy and therefore omitted.

Lemma 7. *Let $\{X_n,\, n \ge 1\}$ be a sequence of independent random variables and let $\{N_n,\, n \ge 1\}$ be a sequence of positive integer-valued random variables independent of $\{X_n,\, n \ge 1\}$ and satisfying (2.10). Assume for arbitrary $\tau > 0$, some sequence of real numbers $\{a_k,\, k \ge 1\}$ and nondecreasing sequence of positive real numbers $\{s_n,\, n \ge 1\}$,*

$$\sum_{j=1}^{n}\Big(b_j + \int_{-\infty}^{\infty} x/(1+x^2)\, dF_j(x+b_j) - a_j\Big)/s_n \to \gamma, \quad \text{as} \;\; n \to \infty, \tag{3.4}$$

and uniformly on compact sets with respect to t

$$\begin{aligned} &\oint_{-\infty}^{\infty} (e^{itx/s_n} - 1 - itx/((1+x^2)s_n))\, d\sum_{j=1}^{n} F_j(x+b_j) \to \\ &\quad \to \oint_{-\infty}^{\infty} (e^{itx} - 1 - itx/(1+x^2))\;(1+x^2)/x\, G(dx), \quad \text{as} \;\; n \to \infty, \end{aligned} \tag{3.5}$$

where

$$F_j(x) = P[X_j < x], \quad b_j = \int_{|x|<\tau} x\, dF_j(x), \quad j \ge 1,$$

and $G(\cdot)$ is nondecreasing bounded function. Then uniformly on compact sets

$$\begin{aligned} J_n(t) = \Big|&E \exp\{it\sum_{j=1}^{N_n}(b_j + \\ &+ \int_{-\infty}^{\infty} x/(1+x^2)\, dF_j(x+b_j) - a_j)/s_{N_n}(s_{N_n}/s_n) + it(L_{N_n} - L_n)/s_n + \\ &+ \oint_{-\infty}^{\infty}(e^{itx/s_{N_n}(s_{N_n}/s_n)} - 1 - itx(s_{N_n}/s_n)/((1+x^2)s_{N_n}))\, d\sum_{j=1}^{N_n} F_j(x+b_j)\} - \\ &- E\exp\{it\gamma(s_{N_n}/s_n) + it(L_{N_n} - L_n)/s_n + \\ &+ \oint_{-\infty}^{\infty}(e^{itx(s_{N_n}/s_n)} - 1 - itx(s_{N_n}/s_n)/(1+x^2))(1+x^2)/x\, dG(x)\}\Big| \to 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{as } n \to \infty, \end{aligned}$$

where

$$L_n = \sum_{j=1}^{n} a_j, \quad L_{N_n} = \sum_{j=1}^{N_n} a_j, \quad n \ge 1.$$

Proof: Let us remark that for every $\varepsilon > 0$

$$P[|\sum_{j=1}^{N_n}(b_j + \int_{-\infty}^{\infty} x/(1+x^2)\,dF_j(x+b_j) - a_j)/s_{N_n} - \gamma| > \varepsilon] \leq$$

$$\leq P[N_n \leq \alpha(n)] + \sup_{k_n \geq \alpha(n)} |\sum_{j=1}^{k_n}(b_j +$$

$$+ \int_{-\infty}^{\infty} x/(1+x^2)\,dF_j(x+b_j) - a_j)/s_{k_n} - \gamma|/\varepsilon \to 0 \quad \text{as} \quad n \to \infty,$$

where $\{\alpha(n),\, n \geq 1\}$ is defined in (2.10). Similarly

$$P[\sup_{|t|<K_1} |\oint_{-\infty}^{\infty} (e^{itx/s_{N_n}} - 1 - itx/((1+x^2)s_{N_n}))\,d\sum_{j=1}^{N_n} F_j(x+b_j) -$$

$$- \oint_{-\infty}^{\infty} (e^{itx} - 1 - itx/(1+x^2))(1+x^2)/x\,dG(x)| > \varepsilon] \to 0 \quad \text{as } n \to \infty.$$

On the other hand, for each positive number K_i, ε_i, $i = 1, 2$, we have

$$\sup_{|t|<K_1} J_n(t) \leq P[|s_{N_n}/s_n| > K_2] +$$

$$+ 2P[|\sum_{j=1}^{N_n}(b_j + \int_{-\infty}^{\infty} x/(1+x^2)\,dF_j(x+b_j) - a_j)/s_{N_n} - \gamma| > \varepsilon_1/K_1] +$$

$$+ 2P[\sup_{|y|<K_1K_2} |\oint_{-\infty}^{\infty} (e^{iyx/s_{N_n}} - 1 - iyx/((1+x^2)s_{N_n}))\,d\sum_{j=1}^{N_n} F_j(x+b_j) -$$

$$- \oint_{-\infty}^{\infty} (e^{iyx} - 1 - iyx/(1+x^2))\ (1+x^2)/x\,dG(x)| > \varepsilon_2] + 2\varepsilon_1 + 2\varepsilon_2, \quad n \geq 1.$$

Let now $K_1 > 1$ and ε be arbitrary positive numbers and let n_1 be such that for every $n \geq n_1$

$$P[|\sum_{j=1}^{N_n}(b_j + \int_{-\infty}^{\infty} x/(1+x^2)\,dF_j(x+b_j) - a_j)/s_{N_n} - \gamma| > \varepsilon/(9K_1)] \leq \varepsilon/9.$$

Now we put K_2 such that for every $n \geq n_1$

$$P[|s_{N_n}/s_n| > K_2] \leq \varepsilon/9,$$

and n_2 such that for every $n \geq n_2$

$$P[\sup_{|y|<K_1K_2} |\oint_{-\infty}^{\infty} (e^{iyx/s_{N_n}} - 1 - iyx/((1+x^2)s_{N_n}))\,d\sum_{j=1}^{N_n} F_j(x+b_j) -$$

$$- \oint_{-\infty}^{\infty} (e^{iyx} - 1 - iyx/(1+x^2))\ (1+x^2)/x\,dG(x)| > \varepsilon/9] \leq \varepsilon/9.$$

Thus for every $n \geq \max(n_1, n_2)$

$$\sup_{|t|<K_1} J_n(t) \leq \varepsilon/9 + 2\varepsilon/9 + 2\varepsilon/9 + 2\varepsilon/9 + 2\varepsilon/9 = \varepsilon,$$

which ends the proof. □

4. Proofs.

Proof of Theorem 1: At first we prove that (2.6) ⇒ (2.7). Let $\{U_n,\, n \geq 1\}$ be a sequence of independent random variables and independent of $\{N_n,\, n \geq 1\}$ and such that

$$\int e^{itx} \mathfrak{L}(U_n)\,(dx) = \exp\{it(b_n + \int_{-\infty}^{\infty} x/(1+x^2)\, F_n(dx + b_n)) + \\ + \oint (e^{itx} - 1 - itx/(1+x^2))\; F_n(dx + b_n)\},$$

where

$$b_n = \int_{|x|<1} x\, dF_n(x), \quad F_n(x) = P[X_n < x], \quad n \geq 1.$$

By Lemma 1 we may and do assume that $\{X_n,\, n \geq 1\}$ and $\{N_n,\, n \geq 1\}$ are independent. Note that by Theorem 4 [7, Chapter IV, § 2, p. 115] and Lemma 5, the assumptions of Lemma 7 hold. By Lemma 7 it is enough to prove that

$$I_n(t) = |E\exp\{it(V_{N_n} - L_n)/s_n\} - E\exp\{it(S_{N_n} - L_n)/s_n\}| \to 0, \quad \text{as} \quad n \to \infty,$$

uniformly on compact sets with respect to t, where

$$V_n = \sum_{j=1}^{n} U_j.$$

Let C and ε be arbitrary positive numbers. Let $n_1 \in \mathbb{N}$ be such that

$$P[N_n < \alpha(n)] < \varepsilon/3,$$

for every $n \geq n_1$. Here, and in what follows, $\{\alpha(n),\, n \geq 1\}$ is defined in Lemma 5. By (2.6) we may put C_ε such that

$$P[|s_{N_n}/s_n| > C_\varepsilon] \leq \varepsilon/3,$$

for every $n \geq n_1$. By (3.5) and (3.6) it is possible to choose $n_2 \in \mathbb{N}$ such that

$$\sup_{|u|<CC_\varepsilon}\ \sup_{j:j>\alpha(n)} |E\exp\{iu(S_j - L_j)/s_j\} - E\exp\{iu(V_j - L_j)/s_j\}| < \varepsilon/3,$$

for every $n \geq n_2$. Thus

$$\sup_{|yt|<C} I_n(t) \leq \int_{0<x<C_\varepsilon} \sup_{|t|<C} \sup_{j:j>\alpha(n)} |E\exp\{itx(S_j - L_j)/s_j\} - E\exp\{itx(V_j - L_j)/s_j\}| + P[N_n < \alpha(n)] + P[s_{N_n}/s_n > C_\varepsilon] < \varepsilon, \quad \text{for } \ n > \max(n_1, n_2).$$

Since the left hand side of the above inequality is independent of ε, we have

$$\lim_{n\to\infty} \sup_{|t|<C} I_n(t) = 0.$$

Thus the proof that (2.6) $\Rightarrow$ (2.7) is ended.

Assume now that (2.7) holds. Then, by Lemma 3, the sequence $\{s_{N_n}/s_n,\ n \geq 1\}$ is tight. Moreover, by Lemma 1 and Lemma 4, the sequence $\{(Z_{N_n} - L_n)/s_n,\ n \geq 1\}$ and $\{(Z_{N_n} - L_{N_n})/s_n,\ n \geq 1\}$ are tight, too, where $\{Z_n,\ n \geq 1\}$ is the sequence defined in Lemma 1. Thus by Lemma 2 (a) the sequence $\{(L_{N_n} - L_n)/s_n,\ n \geq 1\}$ is also tight, so that Lemma 2 (c) implies the tightness of the sequence $\{(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n),\ n \geq 1\}$. □

Proof of Theorem 2: The implication (2.14) $\Rightarrow$ (2.15) can be proved similarly as the implication (2.5), (2.6) $\Rightarrow$ (2.7). In this case, let $\{U_n,\ n \geq 1\}$ be a sequence of independent random variables and independent of $\{N_n,\ n \geq 1\}$ and such that $\mathfrak{L}(U_n) = G_{\alpha,(c_{1,n}-c_{2,n})e_2,0,(c_{1,n}+c_{2,n})e_1}(\cdot)$, $n \geq 1$, then

$$\begin{aligned}
&E\exp\{it(\sum_{j=1}^{N_n} U_j - L_n)/s_n\} = E\exp\{-|t|^\alpha(s_{N_n}^\alpha/s_n^\alpha + i\,\mathrm{sgn}(t)\omega(\alpha, \beta_{N_n}/s_n^\alpha, t)) + \\
&+ it(L_{N_n} - L_n)/s_n\} = E\exp\{-|t|^\alpha(s_{N_n}^\alpha/s_n^\alpha + i\,\mathrm{sgn}(t)(\beta_n/s_n^\alpha)\omega(\alpha, 1, t)) - \\
&- |t|^\alpha i\,\mathrm{sgn}(t)((\beta_{N_n} - \beta_n)/s_n^\alpha)\omega(\alpha, 1, t) + it(L_{N_n} - L_n)/s_n\} \to \\
&\to \iiint_{\mathbb{R}^3} \exp\{-|t|^\alpha(x + i\,\mathrm{sgn}(t)\omega(\alpha, \beta, t)) - \\
&- |t|^\alpha i\,\mathrm{sgn}(t)\omega(\alpha, 1, t)y + itz\}\, A(dx,\, dy,\, dz),
\end{aligned}$$

as $n \to \infty$. We omit further details.

The second part of Theorem 2 can also be obtained similarly as the second part of Theorem 1. Namely, as in Theorem 1, we prove that the sequence $\{(s_{N_n}/s_n, (L_{N_n} - L_n)/s_n),\ n \geq 1\}$ is tight. Thus the sequence $\{(s_{N_n}^\alpha/s_n^\alpha, (L_{N_n} - L_n)/s_n),\ n \geq 1\}$ is tight, too. Now (2.15) follows, if we show that the sequence $\{(\beta_{N_n} - \beta_n)/s_n^\alpha,\ n \geq 1\}$ is tight. But this fact follows from the tightness of the sequence $\{s_{N_n}^\alpha/s_n^\alpha,\ n \geq 1\}$. Namely, we have

$$|\beta_n/s_n^\alpha| \leq 1, \quad |\beta_{N_n}/s_{N_n}^\alpha| \leq 1 \ \text{ a.s.}$$

and

$$|\beta_{N_n} - \beta_n|/s_n^\alpha \leq s_{N_n}^\alpha/s_n^\alpha + 1 \ \text{ a.s.}$$

Hence the proof of Theorem 2 is completed. □

Proof of Theorem 3: The implication (2.21) ⇒ (2.22) follows from the first part of Theorem 1 as the Gaussian law is the special case of Levy laws. The tightness of sequence defined on the left hand side of (2.21) follows from Theorem 1, too. Assume that

$$(s_{N_{n'}}/s_{n'}, (L_{N_{n'}} - L_{n'})/s_{n'}) \xrightarrow{D} A'(\cdot,\cdot) \text{ as } n' \to \infty$$

and

$$(s_{N_{n''}}/s_{n''}, (L_{N_{n''}} - L_{n''})/s_{n''}) \xrightarrow{D} A''(\cdot,\cdot) \text{ as } n'' \to \infty.$$

Then applying two times the implication (2.21) ⇒ (2.22), which is already proved, we get

$$\widehat{\Psi}(t) = \iint_{\mathbb{R}^2} \exp(-t^2x/2 + ity)\, A'(dx,\, dy) = \iint_{\mathbb{R}^2} \exp(-t^2x/2 + ity)\, A''(dx,\, dy).$$

By Lemma 6, $A' = A''$, which ends the proof of Theorem 3. □

Corollaries 1, 2 and 3 easily follow from Theorems 2 and 3, respectively. We note only that if

$$(N_n - n)/n^{1/\alpha} \xrightarrow{D} \text{ (some) } A(\cdot), \text{ as } n \to \infty, \quad 0 < \alpha < 2,$$

then

$$N_n/n \xrightarrow{P} 1, \text{ as } n \to \infty.$$

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