On direct sums of $\mathcal{B}^{(1)}$-groups

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Abstract. A necessary and sufficient condition is given for the direct sum of two $\mathcal{B}^{(1)}$-groups to be (quasi-isomorphic to) a $\mathcal{B}^{(1)}$-group. A $\mathcal{B}^{(1)}$-group is a torsionfree Abelian group that can be realized as the quotient of a finite direct sum of rank 1 groups modulo a pure subgroup of rank 1.

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All groups in the following are Abelian and of finite rank.

Let $\mathcal{B}^{(n)}$ be the class of groups that can be realized as quotients of a completely decomposable group modulo a pure subgroup of rank n. Any such realization is called a *representation* of the group. The union of all classes $\mathcal{B}^{(n)}$ for $n = 1, 2, 3, \dots$ is the well known class of Butler groups. In [FM] the class $\mathcal{B}^{(1)}$ is investigated, showing that the major quasi-isomorphism-invariant properties of a $\mathcal{B}^{(1)}$-group are easily recognized on its *representation type*, that is the isomorphy class of the completely decomposable group occurring as the numerator in the representation. The representation type itself determines the group up to quasi-isomorphism. In this context, "quasi-isomorphic" means "isomorphic to a subgroup of finite index"; this weaker form of isomorphism is the most natural one for a first broad study of classes of torsionfree groups of finite rank, and we will assume it as our standard approach in this paper. For more on quasi-isomorphism, we refer to [F II].

An example is given in [FM] to show that the direct sum of two $\mathcal{B}^{(1)}$-groups need not be quasi-isomorphic to a $\mathcal{B}^{(1)}$-group. We give here a necessary and sufficient condition for this to happen. Following the philosophy of [FM], the condition will consist operatively of a simple check to be performed on the representation types of the groups. The necessary results from [FM] will be quoted without proof; the only other result needed is the main result in [H]. In the references, we quote some other papers, both published and in printing, dealing with $\mathcal{B}^{(1)}$-groups from a similar angle.

Let

$$G = \sum_{i=1}^{m} \langle g_i \rangle_* \qquad (g_i \in G)$$

be a $\mathcal{B}^{(1)}$-group, with

$$t_i = t(g_i)$$

the type of g_i (that is the isomorphy class of $\langle g_i\rangle_*$), and

$$T = T(G) = \{t_1, \dots, t_m\}$$

the *representation type* of G. (By this we mean that G is the quotient of the outer direct sum $\bigoplus_{i=1}^m \langle g_i\rangle_*$ modulo a pure subgroup of rank 1.) Note that G may have different representation types; see [FM, Example 4.1].

We first reduce the problem to regular $\mathcal{B}^{(1)}$-groups. From [FM, 1] we know that all $\langle g_i\rangle_*$ whose type t_i is not $\geq \inf\{t_j \mid j \neq i\}$ are direct summands of G. If we drop them, while maintaining the above notation, T becomes *regular*, that is the infimum if its types coincides with the infimum of all but (any) one of them. Then by [FM, 1.2] we may suppose G itself is *regular*, which means that the only relation holding in G is $\sum_{i=1}^m g_i = 0$ and its consequences.

We now define a set T_I of types of elements of G. As in [FM, 2.3], let

$$I(G) = I = \{1, \dots, m\}$$

be the index set of the representation, and for $E \subset I$[1] set

$$\tau_E = \bigwedge_{i \in E} t_i$$

(thus in particular $\tau = \tau_I$ is the *minimum type* of G). Then

$$t_E = \tau_E \vee \tau_{I \setminus E}$$

is the type of

$$g_E = \sum_{i \in E} g_i.$$

Finally set

$$T_I = \{t_E \mid E \subset I,\ E \neq \{i\},\ E \neq I \setminus \{i\} \ \text{ for each } \ i \in I\}.$$

(For the $\mathcal{B}^{(1)}$-group $H = \sum_{j=1}^n \langle h_j\rangle_*$ the notation will be: $u_j = t(h_j)$, $U = T(H) = \{u_1, \dots, u_n\}$, $J = I(H) = \{1, \dots, n\}$, and, for $F \subset J$, $\upsilon_F = \bigwedge_{j \in F} u_j$ with $\upsilon = \upsilon_j$ the minimum type of H; then $u_F = \upsilon_F \vee \upsilon_{J \setminus F}$ is the type of $h_F = \sum_{j \in F} h_j$. U_J is defined similarly to T_I.)[2]

If G is not strongly decomposable, by 3.2 and 3.3 of [FM] quasi-decomposability of G is signalled by a type $t_E = t(\sum_{i \in E} g_i) \in T_I$ matching or exceeding some type

[1] The symbols $\subset$, $\supset$ denote proper containment.
[2] υ = greek u.

$t_i \in T$; by the symmetry of the definition of t_E we may suppose $i \notin E = \{1, \dots, k\}$ (say, for some $k < m - 1$). This (resorting, if necessary, to a quasi-isomorphic image) will then yield

$$G = G_E \oplus G_F, \quad \text{with} \quad I = E \mathbin{\dot\cup} \{i\} \mathbin{\dot\cup} F$$

where[3]

$$G_E = \sum_{i \in E} \langle g_i \rangle_* + \langle g_{I \setminus E} \rangle_*$$

is again a $\mathcal{B}^{(1)}$-group with regular representation type T' and index set I'

$$T' = T(G_E) = \{t_1, \dots, t_k, t_E\}$$
$$I' = I(G_E) = \{1, \dots, k, E\}.$$

The types signalling a possible splitting in G_E are again of the form $t(\sum_{j \in K} g_j)$ for $K \subset I'$; this is t_K, if $K \subset \{1, \dots, k\}$; $t_{K \cup (I \setminus E)}$ otherwise. Moreover, such a type must either be $\geq t_j$ for some $j = 1, \dots, k$ or $\geq t_E$ which was $\geq t_i$. Hence a quasi-splitting of G_E is already signalled in G by a type (t_K, or $t_{K \cup (I \setminus E)}$) of T_I. Since in a finite number of steps we are bound to reach the uniquely determined quasi-decomposition of G into strongly indecomposable summands [FM, 3.5], clearly the partial order of T_I yields all the information needed to decompose G.

Let us first consider a special situation in which the solution is simplest.

Lemma 1. *Let G, H have representation types T resp. U with $T \cap U \neq \emptyset$. Then $G \oplus H$ is a $\mathcal{B}^{(1)}$-group.*

Proof: In this case, setting $H = \sum_{j=1}^{n} \langle h_j \rangle_*$, and supposing g_m and h_1 are the elements with the same type (and, without loss of generality, characteristic), we have

$$G \oplus H \geq K = \langle g_1 \rangle_* + \dots + \langle g_{m-1} \rangle_* + \langle g_m + h_1 \rangle_* + \langle h_2 \rangle_* + \dots + \langle h_n \rangle_* \,.$$

Now $h_1 = (g_1 + \dots + g_{m-1}) + (g_m + h_1)$, and the characteristics of the three elements are the same, therefore $K \geq \langle h_1 \rangle_*$. By symmetry this holds for $\langle g_m \rangle_*$ as well, therefore K contains both G and H. □

This result generalizes to a necessary and sufficient condition.

Define the type t to be a *basic type* of G if t belongs to some representation type T of G.

Theorem 1. *Let G, H be regular $\mathcal{B}^{(1)}$-groups with minimum types τ resp. υ. Then $G \oplus H$ is a $\mathcal{B}^{(1)}$-group if and only if*

(∗) *there are basic types t of G and u of H such that $t \vee u \leq \tau \vee \upsilon$.*

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[3] $\dot\cup$ means disjoint union.

Proof: For sufficiency, suppose without loss of generality $t = t_m$ and $u = u_1$, and consider the group $K = \langle g_1 \rangle_* + \cdots + \langle g_{m-1} \rangle_* + \langle g_m + h_1 \rangle_* + \langle h_2 \rangle_* + \cdots + \langle h_n \rangle_* \leq G \oplus H$. Clearly K is a regular $\mathcal{B}^{(1)}$-group, $h_1 = g_1 + \cdots + g_{m-1} + (g_m + h_1)$, and the type u_1 of h_1 in H is greater than or equal to the type of h_1 in K, namely $(t_1 \wedge \cdots \wedge t_{m-1} \wedge (t_m \wedge u_1)) \vee (u_2 \wedge \cdots \wedge u_n) = (\tau \wedge u_1) \vee \upsilon$ (by the regularity of U) $= (\tau \vee \upsilon) \wedge (u_1 \vee \upsilon)$. It is easy to verify that this equals u_1 if and only if $u_1 \leq \tau \vee \upsilon$. This proves quasi-equality of K and $G \oplus H$; but since, for a suitable choice of the elements g_i in $\langle g_i \rangle_*$ and h_j in $\langle h_j \rangle_*$, the above equalities will be satisfied by their characteristics [FM, 2.1], we get $K = G \oplus H$.

Necessity requires some deeper probing, which was done in [H]. There it is proved that every summand of a $\mathcal{B}^{(1)}$-group G' is quasi-isomorphic to one of the form G'_E for some $E \subset I$, so that we may restrict our consideration to the standard situation $G' = G'_E \oplus G'_F$ with $I = E \overset{\bullet}{\cup} \{i\} \overset{\bullet}{\cup} F$. Here $g_i = g_{I \setminus E} + g_{I \setminus F}$ entails $t_i = t_E \wedge t_F$; therefore $t_F = (\tau_E \wedge t_i) \vee \tau_F = (\tau_E \wedge t_F) \vee \tau_F = (\tau_E \vee \tau_F) \wedge (t_F \vee \tau_F)$ yields $t_F \leq \tau_E \vee \tau_F$ (note, from above, that t_F is indeed a basic type of G'_F). This is the desired conclusion, since by the regularity of G'_E the infimum of its types is indeed τ_E, and since the same conclusion can be drawn for t_E. □

Observation 1. A representation type of $G \oplus H$ is $\{t_1, \ldots, t_{m-1}, t_m \wedge u_1, u_2, \ldots, u_n\}$.

Observation 2. An equivalent condition for the quasi-decomposability of a $\mathcal{B}^{(1)}$-group G' is for it to have a regular representation type $\{t_1, \ldots, t_{m-1}, t, u_2, \ldots, u_n\}$ with $t \leq \tau \vee \upsilon$ (where $\tau = \bigwedge_{j=1}^{m-1} t_i$ and $\upsilon = \bigwedge_{j=2}^{n} u_i$): then in the above notation $t_E = \tau \vee (\upsilon \wedge t) = (\tau \vee \upsilon) \wedge (\tau \vee t) = (\tau \vee t) \geq t$. Here $t_F = \upsilon \vee (\tau \wedge t) = (\upsilon \vee t)$, hence the two new types needed to complete the representations of G_E and G_F are $t_m = \tau \vee t$ and $u_1 = \upsilon \vee t$.

In the special case where H is completely decomposable, $G \oplus H$ is always a $\mathcal{B}^{(1)}$-group, having as its only relation the one holding in G. If we want it to be regular, though, the following restriction applies:

Corollary. *Let G be a regular $\mathcal{B}^{(1)}$-group, H completely decomposable. $G \oplus H$ is a regular $\mathcal{B}^{(1)}$-group if and only if each extractible type of H is greater than or equal to some basic type of G.*

Proof: Consider first the case where H is of rank 1 and type t. A regular representation for H has representation type $\{u_1, u_2\}$ where $u_1 = u_2 = t = \upsilon$. $\tau \leq t$ is required for the regularity of $G \oplus H$. $t \leq \tau \vee \upsilon$ is trivially satisfied, while $t_i \leq \tau \vee t = t$ remains the only condition. The extension to the finite rank case is now immediate. □

Summing up, we get

Theorem 2. *Let A, B be $\mathcal{B}^{(1)}$-groups, $A = G \oplus Y$, $B = H \oplus Z$, where Y, Z are completely decomposable and G, H are regular $\mathcal{B}^{(1)}$-groups. $A \oplus B$ is a $\mathcal{B}^{(1)}$-group if and only G and H satisfy $(*)$.*

Since the interplay between representation types (mirroring the structural properties of $\mathcal{B}^{(1)}$-groups) is by no means transparent, we give now a different interpretation to the condition $(*)$, rephrasing it in terms of types:

$(**)$ there are types t in T and u in U such that $t \vee u \leq \tau \vee \upsilon$.

We need the following lemma, whose proof is the same as the proof of [FM, 3.1]:

Lemma 2. *In the above notation, if G is a $\mathcal{B}^{(1)}$-group and $t_i \geq t_j$ for some $i \neq j \in I$, then $G = \langle g_i \rangle_* \oplus G'$, where G' is a $\mathcal{B}^{(1)}$-group, and is regular if G was regular.*

Theorem 3. *Let the $\mathcal{B}^{(1)}$-groups G, H have regular representation types T resp. U. If T and U satisfy $(**)$, a regular $\mathcal{B}^{(1)}$-group A with representation type $T \overset{\bullet}{\cup} U$ is quasi-isomorphic to $G \oplus H \oplus R$, where R is a rank 1 group of type $\tau \vee \upsilon$.*

PROOF: Let A be the quotient of the outer direct sum $\bigoplus_{i=1}^m \langle g_i \rangle_* \oplus \bigoplus_{j=1}^n \langle h_j \rangle_*$ modulo the rank 1 subgroup $\langle \sum_{i=1}^m g_i + \sum_{j=1}^n h_j \rangle_*$. A is then a regular $\mathcal{B}^{(1)}$-group, so the types of the elements g_i, h_j in A are the same they have in G resp. H; and $\tau \vee \upsilon$ is the type of the element $a_I = \sum_{i=1}^m g_i$ of A. As before, set $t = t_m$ and $u = u_1$. The part of the condition $(*)$ requiring $u_1 \leq \tau \vee \upsilon$ entails a quasi-splitting $A = G' \oplus H'$, where the representation type of G' is $T' = \{t_1, \dots, t_m, \tau \vee \upsilon\}$ and the one of H' is $U' = \{u_2, \dots, u_n, (\tau \vee \upsilon) \wedge u_1\}$. The last type is in fact $= u_1$, thus H' is quasi-isomorphic to H. As for G', $t_m \leq \tau \vee \upsilon$ entails by Lemma 2 the splitting of G' into $G \oplus R$, where R is a rank 1 group of type $\tau \vee \upsilon$. Therefore A is quasi-isomorphic to $G \oplus H \oplus R$. □

Observation 3. From Observation 2 we see that A has, besides $T \overset{\bullet}{\cup} U$, also the representation type $\{t_1, \dots, t_{m-1}, t_m \wedge u_1, u_2, \dots, u_n, \tau \vee \upsilon\}$.

As a last remark, we note that the property of being a summand of a $\mathcal{B}^{(1)}$-group is not "translation invariant" on the typeset of G. In fact, if the minimum type of the $\mathcal{B}^{(1)}$-group G is the type of $\mathbb{Z}$, and $G \oplus H$ is a $\mathcal{B}^{(1)}$-group, then the condition and Lemma 2 imply that, if H is itself a $\mathcal{B}^{(1)}$-group, it must be quasi-decomposable; while this need not be true for a $\mathcal{B}^{(1)}$-group G' with a higher minimum type.

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