# Topos based homology theory

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*Abstract.* In this paper we extend the Eilenberg-Steenrod axiomatic description of a homology theory from the category of topological spaces to an arbitrary category and, in particular, to a topos. Implicit in this extension is an extension of the notions of homotopy and excision. A general discussion of such homotopy and excision structures on a category is given along with several examples including the interval based homotopies and, for toposes, the excisions represented by "cutting out" subobjects. The existence of homology theories on toposes depends upon their internal logic. It is shown, for example, that all "reasonable" homology theories on a topos in which De Morgan's law holds are trivial. To obtain examples on non-trivial homology theories we consider singular homology based on a cosimplicial object. For toposes singular homology satisfies all the axioms except, possibly, excision. We introduce a notion of "tightness" and show that singular homology based on a sufficiently tight cosimplicial object satisfies the excision axiom. Characterizations of various types of tight cosimplicial objects in the functor topos $\mathrm{Sets}^C$ are given and, as a result, a general method for constructing non-trivial homology theories is obtained. We conclude with several explicit examples.

*Keywords:* singular homology, homotopy, excision, topos, interval

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This paper may be viewed as a contribution to the foundation of E.S. (Eilenberg-Steenrod axiomatic) homology theory. To understand such a homology theory in a topos based topological category one must first understand it in the underlying topos, represented by the discrete objects. In order to formulate the E.S. axioms in a topos, or in any category, two structures, namely, "homotopy" and "excision" must be present. A general discussion of these structures, along with examples, is given, first for categories in general, and then for toposes. In the latter case the internal logic of the topos is seen to be central to the existence of non-trivial homology theories. It is proved, for example, that all "reasonable" homology theories on a De Morgan topos are trivial. For examples of non-trivial theories, it is shown that "singular homology" based on a cosimplicial object satisfies the E.S. axioms under certain "tightness" conditions on the cosimplicial object. Characterizations of various types of tight cosimplicial objects, including internal categories and intervals, in a topos and in the functor topos $\mathrm{Sets}^C$ are given. These considerations lead to a method for constructing non-trivial homology theories, including non-standard homology theories for topological spaces, as well as providing a uniform approach to such classical notions as singular homology for topological spaces and homology of simplicial sets. Moreover, the role of "logic" is brought out by the fact that for topological categories based on a De Morgan topos (e.g. Set-based) the "topology" carries the "homology", i.e. discrete objects are homologically trivial, while for simplicial topological categories, for example, this need not be so.

## 1. Homology in a category.

We begin with a discussion of structures on a category which enable one to describe homology theories for it in a manner analogous to the usual Eilenberg-Steenrod axiomatic description of homology for the category of topological spaces. One basic concept is that of "homotopy". For a category $E$ let $PE$ be the category of pairs in $E$ (a pair $(Y, A)$ consists of an object $Y$ of $E$ and a subobject $A \subset Y$ (i.e. a mono) and a morphism $(Y, A) \to (Z, B)$ of pairs is any map $Y \to Z$ of $E$ for which $A \subset Y \to Z$ factors through $B \subset Z$). By a homotopy structure on a subcategory $E_1$ of $PE$ we mean a congruence on $E_1$ in the sense of [18, p. 52]. Thus a homotopy structure on $E_1$ consists of an equivalence relation $\sim$ on each of the hom-sets of $E_1$ that is preserved by composition, i.e. if $f_0 \sim f_1$ then $kf_0 \sim kf_1$ and $f_0 g \sim f_1 g$ for all maps $k, g$ of $E_1$ for which the indicated compositions are defined. Note that a homotopy structure on $E_1$ restricts to a homotopy structure on any subcategory of $E_1$.

An important example of a homotopy structure on $PE$, when $E$ has finite products, is as follows: Let $\partial_i : X_0 \to X_1$, $i = 0, 1$, be a pair of maps in $E$. For $f_i : (Y, A) \to (Z, B)$, $i = 0, 1$, define $f_0 \sim f_1$ if there is a map $h : (Y \times X_1, A \times X_1) \to (Z, B)$ so that $f_i \pi_1 = h(\mathrm{id}_Y \times \partial_i)$, $i = 0, 1$, where $\pi_1 : (Y \times X_0, A \times X_0) \to (Y, A)$ is projection on the first factor. In this case we refer to $h$ as a direct homotopy from $f_0$ to $f_1$. Clearly if $h$ is a direct homotopy from $f_0$ to $f_1$ then $kh$ and $h(g \times \mathrm{id}_{X_1})$ are direct homotopies from $kf_0$ to $kf_1$ and from $f_0 g$ to $f_1 g$ respectively. Although the relations just defined are reflexive and are preserved by composition, they are generally not equivalence relations. However, since the equivalence relations they generate are also preserved by composition, it readily follows that each pair $\partial_i : X_0 \to X_1$, $i = 0, 1$, induces a homotopy structure on each subcategory of $PE$. We refer to such homotopy structure as representable. A representable homotopy is said to be relational if it can be represented by a relation, i.e. by a pair $\partial_i : X_0 \to X_1$, $i = 0, 1$, with $\langle \partial_0, \partial_1 \rangle : X_0 \to X_1 \times X_1$ a mono. Among the relational homotopies is the classical notion of homotopy in the category of topological pairs (all $(Y, A)$ where $A$ is a subspace of $Y$) in which $\partial_i : X_0 \to X_1$ are the inclusions $1 \to I$ of the endpoints into the standard unit interval $I = [0, 1]$. In this case direct homotopy already is an equivalence relation. Note that the category of topological pairs is a full subcategory of, but not equal to, $P(\mathrm{Top})$ since a mono $A \to Y$ in Top is not necessarily an embedding.

For another example of homotopy structure let $D$ be a family of objects in a category $E$. Define, for $f_i : (Y, A) \to (Z, B)$, $i = 0, 1$, $f_0 \sim f_1$ if $f_0 = f_1$ or if there exists a map $X \to Z$, with $X \in D$, through which both $f_0$ and $f_1$ factor. It is readily seen that $\sim$ is a reflexive, symmetric relation that is preserved by composition. It may be transitive depending on properties of $D$. In any case the equivalence relations generated by the $\sim$'s do define a homotopy structure on $PE$. For example, if $E = \mathrm{Top}$ and $D$ consists of all the compact (or connected or finite) spaces then $\sim$

already is a homotopy structure on topological pairs.

Another basic concept is that of "excision". By an excision structure Exc on a category $E$ having pullbacks we mean an indexed family $\mathrm{Exc}\,(Y)$, $Y \in E$, where $\mathrm{Exc}\,(Y)$ is a class of ordered pairs $(Y_0, Y_1)$ of subobjects of $Y$ that together cover $Y$ (i.e. the inclusions $Y_i \to Y$, $i = 0, 1$, are jointly epi). Further we require that, for $f : Y \to Z$ and $(Z_0, Z_1) \in \mathrm{Exc}\,(Z)$, the pullback pair $(f^*(Z_0), f^*(Z_1))$ be in $\mathrm{Exc}\,(Y)$. Note that since $\mathrm{Exc}\,(Y)$ is partially ordered by inclusion and pullback preserves monos, an excision structure on $E$ is an indexed category (of partially ordered classes) on $E$ in the sense of [24, p. 10].

The classical example of an excision structure on the category $E = \mathrm{Top}$ is the one given by $\mathrm{Exc}\,(Y) = \{(Y_0, Y_1) \mid \text{where the interiors of } Y_i \text{ together cover } Y\}$. Recall that in this case the excision axiom for homology states that if $U \subset A \subset Y$ and (closure $U$) $\subset$ (interior $A$) then the inclusion $(Y - U, A - U = (Y - U) \cap A) \to (Y, A)$ induces an isomorphism on homology. Equivalently, since the condition (closure $U$) $\subset$ (interior $A$) gives exactly that $(Y_0 = Y - U, Y_1 = A) \in \mathrm{Exc}\,(Y)$, the excision axiom asserts that for any $(Y_0, Y_1) \in \mathrm{Exc}\,(Y)$, the inclusion map $(Y_0, Y_0 \cap Y_1) \to (Y, Y_1)$, i.e. the excision of $U = Y - Y_0$ from the pair $(Y, Y_1)$, induces an isomorphism on homology.

Another example of an excision structure on Top is given by $\mathrm{Exc}\,(Y) = \{(Y_0, Y_1) \mid Y_0$ is a regular ($Y_0 =$ interior (closure $Y_0$), cf. [4, # 22, p. 92]) open subset of $Y$, $Y_1$ is an open subset of $Y$, and $Y_0 \cup Y_1 = Y\}$. This defines an excision structure since both regularity and openness are preserved by pullback. We denote this structure by RExc.

A closely related, important example of an excision structure is one in which $E = \mathrm{Shv}\,(B)$, the category of sheaves on a topological space $B$ (i.e. of local homeomorphisms into $B$). Since the subsheaves of a sheaf $Y \to B$ correspond to the open subsets of the total space $Y$, $\mathrm{Exc}\,(Y \to B) = \mathrm{RExc}\,(Y)$ clearly defines an excision structure on $E$. In this case the map $(Y_0, Y_0 \cap Y_1) \to (Y, Y_1)$ represents the excision of the "complement", in $\mathrm{Shv}\,(B)$, of $Y_0$, i.e. of $U =$ interior $(Y - Y_0)$, from the pair $(Y, Y_1)$. That $Y_0$ is the sheaf complement of $U$, i.e. that $Y_0 =$ interior $(Y - U)$, follows from the regularity of $Y_0$. This excision structure will be generalized in § 2.

Let $E$ be a category with an initial object 0. We call a subcategory $E_1$ of $PE$ admissible if whenever a pair $(Y, A)$ is in $E_1$ then so is the sequence $(A, 0) \to (Y, 0) \to (Y, A)$ of objects and maps (induced by $A \subset Y$). In practice, an admissible category $E_1$ may be determined by the type of mono $A \subset Y$ for $(Y, A)$ in $E_1$ such as equalizer, strict or strong ([29, p. 703]) or, if $E$ is a topological category ([10, p. 128]), initial. The last case includes the classical admissible category of topological pairs. Conditions may also be put on the kind of objects $Y, A$ in $E_1$. For example, if $E = \mathrm{Top}$, one has the admissible categories of compact pairs, cellular pairs and triangulable pairs.

We are now in a position to describe what we mean by a homology theory on an admissible category. To this end let Exc be an excision structure on a category $E$ having finite limits and an initial object 0 and let $\sim$ be a homotopy structure on an admissible category $E_1$ of $E$.

A generalized homology theory $(H_*, \partial_*)$ for $(E_1, \sim, \mathrm{Exc})$ consists of the following:

(a) A functor $H_*$ from $E_1$ to the category of graded abelian groups and homomorphisms of degree 0 (i.e. $H_*(Y,A) = \{H_n(Y,A)\}$).

(b) A natural transformation $\partial_*$ of degree $-1$ from $H_*$ on $(Y,A) \in E_1$ to $H_*$ on $(A,0)$ (i.e. $\partial_*(Y,A) = \{\partial_n(Y,A) : H_n(Y,A) \to H_{n-1}(A)\}$, where, as usual, $H_*(A)$ denotes $H_*(A,0)$). These satisfy the following axioms:

(1) Exactness. For any pair $(Y,A) \in E_1$ the inclusions $(A,0) \to (Y,0) \to (Y,A)$ induced sequence $\cdots \to H_n(A) \to H_n(Y) \to H_n(Y,A) \xrightarrow{\partial_n} H_{n-1}(A) \to \dots$ is exact.

(2) Homotopy. If $f_0 \sim f_1$ for $(f_i : (Y,A) \to (Z,B)) \in E_1$, $i = 0,1$, $H_*(f_0) = H_*(f_1)$.

(3) Excision. For any $Y$ and $(Y_0, Y_1) \in \mathrm{Exc}(Y)$ if the inclusion map $(Y_0, Y_0 \cap Y_1) \to (Y, Y_1)$ is in $E_1$ then it induces an isomorphism on homology, where $Y_0 \cap Y_1$ denotes the pullback of $Y_0 \subset Y$ and $Y_1 \subset Y$.

A generalized homology theory is called homology theory if the following axiom also holds.

(4) Dimension. If 1 is a terminal object of $E$ and $(1,0) \in E_1$ then $H_n(1) = 0$ if $n \neq 0$.

As usual, by requiring $H_*$ to be contravariant and $\partial_*$ to be of degree $+1$, and by modifying the axioms accordingly, we obtain the notion of a (generalized) cohomology theory. These concepts clearly include the classical notion of (co)homology for topological pairs.

In general, $H_*$ need not preserve coproducts, but in the following useful (see 2.1, 2.2) special case it does.

**Lemma 1.1.** *Let $H_*$, as in (a), satisfy the excision axiom and satisfy exactness at $(Y,0)$ for any pair $(Y,A)$ in $E_1$. If subobjects $Y_i \to Y$ with $(Y_i, Y_{1-i}) \in \mathrm{Exc}(Y)$ and $(Y, Y_i) \in E_1$, $i = 0,1$, define $Y$ as a disjoint (i.e. $Y_0 \cap Y_1 = 0$) coproduct then the inclusion induced map $H_*(Y_0) \oplus H_*(Y_1) \to H_*(Y)$ is an isomorphism.*

PROOF: In the commutative diagram

$$\begin{tikzcd}
H_n(Y_0) \arrow[dd] \arrow[dr] & & H_n(Y_1) \arrow[dd] \arrow[dl] \\
& H_n(Y) \arrow[dl] \arrow[dr] & \\
H_n(Y, Y_1) & & H_n(Y, Y_0)
\end{tikzcd}$$

in which all maps are inclusion induced, the vertical maps, being induced by $(Y_0, Y_0 \cap Y_1 = 0) \to (Y, Y_1)$ and $(Y_1, 0) \to (Y, Y_0)$, are, by excision, isomorphisms, and the two diagonal sequences are, by assumption, exact at $H_n(Y)$. Hence, by a standard result on abelian groups, e.g. [11, p. 39, Lemma 7.1], $H_n(Y_0) \oplus H_n(Y_1) \to H_n(Y)$ is an isomorphism and the result follows. □

## 2. Homology in a topos.

By a homotopy structure on a topos $E$ we shall mean a homotopy structure on all of $PE$. We shall be concerned mostly with representable homotopies. Note that, in a topos, representable homotopies are always relational since the epi-mono factorization $\langle \partial_0, \partial_1 \rangle : X_0 \to X_0' \xrightarrow{\partial'} X_1 \times X_1$ defines a relation $\partial_i' : X_0' \to X_1$ that represents the same homotopy structure as $\partial_i : X_0 \to X_1$. Important among these homotopy structures are the ones represented by coreflexive pairs, i.e. by those $\{\partial_i\}$ for which there is an $s : X_1 \to X_0$ with $\partial_i s = \mathrm{id}$, $i = 0, 1$, and by the disjoint relations, i.e. by those relations $\{\partial_i\}$ with (image $\partial_0$) $\cap$ (image $\partial_1$) $= 0$. Included in both of these classes are the homotopies represented by intervals, i.e. by linearly ordered objects with disjoint minimum and maximum points.

We next consider excision structures on a topos. Recall that a topos comes equipped with a negation operation $\neg$ ($\neg : \Omega \to \Omega$ classifies the classifying map $1 \to \Omega$ of $0 \subset 1$). If $A \subset Y$ is classified by $f : Y \to \Omega$ then $\neg A \subset Y$ is, by definition, the subobject classified by $\neg f : Y \to \Omega$. $\neg A$ may be viewed as the largest subobject of $Y$ disjoint from $A$. For subobjects $A, B$ of $Y$, $A \cap \neg B$ is often denoted, as for Sets, by $A - B$. A subobject $R \subset Y$ is said to be <u>regular</u> if $\neg\neg R = R$ ([12, 5.16]). Since regular subobjects and epi maps in a topos are both preserved by pullback it readily follows that $\mathrm{Exc}\,(Y) = \{(R, A) \mid R$ is a regular subobject of $Y$, $A \subset Y$, and $R \cup A \approx Y\}$ defines an excision structure. We refer to it as the regular excision structure and denote it by $\mathrm{RExc}$. Note that, for $(R, A) \in \mathrm{RExc}\,(Y)$, $\neg R = \neg R \cap Y = \neg R \cap (R \cup A) = (\neg R \cap R) \cup (\neg R \cap A) = \neg R \cap A$, i.e. $\neg R \subset A$. Thus for any excision structure contained in the regular excision structure (i.e. <u>subregular</u> excision structure) the map $(R, R \cap A) \to (Y, A)$ appearing in the excision axiom represents "excision" of the subobject $U = \neg R$ of $A$ (or of any $U \subset Y$ with $\neg U = R$ and $U \subset A$) from the pair $(Y, A)$ in the sense that $(R, R \cap A) = (\neg U, \neg U \cap A) = (Y - U, A - U)$.

By way of example, the excision structure on the topos $\mathrm{Shv}\,(B)$ described in § 1 is the regular one. Among the subregular excision structures on any topos are those determined by topologies ([12, Chapter 3]) on the topos. More explicitly let $j : \Omega \to \Omega$ be a topology on a topos and let, for $A \subset Y$, $jA$ be the $j$-closure of $A$ in $Y$ (if $f : Y \to \Omega$ classifies $A \subset Y$ then $jf$ classifies $jA \subset Y$). Since $j$-closure is preserved by pullback, ([12, 3.14]) it follows that $\mathrm{Exc}\,(Y) = \{(R, A) \mid R \subset Y,\ A \subset Y,\ R \cup A \approx Y$, there is $U \subset Y$ with $\neg U = r$ and $jU \subset A\}$ defines a subregular excision structure. In this case $(R, R \cap A) \to (Y, A)$ represents the excision of a subobject $U \subset A$ for which $jU \subset A$. This is similar to the classical topological situation in which one considers excisions of $U$ from an open set $A$ when (closure $U$) $\subset A$. Further, $\mathrm{Exc}\,(Y) = \{(R, A) \mid R \subset Y,\ A \subset Y,\ R \cup A \approx Y$, there is $U \subset Y$ with $\neg U = R$ and $jU = U \subset A\}$ is easily shown to be an excision structure for which $(R, R \cap A) \to (Y, A)$ represents excision of a $j$-closed subobject. More generally if $P$ is a universal (i.e. preserved by pullback) property of subobjects then $\mathrm{Exc}\,(Y) = \{(R, A) \mid R \subset Y,\ A \subset Y,\ R \cup A \approx Y$, there is $U \subset Y$ with $\neg U = R$ and $U \subset A$ having $P\}$ defines a subregular excision structure for which $(R, R \cap A) \to (Y, A)$ represents excision of $P$-subobjects of $A$. Two other excision structures of note are the maximum structure MExc given by $\mathrm{MExc}(Y) = \{(Y_0, Y_1) \mid Y_i \subset Y, Y_0 \cup Y_1 \approx Y\}$ and the symmetric (Exc is called symmetric if $(Y_0, Y_1) \in \mathrm{Exc}(Y)$ iff $(Y_1, Y_0) \in \mathrm{Exc}(Y)$) regular structure SRExc given by $\mathrm{SRExc}(Y) = \{(R_0, R_1) \mid (R_0, R_1) \in \mathrm{MExc}(Y)$ and $R_i$ is regular, $i = 0, 1\}$. Clearly MExc is the largest excision structure on $E$, RExc is the largest excision structure for which $(R, R \cap A) \to (Y, A)$, $(R, A) \in \mathrm{Exc}(Y)$, $Y \in E$, represents excision of a subobject, and SRExc is the largest symmetric subregular excision structure.

When referring to homology theories on a topos $E$, unless stated otherwise, we shall mean a homology theory on the admissible category $PE$ relative to some given, but not necessarily mentioned, homotopy structure and excision structure that contains RExc.

**Proposition 2.1.** *If $(H_*, \partial_*)$ is a homology theory on a topos $E$ then the inclusion induced map $H_*(Y_0) \oplus H_*(Y_1) \to H_*(Y_0 \amalg Y_1)$ is an isomorphism for all objects $Y_i$, $i = 0, 1$, of $E$.*

PROOF: This follows from 1.1 since coproducts in a topos are disjoint ([12, 1.57]) and $(Y_i, Y_{1-i}) \in \mathrm{RExc}(Y_0 \amalg Y_1)$, in fact $Y_i = \neg Y_{1-i}$, $i = 0, 1$. □

Every topos has at least one homology theory on it, namely, the trivial homology theory in which $H_*(Y, A) = 0$ for all $(Y, A)$. Depending upon the homotopy structure, it may have no others. By the trivial homotopy structure on a topos we mean the one in which every pair of parallel maps is homotopic.

**Corollary 2.2.** *Any homology theory on a topos with the trivial homotopy structure is trivial.*

PROOF: For any object $Y$, the first factor inclusion $Y \to Y \amalg Y$ is, since the homotopy structure is trivial, a homotopy equivalence (the fold map $\mathrm{id} + \mathrm{id} : Y \amalg Y \to Y$ is a homotopy inverse). Thus, by the homotopy axiom and 2.1, the first factor inclusion $H_*(Y) \to H_*(Y \amalg Y) \approx H_*(Y) \oplus H_*(Y)$ is an isomorphism and consequently $H_*(Y) = 0$. It now follows from the exactness axiom that $H_*$ is trivial. □

In view of 2.2 it is useful to have conditions under which various homotopy structures on a topos are trivial. We call a topos strongly homotopically trivial if all of the homotopy structures that can be represented by a disjoint relation (i.e. with $\partial_0(X_0) \cap \partial_1(X_0) = 0$) are trivial.

**Lemma 2.3.** *If all regular subobjects in a topos have complements (i.e. if $R \subset Y$ is regular then $R \amalg \neg R \approx Y$) then it is strongly homotopically trivial.*

PROOF: Given $\partial_i : X_0 \to X_1$ with $\partial_0(X_0) \cap \partial_1(X_0) = 0$ then $\partial_0(X_0) \subset \neg\partial_1(X_0) = X^0$, $\partial_1(X_0) \subset \neg X^0 = X^1$ and, since $X^0$ is regular, $X^0 \amalg X^1 \approx X_1$. The result now follows since any pair of maps $f_i : (Y, A) \to (Z, B)$, $i = 0, 1$, are rendered directly homotopic by the map $h = (f_0 + f_1)(p^0 \amalg p^1) : Y \times X_1 \approx Y \times X^0 \amalg Y \times X^1 \to Y \amalg Y \to Z$, where $p^i : Y \times Y^i \to Y$ is the first factor projection. □

A topos $E$ is said to be homotopically trivial if all the homotopy structures on $E$ represented by intervals are trivial and to be locally (strongly) homotopically trivial if $E/Y$ is (strongly) homotopically trivial for all $Y \in E$. Note that in [23] a topos $E$ was called homotopically trivial if all the congruences on $E$ defined, as above, by intervals are trivial while here we require all such congruences on $PE$ to be trivial. However, in view of 1.1 [23] the two notions are readily shown, as in the proof of 2.3, to be equivalent. Moreover, we have the following:

**Proposition 2.4.** *Let $E$ be a topos. Among the following statements about $E$ we have: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Further, if $E$ satisfies* SG *(i.e. if $E$ is generated by the subobjects of the terminal) then all the statements are equivalent.* (1) *$E$ is locally homotopically trivial.* (2) *$E$ is locally strongly homotopically trivial.* (3) *$E$ is strongly homotopically trivial.* (4) *$E$ is homotopically trivial.*

Proof: The implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. To show (1) $\Rightarrow$ (2) note that if (1) holds then, by [23, 1.5 (1), (2)], regular subobjects in $E$ have complements. Since this condition is clearly local (see the proof of [23, 1.5]), regular subobjects in $E/Y$ have complements for all $Y \in E$ and thus, by 2.3, (2) holds. Finally if SG holds then, by [23, 1.8], (4) $\Rightarrow$ (1) and the result follows. $\square$

We call a topos homologically (strongly homologically) trivial if any homology theory on it for any homotopy structure represented by an interval (a disjoint relation) is trivial. The corresponding local notions are defined as usual. The results 2.2 and 2.4 combine to show how the logic of a topos, especially De Morgan's law, relates to homology.

**Corollary 2.5.** *Any topos in which De Morgan's law holds is locally strongly homologically trivial.*

Proof: The result is a direct consequence of 2.2 nd 2.4 in view of the fact ([23, 1.5]) that De Morgan's law holds in a topos iff the topos is locally homotopically trivial. $\square$

There are many conditions ([13], [14]) on a topos that are equivalent to (e.g. $\Omega$ is a Stone lattice, maximal ideals in internal commutative unitary rings are prime, Dedekind reals are conditionally order-complete) or, at least imply (e.g. Axiom of Choice, Booleaness), the validity of De Morgan's law and consequently imply that certain important homology theories are trivial. We thus know where not to look for examples of nontrivial homology theories based on homotopy structures represented by intervals or disjoint relations. To obtain examples of nontrivial homology theories we turn to a generalization of classical singular homology theory for topological spaces.

## 3. Singular homology in a category.

For a category $E$ let Cosimpl$(E)$ (Simpl$(E)$) be the functor category $E^{\Delta}$ ($E^{\Delta^{op}}$) of cosimplicial (simplicial) objects in $E$, where $\Delta$ is the skeletal category of finite nonempty linearly ordered sets $[n] = \{0 \leq \cdots \leq n\}$ and order preserving maps. As usual, we view a cosimplicial (simplicial) object as a system $\{(X_n),\ n = 0, 1, \ldots, \partial_i, \sigma_i\}$ ($\{(X_n),\ n = 0, 1, \ldots, d_i, s_i\}$) of objects $X_n$ and maps $\partial_i : X_{n-1} \to X_n$, $\sigma_i : X_n \to X_{n-1}$ ($d_i : X_n \to X_{n-1}$, $s_i : X_{n-1} \to X_n$) satisfying certain identities ([18, § 5, p. 171]). There is a sequence of functors $(\mathrm{Cosimpl}(E)^{op}) \times$

$E \xrightarrow{S}$ Simpl (Sets) $\xrightarrow{F^*}$ Simpl (Abelian Groups) $\xrightarrow{C}$ Chain Complexes $\xrightarrow{H_*}$ Graded Abelian Groups, where $S$, the singular functor, is given by $S(X_*, Y)\,[n] = E(X_n, Y)$, $F^*$ is induced by the free functor $F(U \mapsto$ free abelian group generated by $U)$ : Sets $\to$ Abelian Groups, $C(G_*) = \{G_n, \partial_n = \sum_{i=0}^{n}(-1)^i d_i\}$, and $H_* = \{\mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n-1})\}$ is the homology functor. Moreover, the correspondence $(X_*, (Y, A)) \mapsto H_*\,(CF^*S(X_*, (Y, A)))$, where $CF^*S(X_*, (Y, A))$ is defined by the short exact sequence $0 \to CF^*S(X_*, A) \to CF^*S(X_*, Y) \to CF^*S(X_*, (Y, A)) \to 0$ of chain complexes, induces an extension $H_* : (\mathrm{Cosimpl}\,(E)^{op}) \times PE \to$ Graded Abelian Groups of the composite of the foregoing sequence of functors. We refer to $H_*(X_*, (Y, A))$ as the (graded) $X_*$-singular homology groups of $(Y, A)$. It clearly coincides with the classical singular homology group of a topological pair $(Y, A)$ when $X_* = \{\Delta_n\}$ is the cosimplicial space of affine simplexes. Each triple $B \subset A \subset Y$ defines a short exact sequence $0 \to CF^*S(X_*, (A, B)) \to CF^*S(X_*, (Y, B)) \to CF^*S(X_*, (Y, A)) \to 0$ of chain complexes that in turn induces, by standard homological results (e.g. [17, § 4, p. 44–45], [25, Lemma 3, p. 181]), a natural connecting map $\partial_n(X_*) : H_n(X_*, (Y, A)) \to H_{n-1}(X_*, (A, B))$, $n \geq 1$, that, in case $B = 0$, renders exactly the long exact sequence of the exactness axiom (note that $H_n(X_*, -) = 0$ if $n < 0$). In general $H_*CF^*S(X_*, 0)$ need not be zero (e.g. in a pointed category $0 \approx 1$) and consequently, in contrast to the topological case, $H_*CF^*S(X_*, Y) \to H_*CF^*S(X_*, (Y, 0))$ need not be an isomorphism. It is an isomorphism, however, if 0 is strict (i.e. if any map $Y \to 0$ is an isomorphism) and $X_0 \neq 0$, for then $S(X_*, 0) = \emptyset$ and, by definition, $CF^*S(X_*, Y) \approx CF^*S(X_*, (Y, 0))$. Moreover, in this case $S(X_*, 1)$ is the terminal simplicial set and the usual algebraic computation (e.g. [17, Proposition 7.3, p. 57]) shows $H_n(X_*, (1, 0)) = 0$ if $n \neq 0$ and is infinite cyclic otherwise. Note that if $X_* = 0$ then $S(X_*, A) \approx S(X_*, Y)$ and consequently $H_*(X_*, (Y, A)) = 0$. In summary we have the following:

**Lemma 3.1.** *For any* $X_* \in \mathrm{Cosimpl}\,(E)$ *the associated* $X_*$*-singular homology satisfies the conditions* (a) *and* (b) *as well as the exactness axiom of a homology theory. Moreover, if* 0 *is strict then the dimension axiom also holds, and if* $X_0 \neq 0$ *then* $H_0(X_*, 1)$ *is infinite cyclic.*

If, in the definition of $H_*(X_*, (Y, A))$, the free functor $F$ : Sets $\to$ Abelian Groups is replaced by $U \mapsto F(U) \otimes G$ ($U \mapsto \mathrm{Hom}(F(U), G)$), where $G$ is an abelian group, then the resulted graded groups are referred to as the $X_*$-singular homology (cohomology) groups with coefficient $G$. We note in passing that if $E$ has finite products then $S(X_*, Y \times Z) \approx S(X_*, Y) \times S(X_*, Z)$ and consequently, since the Eilenberg-Zilber theorem ([17, § 8, p. 238]), the Kunneth formula ([17, § 10, p. 166]), and the Universal Coefficient theorems ([17, § 4, p. 76, § 11, p.170]) are homological in nature, the usual methods for computing singular homology and cohomology of products with various coefficient groups extend to the $X_*$-singular situation as well.

We next turn to homotopy. So far, the maps $\sigma_i$ of the cosimplicial objects have not been used but they will play a central role in our study of homotopy. Recall that a homotopy $h_* : f_* \to g_*$ of cosimplicial maps $f_*, g_* : X_* \to X'_*$ consists of a system of maps $\{h_i : X_n \to X'_n,\ i = 0, \ldots, n;\ n = 0, 1, \ldots\}$ that satisfies the following set of identities, which are dual to those defining simplicial homotopies

([5, p. 10]): $h_0\partial_0 = f_n$, $h_n\partial_{n+1} = g_n$

$$\begin{cases} h_j\partial_i = \partial_i h_{j-1} & i < j \\ h_{j+1}\partial_{j+1} = h_j\partial_{j+1} & \\ h_j\partial_i = \partial_{i-1}h_j & i > j+1 \end{cases} \quad \text{and} \quad \begin{cases} h_j\sigma_i = \sigma_i h_{j+1} & i \le j \\ & \\ h_j\sigma_i = \sigma_{i-1}h_j & i > j \end{cases}$$

Before considering the homotopy axiom we give a technical result.

**Lemma 3.2.** *Let $X_* \in \mathrm{Cosimpl}\,(E)$. If $E$ has finite products then there is a cosimplicial map $\varepsilon_* : X_* \to X_* \times X_0 = \{X_n \times X_0,\ \partial_i \times 1,\ \sigma_i \times 1\}$ such that:*
*(1) $\pi_1\varepsilon_* = \mathrm{id}$, where $\pi_1 : X_* \times X_0 \to X_*$ is first factor projection.*
*(2) There is a cosimplicial homotopy $h_* : (1 \times \partial_0)\varepsilon_* \to (1 \times \partial_1)\varepsilon_* : X_* \xrightarrow{\varepsilon_*} X_* \times X_0 \xrightarrow{1\times\partial_1} X_* \times X_1$.*

Proof: A direct calculation, using the cosimplicial identities among the $\partial$'s and $\sigma$'s shows that the maps $\sum_n = \sigma_0\sigma_1 \dots \sigma_{n-1} : X_n \to X_{n-1} \to \dots \to X_1 \to X_0$, if $n \ge 1$, and $\sum_0 = \mathrm{id} : X_0 \to X_0$, define a cosimplicial map $\sum_* : X_* \to X_0$ where $X_0$ is viewed as the constant cosimplicial object with all $\partial$'s and $\sigma$'s identities. The cosimplicial map $\varepsilon_* = \langle 1, \sum_* \rangle : X_* \to X_* \times X_0$ clearly satisfies (1). Similarly one verifies that $\{h_i = \langle \sigma_i, \sigma_0 \dots \hat{\sigma}_i \dots \sigma_n \rangle : X_{n+1} \to X_n \times X_1,\ i = 0, \dots, n;\ n = 0, 1, \dots\}$, where $\hat{\sigma}_i$ denotes the omission of $\sigma_i$ from the composition of $\sigma$'s and $\sigma_0 \dots \hat{\sigma}_i \dots \sigma_n = \mathrm{id} : X_1 \to X_1$ if $n = i = 0$, defines the desired homotopy in (2). For example, from $\sigma_j\partial_i = \partial_i\sigma_{j-1}$ if $i < j$ and $\sigma_i\partial_i = \mathrm{id}$, we have, for $n > 0$, $h_0\partial_0 = \langle \sigma_0, \sigma_1 \dots \sigma_n \rangle \partial_0 = \langle \sigma_0\partial_0, \sigma_1 \dots \sigma_n\partial_0 \rangle = \langle 1, \partial_0\sigma_0 \dots \sigma_{n-1} \rangle = (1 \times \partial_0)\langle 1, \sum_n \rangle = (1 \times \partial_0)\varepsilon_n$, while $\sigma_j\partial_i = \partial_{i-1}\sigma_j$ if $i > j+1$ and $\sigma_i\partial_{i+1} = \mathrm{id}$ give $h_n\partial_{n+1} = \langle \sigma_n, \sigma_0 \dots \sigma_{n-1} \rangle \partial_{n+1} = \langle \sigma_n\partial_{n+1}, \sigma_0 \dots \sigma_{n-1}\partial_{n+1} \rangle = \langle 1, \partial_1\sigma_0 \dots \sigma_{n-1} \rangle = (1 \times \partial_1)\langle 1, \sum_n \rangle = (1 \times \partial_1)\varepsilon_n$. □

We can now verify a homotopy axiom for $X_*$-singular homology.

**Lemma 3.3.** *$X_*$-singular homology satisfies the homotopy axiom relative to the homotopy structure represented by the coreflexive pair $\partial_i : X_0 \to X_1$, $i = 0, 1$.*

Proof: It clearly suffices (see [25, Theorem 2, p. 163]) to show that any pair of directly homotopic maps $(Y, A) \to (Z, B)$ induce chain homotopic maps $CF^*S(X_*, (Y, A)) \to CF^*S(X_*, (Z, B))$ and for this it suffices, by passing to quotients, to consider the case of directly homotopic maps $f_i : Y \to Z$, $i = 0, 1$. To this end suppose $h : Y \times X_1 \to Z$ is a direct homotopy of $f_0$ to $f_1$, i.e. $f_i\pi_1 = h(1 \times \partial_i)$, $i = 0, 1$. In the following diagram of simplicial sets, in which $\varepsilon_*$ is as in 3.2 and $P_i$ is induced by $(g_n : X_n \to Y) \mapsto (g_n \times 1 : X_n \times X_i \to Y \times X_i)$, i=0,1, the middle square obviously commutes, the top square commutes since $(1 \times \partial_i)(g_n \times 1) = (g_n \times 1)(1 \times \partial_i)$, and the bottom square commutes by the definition of $h$.

$$\begin{array}{ccc}
S(X_*, Y) & \xrightarrow{P_1} & S(X_* \times X_1, Y \times X_1) \\
\Big\downarrow{\scriptstyle P_0} & & \Big\downarrow{\scriptstyle S(1\times\partial_i,1)} \\
S(X_* \times X_0, Y \times X_0) & \xrightarrow{S(1,1\times\partial_i)} & S(X_* \times X_0, Y \times X_1) \\
\Big\downarrow{\scriptstyle S(\varepsilon_*,1)} & & \Big\downarrow{\scriptstyle S(\varepsilon_*,1)} \\
S(X_*, Y \times X_0) & \xrightarrow{S(1,1\times\partial_i)} & S(X_*, Y \times X_1) \\
\Big\downarrow{\scriptstyle S(1,\pi_1)} & & \Big\downarrow{\scriptstyle S(1,h)} \\
S(X_*, Y) & \xrightarrow{S(1,f_i)} & S(X_*, Z)
\end{array}$$

Moreover, by 3.2 (1), $S(1,\pi_1)S(\varepsilon_*,1)P_0(g_n) = \pi_1(g_n \times 1)\varepsilon_* = g_n(\pi_1\varepsilon_*) = g_n$, i.e. $S(\varepsilon_*,\pi_1)P_0 = \mathrm{id}$. Hence $S(1,f_i) = S(1,f_i)S(\varepsilon_*,\pi_1)P_0 = S(1,h)S(\varepsilon_*,1)S(1\times\partial_i,1)P_1 = S(1,h)S((1\times\partial_i)\varepsilon_*,1)P_1$. Since $S(-,-)$ is contravariant in the first variable, 3.2 (2) implies that $S((1\times\partial_0)\varepsilon_*,1)$ and $S((1\times\partial_1)\varepsilon_*,1)$, and consequently $S(1,f_0)$ and $S(1,f_1)$ are simplicially homotopic. Since simplicially homotopic maps are preserved by $F^*$ and transformed to chain homotopic maps by $C$, $CF^*S(1,f_0)$ and $CF^*S(1,f_1)$ are chain homotopic and the result follows. □

Lemma 3.3 clearly gives the classical homotopy axiom for singular homology theory when $X_* = \{\Delta_n\}$ is the cosimplicial space of affine simplexes since the associated homotopy structure is the one represented by the unit interval $\Delta_1$. The usual proofs ([1], [3], [6], [8], [17], [25],[26], [27], [28]) of the homotopy axiom generally rely on analytic and geometric properties of the $\Delta_n$'s (e.g. barycentric coordinates, convexity of $\Delta_n$ and $\Delta_n \times \Delta_1$ as subsets of Euclidean space) which may lead one to conclude, incorrectly, that it depends on properties beyond the cosimplicial structure of $\{\Delta_n\}$. In fact, as 3.1 and 3.3 show, the 'topology' essentially enters classical singular homology through the excision axiom, the proof of which does depend on special properties (e.g. subdivision) of $\{\Delta_n\}$ beyond the cosimplicial structure. However, a generalization of those parts of the proof that are essentially homological (depending on the Noether isomorphism theorem, the 5-lemma, and the exactness axiom) give, for a general excision structure Exc on a category $E$, the following:

**Lemma 3.4.** *If for all $Y \in E$ and all $(Y_0, Y_1) \in \mathrm{Exc}\,(Y)$ the inclusion $S(X_*, Y_0) \cup S(X_*, Y_1) \to S(X_*, Y)$ induced map $H_m(CF^*(S(X_*, Y_0) \cup S(X_*, Y_1))) \to H_m(CF^*S(X_*, Y))$ is an isomorphism for $0 \le m \le n$ then $X_*$-singular homology satisfies the* Exc *excision axiom in dimensions $\le n$.*

Proof: An obvious modification of the proof of Theorem 4 [25, p. 188] gives the result. □

We next consider conditions on $X_*$ that ensure the validity of the excision axiom. One could formally require $X_*$ to admit a 'subdivision operation' that would allow for a generalization of the usual proof of excision for topological spaces. We shall consider the special case in which, in effect, no subdivision is needed. To this end call an object $Y$ of $E$ Exc-tight, where Exc is an excision structure on $E$, if for all $(Y_0, Y_1) \in \mathrm{Exc}\,(Y)$ at least one of the monos $Y_0 \subset Y$ or $Y_1 \subset Y$ is an isomorphism and call a (co)simplicial object $X_*$ of $E$ $n$-Exc-tight if $X_n$ is Exc-tight and call it $\infty$-Exc-tight if it is $n$-Exc-tight for $n = 0, 1, \ldots$ .

**Lemma 3.5.** *If $X_*$ is $n$-Exc-tight, for $n$ a positive integer, then the Exc excision axiom holds for $X_*$-singular homology in dimensions $\leq n - 1$.*

Proof: Since $\mathrm{id} = \sigma_0 \partial_0 : X_{n-1} \to X_n \to X_{n-1}$ defines $X_{n-1}$ as a retract of $X_n$, it readily follows that pullback along $\sigma_0$ reflects isomorphisms. Hence if $X_n$ is Exc-tight then $X_{n-1}$ is Exc-tight and thus, by induction, $X_m$ is Exc-tight for $0 \leq m \leq n$. If $g_m \in S(X_*, Y)\ [m] = E(X_m, Y)$ and $(Y_0, Y_1) \in \mathrm{Exc}\,(Y)$ then $(g_m^* Y_0, g_m^* Y_1) \in \mathrm{Exc}\,(X_m)$ and consequently, if $0 \leq m \leq n$, at least one of the maps $g_m^*(Y_i) \subset X_m$ is an isomorphism. Thus $g_m$ factors through $Y_0$ or $Y_1$ and the inclusion $S(X_*, Y_0)\ [m] \cup S(X_*, Y_1)\ [m] \to S(X_*, Y)\ [m]$ is a bijection for $0 \leq m \leq n$. By applying $CF^*$ and taking homology one readily obtains an isomorphism $H_m(CF^*(S(X_*, Y_0) \cup S(X_*, Y_1))) \to H_m(CF^* S(X_*, Y))$ for $m \leq n - 1$. The result now follows from 3.4. □

Combining 3.1, 3.3, and 3.5 we obtain the following:

**Theorem 3.6.** *Let $E$ be a category with finite limits, an initial object $0$, and an excision structure Exc. If $X_*$ is an $\infty$-Exc-tight cosimplicial object of $E$ then $X_*$-singular homology is a generalized homology theory relative to the homotopy structure represented by $\partial_i : X_0 \to X_1$, $i = 0, 1$. Moreover if $0$ is strict it is a homology theory and it is nontrivial if $X_0 \neq 0$.*

There are numbers of ways in which cosimplicial objects can arise in a category. For example if $T$ is a monad on $E$ then any object $Y$ of $E$ has a cosimplicial resolution $Y \to X_*$ (the dual of the simplicial resolution of a comonad ([5, p. 28])) where $X_n = T^{n+1}(Y)$. We shall concentrate, however, on another class of cosimplicial objects; namely, those determined by certain internal categories in $E$. Recall ([12, p. 48]) that if $E$ has finite limits than the category $\mathrm{Cat}\,(E)$ of internal categories in $E$ may be identified, via the nerve functor $\mathrm{Ner}\colon \mathrm{Cat}\,(E) \to \mathrm{Simpl}\,(E)$, with a full subcategory of $\mathrm{Simpl}\,(E)$. For $C \in \mathrm{Cat}\,(E)$, $\mathrm{Ner}\,(C)_n = C_n$ can be viewed as the object of $n$-strings (i.e. composable $n$-tuples of maps) of $C$, $n = 0, 1, \ldots$, where the object of 0-strings $C_0$ is the object of objects of $C$. If $C \in \mathrm{Cat}\,(E)$ is a bounded category, i.e. one with initial and terminal objects, then it is possible to modify the simplicial object $C_* = \mathrm{Ner}\,(C)$, by relabelling and adding maps, to obtain a cosimplicial object $\overline{C}_*$. Explicitly, let $\overline{C}_{n+1} = C_n$, $n = 0, 1, \ldots$, and $\overline{C}_0 = 1$, the terminal of $E$. Take $\sigma_i : \overline{C}_{n+1} \to \overline{C}_n$ to be the map $d_{n-i} : C_n \to C_{n-1}$ of $\mathrm{Ner}\,(C)$, $i = 0, \ldots, n$; $n = 0, 1, \ldots$, and $\sigma_0 : \overline{C}_1 \to \overline{C}_0 = 1$ to be the unique map. Further, let $\partial_i : \overline{C}_n \to \overline{C}_{n+1}$ be the map $s_{n-i} : C_{n-1} \to C_n$ of $\mathrm{Ner}\,(C)$ for $i = 1, \ldots, n$; $n = 1, 2, \ldots$ . Finally the map $\partial_0$ $(\partial_1) : \overline{C}_0 = 1 \to C_0 = \overline{C}_1$ corresponds to a choice of terminal (initial) object, while the map $\partial_0$ $(\partial_{n+1}) : \overline{C}_n \to \overline{C}_{n+1}$, $n = 1, 2, \ldots$, can be interpreted as transforming an $(n-1)$-string to an $n$-string by inserting the unique map to (from) the terminal (initial) object at the end (beginning) of the $(n-1)$-string. The relationship between $\overline{C}_*$ and $C_*$ can be elucidated further by the embedding $\varepsilon : \Delta^{op} \to \Delta$ given by the correspondences $[n] \mapsto [n+1]$, $(\partial_i : [n-1] \to [n]) \mapsto (\sigma_{n-1} : [n+1] \to [n])$ and $(\sigma_i : [n+1] \to [n]) \mapsto (\sigma_{n+1-i} : [n+1] \to [n+2])$, $i = 0, \dots, n$; $n = 0, 1, \dots,$ in that the $\varepsilon$-induced map Cosimpl$(E) \to$ Simpl$(E)$ carries $\overline{C}_*$ to $C_*$. (Note that $\varepsilon$ is just $\Delta(-, [1])$ modulo the equivalence $\Delta([n], [1]) \approx [n+1]$ induced by the correspondence $(f : [n] \to [1] = \{0, 1\}) \mapsto$ cardinal $\{f^{-1}(1)\}$. Moreover, $\varepsilon$ identifies $\Delta^{op}$ with the category of finite intervals, i.e. with the category having $[n]$, $n = 1, 2, \dots,$ as objects and all order preserving maps that also preserve the maximum and minimum points as morphisms (the image of $\varepsilon$ in $\Delta$ omits only the first and last of the maps $\partial_*$ on each level).)

By taking $X_*$ of 3.6 to be $\overline{C}_*$ and noting that $\overline{C}_n = \mathrm{Ner}\,(C)_{n-1}$ we obtain the following:

**Corollary 3.7.** *Let* Exc *be an excision structure on a category $E$ having finite limits and a strict initial object. Each bounded internal category $C$ in $E$ with an $\infty$-*Exc*-tight nerve defines a nontrivial homology theory on $(E, \mathrm{Exc})$ for the homotopy structure represented by the choice $1 \rightrightarrows C_0$ of a terminal and initial object of $C$.*

Note that classical singular homology for topological space arises from a bounded category in Top; namely, the standard unit interval $I = [0, 1]$ viewed as a category, via its linear order, with initial and terminal objects 0 and 1 respectively. It is readily seen that $\overline{I}_n$ coincides with the affine simplex $\Delta_n$.

Examples of bounded categories with $n$-Exc-tight nerves are given in § 5.

## 4. Singular homology in topos.

Certain of the previous results take on a special from when $E$ is a topos. We shall be concerned mainly with singular homology for the excision structures MExc, RExc, and SRExc of § 2. For simplicity in what follows we refer to MExc (resp. RExc, SRExc)-tight objects in a topos as supertight (resp. tight, subtight). The following result is a direct consequence of 3.6 and 3.7 for the structures MExc and RExc in view of the fact that 0 is always strict in a topos ([12, 1.56]).

**Theorem 4.1.** *Any $\infty$-supertight ($\infty$-tight) cosimplicial object $X_*$ in a topos with $X_0 \neq 0$; in particular, any bounded internal category with an $\infty$-supertight ($\infty$-tight) nerve, defines a nontrivial homology theory where homotopy is represented by $\partial_i : X_0 \to X_1$, $i = 0, 1$. Moreover, for any subobjects $Y_0, Y_1$ of $Y$ $(U \subset A \subset Y)$ the inclusion $(Y_0, Y_0 \cap Y_1) \to (Y_0 \cup Y_1, Y_1)$ $((Y - U, A - U) \to (Y, A))$ induces an isomorphism on homology.*

Among the bounded categories $C$ in a topos are the intervals, i.e. those $C$ for which $\langle d_0, d_1 \rangle : C_1 \to C_0 \times C_0$ is mono, $\langle d_i, d_{1-i} \rangle : C_1 \to C_0 \times C_0$, $i = 0, 1$, are jointly epi, and the endpoints are disjoint (cf. [15, p. 257–258]). Note that although this notion of interval can be formulated in any category having finite limits, the usual idea of interval, at least from the point of view of algebraic topology includes further structure sufficient to ensure the existence of an associated 'geometric realization' functor that preserves certain types of limits (cf. [7, Chapter 3]). It is the content of Theorem 7.3 [15] that intervals, defined as above, in Set-toposes admit a left exact geometric realization. The more general situation concerning intervals in toposes with natural number objects and in various related categories is considered in [22]. For discussion of intervals in the category of $k$-spaces see [20], [21].

Among the singular homology theories on a topos, the ones that most closely resemble classical singular homology theory are those based on intervals and, in particular, those based on intervals with $\infty$-tight nerves. We next consider some explicit examples of topos that admit such, among other, homology theories.

## 5. Singular homology in $\mathbf{Sets}^{C}$.

In this section we give a characterization of (super, sub) tight objects in the functor topos $E = \mathrm{Sets}^{C}$, where $C$ is a small category, i.e. $C \in \mathrm{Cat}\,(\mathrm{Sets})$. As a consequence we obtain both a characterization of bounded internal categories (in particular intervals) in $E$ with $n$-(super, sub) tight nerves and a means of constructing explicit examples of the same.

We begin with what may, at first, seem like an unrelated study of certain Exc-tight topological spaces. To this end call a topological space $Y$ supertight (resp. tight, subtight) if for any open cover $\{U, V\}$ (resp. $\{U, V\}$ with $U$ or $V$, with $U$ and $V$, regular) either $U \subset V$ or $V \subset U$. Thus a supertight (tight, subtight) space $Y$ is Exc-tight where $\mathrm{Exc}\,(Y)$ consists of all ordered two element open covers of $Y$ (with the first (both) element(s) regular). In particular a space is tight iff it is RExc-tight. In the following characterization of these spaces, '$-$' denotes closure.

**Lemma 5.1.** *Let $Y$ be a topological space.* (1) *$Y$ is supertight iff for any pair of distinct points $x, y$, $\overline{x} \cap \overline{y} \neq \emptyset$.* (2) *$Y$ is tight iff for any open subset $U \neq \emptyset$ and any point $y$, $\overline{U} \cap \overline{y} \neq \emptyset$.* (3) *$Y$ is subtight iff for any pair of nonempty open subsets $U, V$, $\overline{U} \cap \overline{V} \neq \emptyset$.*

PROOF: (1) If there are points $x, y$ with $\overline{x} \cap \overline{y} = \emptyset$ then $(Y - \overline{x}) \cup (Y - \overline{y}) = Y$ and neither open set $Y - \overline{x}$, $Y - \overline{y}$ is $Y$. Thus $Y$ is not supertight. Conversely if $U \cup V = Y$ and $U \not\subset V$ then there is a point $y \notin V$. For any $x \in V$ there is, by assumption, a point $z \in \overline{x} \cap \overline{y}$. Now $z \notin V$ since $y \notin V$. Consequently $z \in U$ and, since $U$ is open, $x \in U$, i.e. $V \subset U$ and $Y$ is supertight.

(2) If there is an open set $U \neq \emptyset$ and a point $y$ with $\overline{U} \cap \overline{y} = \emptyset$ then $(Y - \overline{U}) \cup (Y - \overline{y}) = Y$ and neither the regular open set $Y - \overline{U}$ nor the open set $Y - \overline{y}$ is $Y$. Thus $Y$ is not tight. Conversely suppose $U \cup V = Y$ with $U$ regular. If $U \neq Y$ then, by regularity $Y - \overline{U} \neq \emptyset$ and thus, by assumption, $\overline{(Y - \overline{U})} \cap \overline{y} \neq \emptyset$ for any point $y \in U$. If $x \in \overline{(Y - \overline{U})} \cap \overline{y}$ then, since $\overline{(Y - \overline{U})} \cap U = \emptyset$, $x \in V$, and since $V$ is open, $y \in V$, i.e. $U \subset V$ and $Y$ is tight.

(3) If there are nonempty open sets $U, V$ with $\overline{U} \cap \overline{V} = \emptyset$ then $(Y - \overline{U}) \cup (Y - \overline{V}) = Y$ and neither regular set $Y - \overline{U}$, $Y - \overline{V}$ is $Y$. Thus $Y$ is not subtight. Conversely suppose $U \cup V = Y$ with both $U$ and $V$ regular. If $U \neq Y \neq V$ then $Y - \overline{U} \neq \emptyset \neq Y - \overline{V}$ and consequently, by assumption, there is a point $y \in \overline{(Y - \overline{U})} \cap \overline{(Y - \overline{V})}$. Since $\overline{(Y - \overline{U})} \cap U = \emptyset = V \cap \overline{(Y - \overline{V})}$, $y \notin U$ and $y \notin V$, a contradiction. Thus $Y$ is subtight. $\square$

It readily follows from 5.1 that no $T_1$ (resp. $T_2, T_3$) space with more than one point can be supertight (resp. tight, subtight). However, a $T_1$ ($T_2$) space can be tight (subtight). For example any infinite set with the cofinite topology is both $T_1$ and tight while Bing's example of a countable, connected, Hausdorff space ([2, p. 474]) is both $T_2$ and subtight. Spaces with generic points (e.g. spectra of rings with prime null radicals ([9, p. 82])) are also tight but are not $T_1$ if they contain more than one point.

Although 5.1 may be used, in conjuction with 3.6, to obtain homology theories on Top, we consider, instead, its application in the study of $\mathrm{Sets}^C$.

For $C \in \mathrm{Cat} = \mathrm{Cat}\,(\mathrm{Sets})$ let $\tau(C) = \{A \mid A \subset C_0$, for all $(f : a \to b) \in C_1$, if $a \in A$ then $b \in A\}$. It is not difficult to see that $\tau(C)$ is a topology on $C_0$ and that the correspondence $C \mapsto (C_0, \tau(C))$ defines a functor $\tau : \mathrm{Cat} \to \mathrm{Top}$. Moreover, each $a \in C_0$ has a minimum neighborhood $m(a) = \{b \mid$ there is $(f : a \to b) \in C_1\}$ in $\tau(C)$. Recall ([12, p. 49]) that objects $F$ of $\mathrm{Sets}^C$ correspond to discrete opfibrations where the total category $F^*$ of $F$ has the following description: its objects are the pairs $(a, x)$ where $x \in F(a)$, $a \in C_0$, and its maps $(a, x) \to (b, y)$ are the maps $f : a \to b$ in $C$ for which $F(f)(x) = y$. Moreover, $F \mapsto F^*$ defines a functor $\mathrm{Sets} \to \mathrm{Cat}$. Composing this with $\tau$ we obtain a functor $t : \mathrm{Sets}^C \to \mathrm{Top}$ that figures in the following characterization of tight objects in $\mathrm{Sets}^C$.

**Lemma 5.2.** *Let $C \in \mathrm{Cat}$. $F \in \mathrm{Sets}^C$ is supertight (tight, subtight) iff $t(F)$ is supertight (tight, subtight) in* Top.

Proof: Clearly there is a bijective correspondence between subobjects $A$ of $F$ and subcategories $A^*$ of $F^*$ that satisfy the following condition:

(1) For any $(f : \alpha \to \beta) \in F^*$, if $\alpha \in A^*$ then $f \in A^*$. Further, since $\neg A$ corresponds to the full subcategory with $(\neg A)^*_0 = \{\alpha \mid$ there is no $(\alpha \to \beta) \in F^*$ with $\beta \in A^*\}$ of $F^*$ (it is easily seen that $(\neg A)^*$ is the largest subcategory of $F^*$ that is disjoint from $A^*$ and satisfies (1)) it follows that $A$ is a regular subobject of $F$ (i.e. $\neg\neg A = A$) iff $A^*$ satisfies both (1) and the following condition:

(2) $\alpha \in A^*$ iff for all $(\alpha \to \beta) \in F^*$ there is $(\beta \to \gamma) \in F^*$ with $\gamma \in A^*$. Moreover it is readily checked that, under $\tau$, the subcategories of $F^*$ satisfying (1) correspond bijectively to the open subsets of $\tau(F^*) = t(F)$ with those also satisfying (2) corresponding to the regular open subsets (note that (2) $\Leftrightarrow$ ((interior $(\overline{\tau(A^*)})) \subset \tau(A^*))$ since (interior $(\overline{\tau(A^*)})) = \{\alpha \mid m(\alpha) \subset \overline{\tau(A^*)} = \{\beta \mid$ there is $(\beta \to \gamma) \in F^*$ with $\gamma \in A^*\}\})$. The desired result now readily follows. $\square$

The following result leads to a more useful form, 5.4, of 5.2.

**Lemma 5.3.** *Let $C \in \mathrm{Cat}$. $\tau(C)$ is subtight (resp. tight, supertight) in* Top *iff every ordered pair $(a, b)$ of objects in $C$ can be embedded in a diagram of the form $a \to c_0 \leftarrow c_1 \to c_2 \leftarrow b$ (resp. $a \to c_0 \leftarrow c_1 \to b$, $a \leftarrow c_1 \to b$).*

Proof: It follows from 5.1 that $\tau(C)$ is subtight (tight, supertight) iff for any pair $a, b$, $\overline{m(a)} \cap \overline{m(b)} \neq \emptyset$ ($\overline{m(a)} \cap \overline{\{b\}} \neq \emptyset$, $\overline{\{a\}} \cap \overline{\{b\}} \neq \emptyset$) where $m(x) = \{y \mid$ there is $x \to y$ in $C\}$ is the minimum neighborhood of $x$ in $\tau(C)$. However, $\overline{m(a)} \cap \overline{m(b)} \neq \emptyset$ iff there is $c_1 \in \overline{m(a)} \cap \overline{m(b)}$ iff there are $c_0, c_1, c_2$ in $C$ with $c_0 \in m(c_1) \cap m(a)$ and

$c_2 \in m(c_1) \cap m(b)$ iff there are diagrams $a \to c_0 \leftarrow c_1$ and $c_1 \to c_2 \leftarrow b$. The result now follows since in the case $\overline{m(a)} \cap \overline{\{b\}} \neq \emptyset$ ($\overline{\{a\}} \cap \overline{\{b\}} \neq \emptyset$) we can clearly take $c_2 = b$ ($c_2 = b$ and $c_0 = a$). It follows from 5.3 that $\tau(C)$ is supertight iff $C^{op}$ is directed in the sense of [18, p. 207] and that $\tau(C)$ is tight iff $\tau(C^{op})$ is tight. Thus, for example, $\tau(C)$ is supertight if $C$ has binary products or an initial object while it is tight if $C$ has either binary products or coproducts or initial or terminal objects. □

**Theorem 5.4.** *Let $C \in \mathrm{Cat}$. A cosimplicial object $F_*$ in $\mathrm{Sets}^C$ is $n$-supertight ($n$-tight, $n$-subtight) iff for all pairs $x \in F_n(a)$, $Y \in F_n(b)$ there are maps $a \xleftarrow{g} c_1 \xrightarrow{h} b$ ($a \xrightarrow{f} c_0 \xleftarrow{g} c_1 \xrightarrow{h} b$, $a \xrightarrow{f} c_0 \xleftarrow{g} c_1 \xrightarrow{h} c_2 \xleftarrow{k} b$) in $C$ and $z \in F_n(c_1)$ with $F_n(g)(z) = x$ ($= F_n(f)(x), = F_n(f)(x)$) and $F_n(h)(z) = y$ ($= y, = F_n(k)(y)$).*

Proof: This is a direct consequence of 5.2, 5.3, the definition of the total category $F_n^*$ of $F_n$, and the fact that $t(F_n) = \tau(F_n^*)$. □

As a first application of 5.4 note that iff the category $C$ has a terminal object and $F \in \mathrm{Sets}^C$ preserves it, then $F^*$ has a terminal and therefore $\tau(F^*)$ is tight. Consequently any cosimplicial object in $(\mathrm{Sets}^C)_*$, the full subcategory of $\mathrm{Sets}^C$ consisting of the terminal preserving functors, is $\infty$-tight in $\mathrm{Sets}^C$. In particular if $C = O(B)^{op}$, where $O(B)$ is the category (lattice) of open subsets of a topological space $B$, i.e. if $\mathrm{Sets}^C$ is the category of presheaves on $B$, then the embedding $\Gamma : \mathrm{Shv}(B) \to \mathrm{Sets}^C$ that identifies a sheaf with its presheaf of local sections factors through $(\mathrm{Sets}^C)_*$ (the terminal of $C$ is the empty set $\emptyset$ and there is a unique section over $\emptyset$). As a consequence we have the following:

**Corollary 5.5.** *Let $B$ be a topological space. Any cosimplicial sheaf on $B$ is $\infty$-tight in the topos of presheaves on $B$.*

For another application of 5.4 note that $\mathrm{Cat}(\mathrm{Sets}^C) \approx \mathrm{Cat}^C$ and consequently internal (bounded) categories $K$ in $\mathrm{Sets}^C$ correspond to functors $K : C \to \mathrm{Cat}$. An interpretation of 5.4, noting that the nerve of $K$ at $c \in C$ is given by $\mathrm{Ner}(K)_n(c) = \{S_c \mid S_c = n\text{-string of maps in } K(c)\}$, readily gives the following:

**Corollary 5.6.** *A (bounded) category $K : C \to \mathrm{Cat}$ in $\mathrm{Sets}^C$ has an $\infty$-subtight nerve iff for any integer $n \geq 0$ and any pair $S_a, S_b$ of $n$-strings of maps in $K(a), K(b)$, $a, b \in C$, respectively, there are maps $a \xrightarrow{f} c_0 \xleftarrow{g} c_1 \xrightarrow{h} c_2 \xleftarrow{k} b$ in $C$ and an $n$-string $S$ in $K(c_1)$ with $K(f)(S_a) = K(g)(S)$ and $K(h)(S) = K(k)(S_b)$. The nerve of $K$ is $\infty$-tight ($\infty$-supertight) iff these conditions hold for $f = \mathrm{id}_a$ ($F = \mathrm{id}_a$, $k = \mathrm{id}_b$).*

Corollary 5.6 together with 4.1 can be used to obtain examples of toposes that admit nontrivial (singular) homology theories. It is not difficult, for example, to contrive small subcategories $C$ of $\mathrm{Cat}_*$, the category with the small bounded categories as objects, and with initial and terminal object preserving functors as morphisms, for which the nerve of the bounded internal category in $\mathrm{Sets}^C$, corresponding to the inclusion $C \subset \mathrm{Cat}_* \subset \mathrm{Cat}$, is $n$-(super, sub) tight. In particular the inclusion $C \subset \mathrm{Int} \subset \mathrm{Cat}_*$ of any small full subcategory $C$ of $\mathrm{Int}$, the category of intervals in

Sets, defines an interval $I \in \mathrm{Int}\,(\mathrm{Sets}^C) \approx \mathrm{Int}^{\,C}$. Moreover if $C$ contains an interval $I_0$ with at least $n+3$ points then $I$ has an $n$-supertight nerve. This readily follows from 5.4, as in the proof of 5.6, where, for intervals, an $n$-string $S_c$ is simply an ordered tuple $(t_0 \leq \cdots \leq t_n)$ of points of $I(c)$. Indeed if $t_* = (t_0 \leq \cdots \leq t_n)$ and $t'_* = (t'_0 \leq \cdots \leq t'_n)$ are ordered tuples in $I_1, I_2 \in C$, respectively, then there is a tuple $s_* = (s_0 < \cdots < s_n)$ in $I_0$ with neither $s_0$ nor $s_n$ an endpoint. Clearly there are maps $I_1 \leftarrow I_0 \rightarrow I_2$ in $C$ carrying $s_*$ to $t_*$ and to $t'_*$, respectively. We have thus proved the following:

**Corollary 5.7.** *Any small full subcategory $C$ of* Int *that contains an interval with at least $n+3$ points determines a topos* $\mathrm{Sets}^C$ *that admits an interval with an $n$-supertight nerve.*

For an application of 5.7 recall that the topos $\mathrm{Sets}^M$ of $M$-sets, where $M$ is monoid, is Boolean iff $M$ in a group ([12, 5.15 (ii)]). Hence, by 2.5, if $M$ is a group then $\mathrm{Sets}^M$ admits no nontrivial homology (singular or otherwise) for any interval based homotopy structure. However, if $M$ is the monoid of all endomorphisms (in Int ) of an infinite interval then, by 5.7, $\mathrm{Sets}^M$ admits an interval with $\infty$-supertight nerve. This and 4.1 gives the following:

**Corollary 5.8.** *If $M$ is the monoid of all interval endomorphisms of an infinite interval then the topos of $M$-sets admits a nontrivial homology theory relative to an interval based homotopy and for which any pair $Y_0, Y_1$ of sub $M$-sets of an $M$-set $Y$ form an excisive couple, i.e. the inclusion $(Y_0, Y_0 \cap Y_1) \rightarrow (Y_0 \cup Y_1, Y_1)$ induces an isomorphism on homology.*

For another application of 5.7 recall, as noted in the discussion preceding 3.7, that the embedding $\varepsilon : C = \Delta^{op} \rightarrow \mathrm{Int}$ identifies $C$ with the full subcategory of Int determined by the intervals $[n]$, $n = 1, 2 \ldots$ . Thus, by 5.7, $\varepsilon$ determines an interval in Simpl (Sets) with $\infty$-supertight nerve. In fact this interval is equivalent to the standard simplicial interval $\Delta_1 = \Delta(-, [1])$ and the associated cosimplicial object $\Delta \rightarrow$ Simpl (Sets) is equivalent to the right Yoneda functor $R_*$, where $R_n = \Delta(-, [n])$. From this, 4.1, and Proposition 6.2 [19, p. 16] we have the following:

**Corollary 5.9.** *The right Yoneda functor $R_* : \Delta \rightarrow$* Simpl (Sets) *defines a nontrivial homology theory in* Simpl (Sets) *relative to simplicial homotopy and for which any pair of subsimplicial sets of a simplicial set forms an excisive couple.*

We conclude with the observation that the $R_*$-singular homology coincides with the classical homology of simplicial sets. To see this recall ([16, p. 98]) that the classical homology of a simplicial set $Y$ is defined as the homology of the chain complex $C(Y)$ (or equivalently, by the normalization theorem ([17, p. 236]), of the associated normalized chain complex), where $C_n(Y)$ is the free abelian group generated by $Y_n$, while the $R_*$-singular homology of $Y$ is determined by the chain complex $CF^*S(R_*, Y)$. The equivalence of the two homologies is a consequence of the Yoneda lemma ([18, p. 61]) which implies that $S(R_*, Y)$ and $Y$ are isomorphic simplicial sets.

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